

# Chapter 9

## Maintaining Mathematical Proficiency

Order the expressions by value from least to greatest.

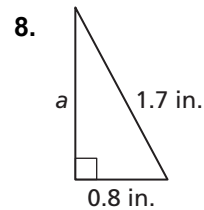
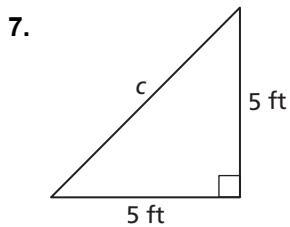
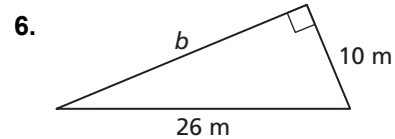
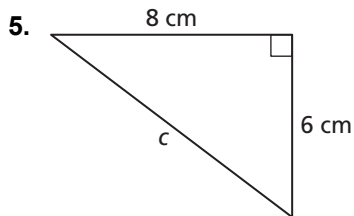
1.  $|-8|, |1 + 4|, |3 - 7|, -|2|$

2.  $|8 - 2|, |3 - 6|, |0|, \left|-\frac{3}{2}\right|$

3.  $|-2^2|, |-9 - 1|, |3 \cdot (-3)|, |-2| + |-1| - |3|$

4.  $|-5 + 15|, |3| - |3 \cdot 4|, |3 - 7|, -|3^3|$

Find the missing side length of the triangle.



**9.1****Right Triangle Trigonometry**

For use with Exploration 9.1

**Essential Question** How can you find a trigonometric function of an acute angle  $\theta$ ?

Consider one of the acute angles  $\theta$  of a right triangle. Ratios of a right triangle's side lengths are used to define the six *trigonometric functions*, as shown.

**Sine**  $\sin \theta = \frac{\text{opp.}}{\text{hyp.}}$

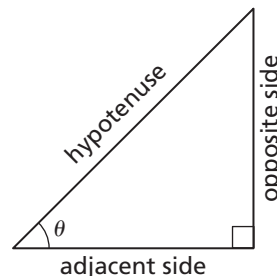
**Cosine**  $\cos \theta = \frac{\text{adj.}}{\text{hyp.}}$

**Tangent**  $\tan \theta = \frac{\text{opp.}}{\text{adj.}}$

**Cotangent**  $\cot \theta = \frac{\text{adj.}}{\text{opp.}}$

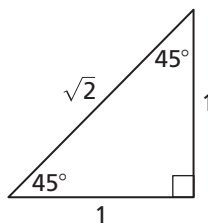
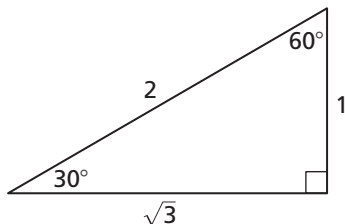
**Secant**  $\sec \theta = \frac{\text{hyp.}}{\text{adj.}}$

**Cosecant**  $\csc \theta = \frac{\text{hyp.}}{\text{opp.}}$



**1 EXPLORATION:** Trigonometric Functions of Special Angles

**Work with a partner.** Find the exact values of the sine, cosine, and tangent functions for the angles  $30^\circ$ ,  $45^\circ$ , and  $60^\circ$  in the right triangles shown.



**9.1** Right Triangle Trigonometry (continued)**2** **EXPLORATION:** Exploring Trigonometric Identities

Work with a partner.

Use the definitions of the trigonometric functions to explain why each *trigonometric identity* is true.

a.  $\sin \theta = \cos(90^\circ - \theta)$

b.  $\cos \theta = \sin(90^\circ - \theta)$

c.  $\sin \theta = \frac{1}{\csc \theta}$

d.  $\tan \theta = \frac{1}{\cot \theta}$

Use the definitions of the trigonometric functions to complete each trigonometric identity.

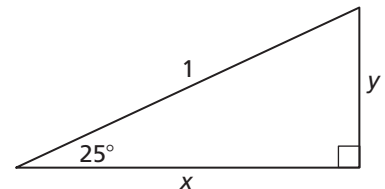
e.  $(\sin \theta)^2 + (\cos \theta)^2 = \underline{\hspace{2cm}}$

f.  $(\sec \theta)^2 - (\tan \theta)^2 = \underline{\hspace{2cm}}$

**Communicate Your Answer**

3. How can you find a trigonometric function of an acute angle  $\theta$ ?

4. Use a calculator to find the lengths  $x$  and  $y$  of the legs of the right triangle shown.



**9.1****Notetaking with Vocabulary**

For use after Lesson 9.1

In your own words, write the meaning of each vocabulary term.

sine

cosine

tangent

cosecant

secant

cotangent

**Core Concepts****Right Triangle Definitions of Trigonometric Functions**

Let  $\theta$  be an acute angle of a right triangle. The six trigonometric functions of  $\theta$  are defined as shown.

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\csc \theta = \frac{\text{hypotenuse}}{\text{opposite}} \quad \sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}} \quad \cot \theta = \frac{\text{adjacent}}{\text{opposite}}$$

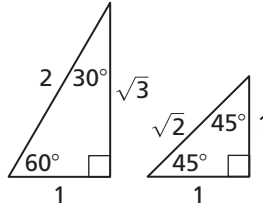
The abbreviations *opp.*, *adj.*, and *hyp.* are often used to represent the side lengths of the right triangle. Note that the ratios in the second row are reciprocals of the ratios in the first row.

$$\csc \theta = \frac{1}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \cot \theta = \frac{1}{\tan \theta}$$

**Notes:**

**9.1** Notetaking with Vocabulary (continued)**Trigonometric Values for Special Angles**

The table gives the values of the six trigonometric functions for the angles  $30^\circ$ ,  $45^\circ$ , and  $60^\circ$ . You can obtain these values from the triangles shown.

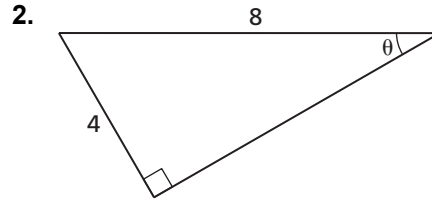
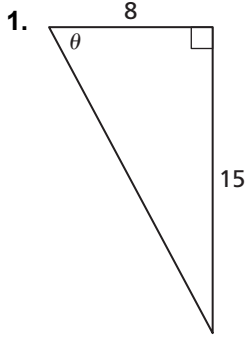


$\theta$	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\cot \theta$
$30^\circ$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$45^\circ$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60^\circ$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

**Notes:**

**9.1** Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1 and 2, evaluate the six trigonometric functions of the angle  $\theta$ .

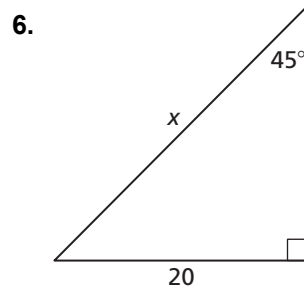
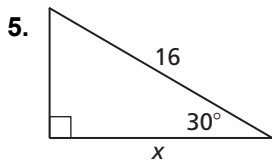


In Exercises 3 and 4, let  $\theta$  be an acute angle of a right triangle. Evaluate the other five trigonometric functions of  $\theta$ .

3.  $\tan \theta = 1$

4.  $\sin \theta = \frac{3}{19}$

In Exercises 5 and 6, find the value of  $x$  for the right triangle.



# 9.2

## Angles and Radian Measure

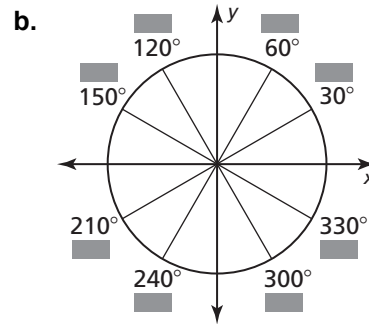
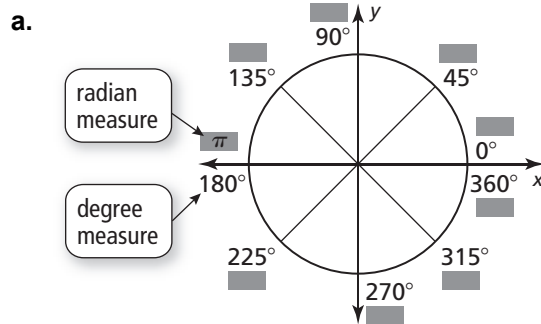
For use with Exploration 9.2

**Essential Question** How can you find the measure of an angle in radians?

Let the vertex of an angle be at the origin, with one side of the angle on the positive  $x$ -axis. The *radian measure* of the angle is a measure of the intercepted arc length on a circle of radius 1. To convert between degree and radian measure, use the fact that  $\frac{\pi \text{ radians}}{180^\circ} = 1$ .

### 1 EXPLORATION: Writing Radian Measures of Angles

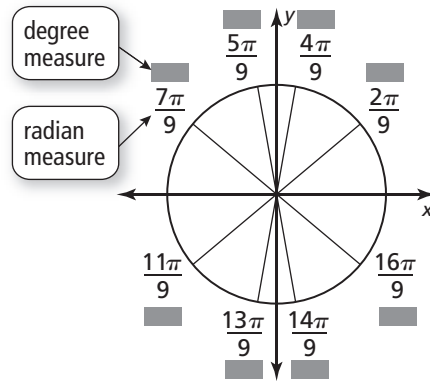
**Work with a partner.** Write the radian measure of each angle with the given degree measure. Explain your reasoning.



**9.2 Angles and Radian Measure (continued)**

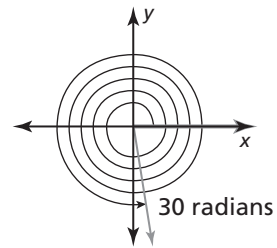
**2 EXPLORATION: Writing Degree Measures of Angles**

**Work with a partner.** Write the degree measure of each angle with the given radian measure. Explain your reasoning.



**Communicate Your Answer**

3. How can you find the measure of an angle in radians?
  
4. The figure shows an angle whose measure is 30 radians. What is the measure of the angle in degrees? How many times greater is 30 radians than 30 degrees? Justify your answers.





**9.2****Notetaking with Vocabulary**

For use after Lesson 9.2

In your own words, write the meaning of each vocabulary term.

initial side

terminal side

standard position

coterminal

radian

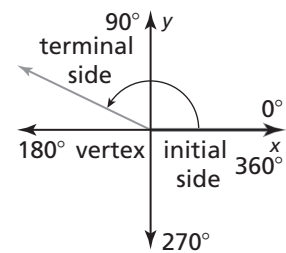
sector

central angle

**Core Concepts****Angles in Standard Position**

In a coordinate plane, an angle can be formed by fixing one ray, called the **initial side**, and rotating the other ray, called the **terminal side**, about the vertex.

An angle is in **standard position** when its vertex is at the origin and its initial side lies on the positive  $x$ -axis.

**Notes:**

**9.2** Notetaking with Vocabulary (continued)

**Converting Between Degrees and Radians**

**Degrees to radians**

Multiply degree measure by

$$\frac{\pi \text{ radians}}{180^\circ}$$

**Radians to degrees**

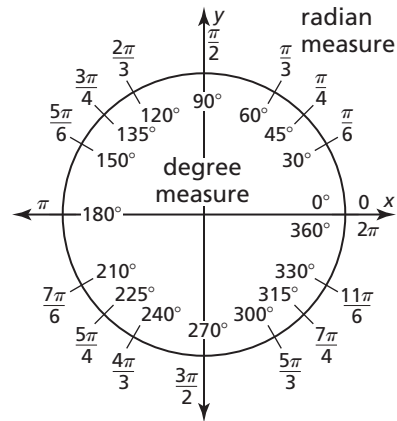
Multiply radian measure by

$$\frac{180^\circ}{\pi \text{ radians}}$$

**Degree and Radian Measures of Special Angles**

The diagram shows equivalent degree and radian measures for special angles from  $0^\circ$  to  $360^\circ$  (0 radians to  $2\pi$  radians).

You may find it helpful to memorize the equivalent degree and radian measures of special angles in the first quadrant and for  $90^\circ = \frac{\pi}{2}$  radians. All other special angles shown are multiples of these angles.

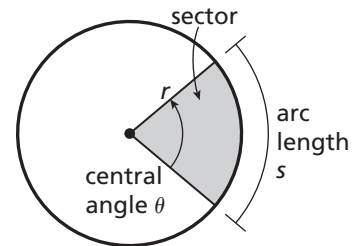


**Arc Length and Area of a Sector**

The arc length  $s$  and area  $A$  of a sector with radius  $r$  and central angle  $\theta$  (measured in radians) are as follows.

**Arc length:**  $s = r\theta$

**Area:**  $A = \frac{1}{2}r^2\theta$

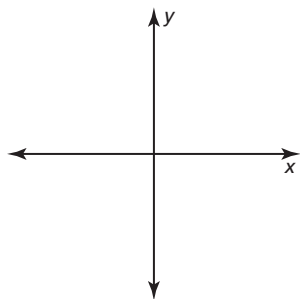


**Notes:**

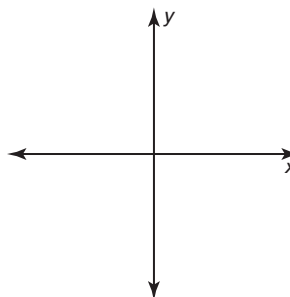
**9.2** Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1 and 2, draw an angle with the given measure in standard position.

1.  $260^\circ$



2.  $-750^\circ$



In Exercises 3–6, find one positive angle and one negative angle that are coterminal with the given angle.

3.  $55^\circ$

4.  $-300^\circ$

5.  $460^\circ$

6.  $-220^\circ$

In Exercises 7–10, convert the degree measure to radians or the radian measure to degrees.

7.  $54^\circ$

8.  $-310^\circ$

9.  $\frac{16\pi}{15}$

10.  $-\frac{2\pi}{5}$

# 9.3

## Trigonometric Functions of Any Angle

For use with Exploration 9.3

**Essential Question** How can you use the unit circle to define the trigonometric functions of any angle?

Let  $\theta$  be an angle in standard position with  $(x, y)$  a point on the terminal side of  $\theta$  and  $r = \sqrt{x^2 + y^2} \neq 0$ . The six trigonometric functions of  $\theta$  are defined as shown.

$$\sin \theta = \frac{y}{r}$$

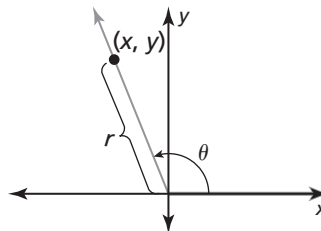
$$\csc \theta = \frac{r}{y}, y \neq 0$$

$$\cos \theta = \frac{x}{r}$$

$$\sec \theta = \frac{r}{x}, x \neq 0$$

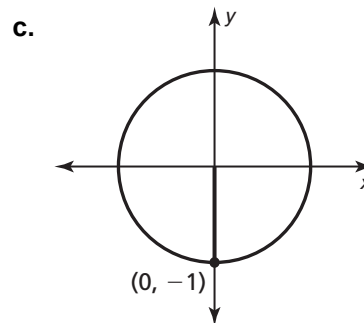
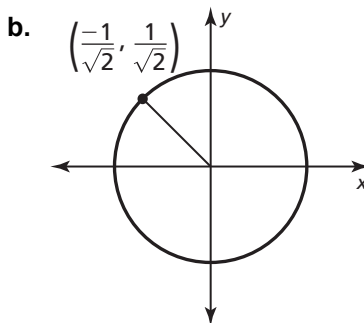
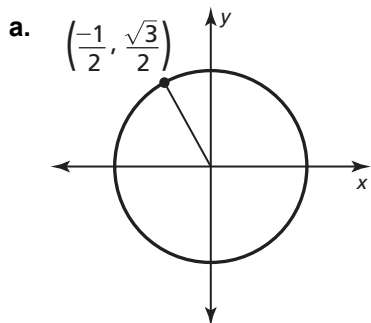
$$\tan \theta = \frac{y}{x}, x \neq 0$$

$$\cot \theta = \frac{x}{y}, y \neq 0$$



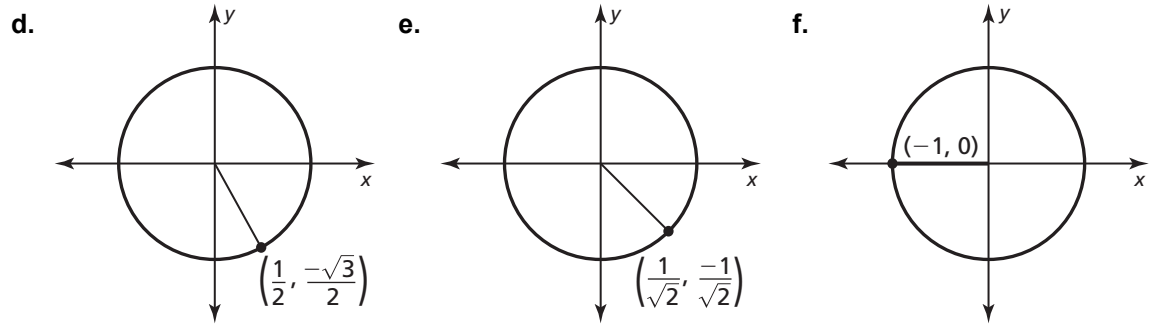
### 1 EXPLORATION: Writing Trigonometric Functions

**Work with a partner.** Find the sine, cosine, and tangent of the angle  $\theta$  in standard position whose terminal side intersects the unit circle at the point  $(x, y)$  shown.



**9.3** Trigonometric Functions of Any Angle (continued)

**1** **EXPLORATION:** Writing Trigonometric Functions (continued)



**Communicate Your Answer**

2. How can you use the unit circle to define the trigonometric functions of any angle?
  
3. For which angles are each function undefined? Explain your reasoning.
  - a. tangent
  - b. cotangent
  - c. secant
  - d. cosecant

**9.3****Notetaking with Vocabulary**

For use after Lesson 9.3

In your own words, write the meaning of each vocabulary term.

unit circle

quadrantal angle

reference angle

**Core Concepts****General Definitions of Trigonometric Functions**

Let  $\theta$  be an angle in standard position, and let  $(x, y)$  be the point where the terminal side of  $\theta$  intersects the circle  $x^2 + y^2 = r^2$ . The six trigonometric functions of  $\theta$  are defined as shown.

$$\sin \theta = \frac{y}{r} \qquad \csc \theta = \frac{r}{y}, y \neq 0$$

$$\cos \theta = \frac{x}{r} \qquad \sec \theta = \frac{r}{x}, x \neq 0$$

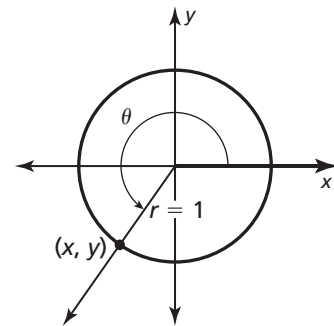
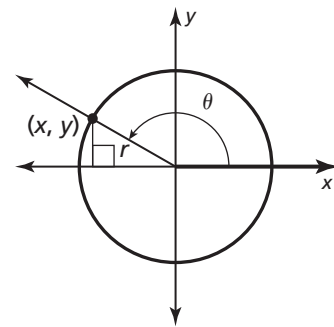
$$\tan \theta = \frac{y}{x}, x \neq 0 \qquad \cot \theta = \frac{x}{y}, y \neq 0$$

These functions are sometimes called *circular functions*.

**The Unit Circle**

The circle  $x^2 + y^2 = 1$ , which has center  $(0, 0)$  and radius 1, is called the **unit circle**. The values of  $\sin \theta$  and  $\cos \theta$  are simply the  $y$ -coordinate and  $x$ -coordinate, respectively, of the point where the terminal side of  $\theta$  intersects the unit circle.

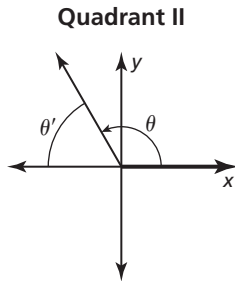
$$\sin \theta = \frac{y}{r} = \frac{y}{1} = y \qquad \cos \theta = \frac{x}{r} = \frac{x}{1} = x$$

**Notes:**

**9.3** Notetaking with Vocabulary (continued)

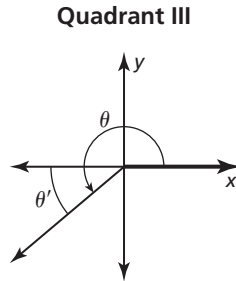
**Reference Angle Relationships**

Let  $\theta$  be an angle in standard position. The **reference angle** for  $\theta$  is the acute angle  $\theta'$  formed by the terminal side of  $\theta$  and the  $x$ -axis. The relationship between  $\theta$  and  $\theta'$  is shown below for nonquadrantal angles  $\theta$  such that  $90^\circ < \theta < 360^\circ$  or, in radians,  $\frac{\pi}{2} < \theta < 2\pi$ .



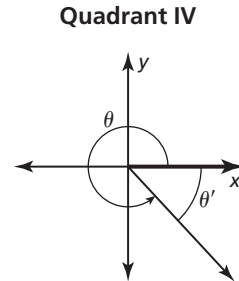
Degrees:  $\theta' = 180^\circ - \theta$

Radians:  $\theta' = \pi - \theta$



Degrees:  $\theta' = \theta - 180^\circ$

Radians:  $\theta' = \theta - \pi$



Degrees:  $\theta' = 360^\circ - \theta$

Radians:  $\theta' = 2\pi - \theta$

**Notes:**

**Evaluating Trigonometric Functions**

Use these steps to evaluate a trigonometric function for any angle  $\theta$ :

**Step 1** Find the reference angle  $\theta'$ .

**Step 2** Evaluate the trigonometric function for  $\theta'$ .

**Step 3** Determine the sign of the trigonometric function value from the quadrant in which  $\theta$  lies.

**Notes:**

**Signs of Function Values**

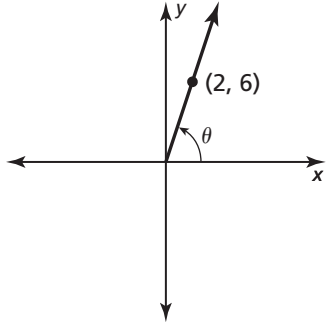
Quadrant II	$y$	Quadrant I
sin $\theta$ , csc $\theta$ : +		sin $\theta$ , csc $\theta$ : +
cos $\theta$ , sec $\theta$ : -		cos $\theta$ , sec $\theta$ : +
tan $\theta$ , cot $\theta$ : -		tan $\theta$ , cot $\theta$ : +
←	+	→
Quadrant III	-	Quadrant IV
sin $\theta$ , csc $\theta$ : -		sin $\theta$ , csc $\theta$ : -
cos $\theta$ , sec $\theta$ : -		cos $\theta$ , sec $\theta$ : +
tan $\theta$ , cot $\theta$ : +		tan $\theta$ , cot $\theta$ : -
-	-	-
x	x	x

**9.3** Notetaking with Vocabulary (continued)

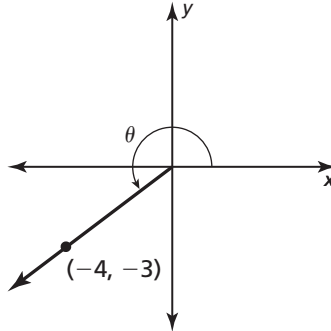
**Extra Practice**

In Exercises 1 and 2, evaluate the six trigonometric functions of  $\theta$ .

1.



2.



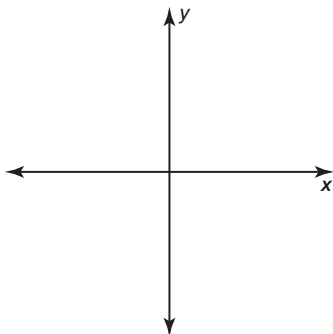
In Exercises 3 and 4, use the unit circle to evaluate the six trigonometric functions of  $\theta$ .

3.  $\theta = -90^\circ$

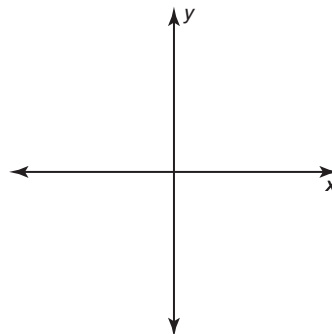
4.  $\theta = 4\pi$

In Exercises 5 and 6, sketch the angle. Then find its reference angle.

5.  $-310^\circ$



6.  $\frac{27\pi}{10}$



7. Evaluate the function  $\csc 150^\circ$  without using a calculator.



# 9.4

## Graphing Sine and Cosine Functions

For use with Exploration 9.4

**Essential Question** What are the characteristics of the graphs of the sine and cosine functions?

### 1 EXPLORATION: Graphing the Sine Function

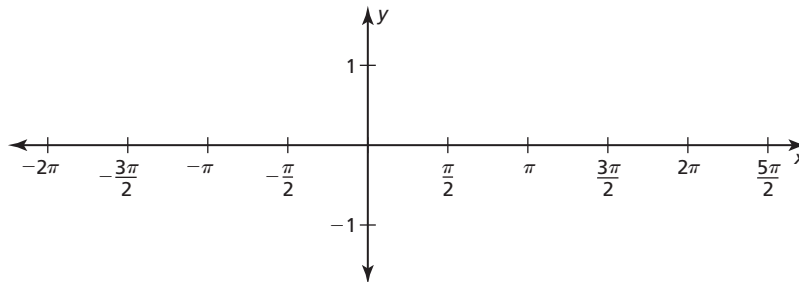
Go to *BigIdeasMath.com* for an interactive tool to investigate this exploration.

Work with a partner.

- a. Complete the table for  $y = \sin x$ , where  $x$  is an angle measure in radians.

$x$	$-2\pi$	$-\frac{7\pi}{4}$	$-\frac{3\pi}{2}$	$-\frac{5\pi}{4}$	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	$0$
$y = \sin x$									
$x$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	$2\pi$	$\frac{9\pi}{4}$
$y = \sin x$									

- b. Plot the points  $(x, y)$  from part (a). Draw a smooth curve through the points to sketch the graph of  $y = \sin x$ .



- c. Use the graph to identify the  $x$ -intercepts, the  $x$ -values where the local maximums and minimums occur, and the intervals for which the function is increasing or decreasing over  $-2\pi \leq x \leq 2\pi$ . Is the sine function *even*, *odd*, or *neither*?

**9.4 Graphing Sine and Cosine Functions (continued)****2 EXPLORATION:** Graphing the Cosine Function

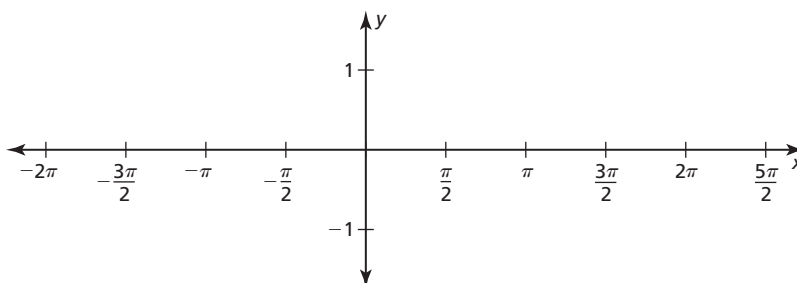
Go to *BigIdeasMath.com* for an interactive tool to investigate this exploration.

Work with a partner.

- a. Complete the table for  $y = \cos x$  using the same values of  $x$  as those used in Exploration 1.

$x$	$-2\pi$	$-\frac{7\pi}{4}$	$-\frac{3\pi}{2}$	$-\frac{5\pi}{4}$	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	$0$
$y = \cos x$									
$x$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	$2\pi$	$\frac{9\pi}{4}$
$y = \cos x$									

- b. Plot the points  $(x, y)$  from part (a) and sketch the graph of  $y = \cos x$



- c. Use the graph to identify the  $x$ -intercepts, the  $x$ -values where the local maximums and minimums occur, and the intervals for which the function is increasing or decreasing over  $-2\pi \leq x \leq 2\pi$ . Is the cosine function *even*, *odd*, or *neither*?

**Communicate Your Answer**

3. What are the characteristics of the graphs of the sine and cosine functions?
4. Describe the end behavior of the graph of  $y = \sin x$ .

**9.4****Notetaking with Vocabulary**

For use after Lesson 9.4

In your own words, write the meaning of each vocabulary term.

amplitude

periodic function

cycle

period

phase shift

midline

**Core Concepts****Characteristics of  $y = \sin x$  and  $y = \cos x$** 

- The domain of each function is all real numbers.
- The range of each function is  $-1 \leq y \leq 1$ . So, the minimum value of each function is  $-1$  and the maximum value is  $1$ .
- The **amplitude** of the graph of each function is one-half of the difference of the maximum value and the minimum value, or  $\frac{1}{2}[1 - (-1)] = 1$ .
- Each function is **periodic**, which means that its graph has a repeating pattern. The shortest repeating portion of the graph is called a **cycle**. The horizontal length of each cycle is called the **period**. The graph of each function has a period of  $2\pi$ .
- The  $x$ -intercepts for  $y = \sin x$  occur when  $x = 0, \pm\pi, \pm2\pi, \pm3\pi, \dots$
- The  $x$ -intercepts for  $y = \cos x$  occur when  $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \dots$

**9.4 Notetaking with Vocabulary (continued)****Amplitude and Period**

The amplitude and period of the graphs of  $y = a \sin bx$  and  $y = a \cos bx$ , where  $a$  and  $b$  are nonzero real numbers, are as follows:

$$\text{Amplitude} = |a| \qquad \text{Period} = \frac{2\pi}{|b|}$$

**Notes:****Graphing  $y = a \sin b(x - h) + k$  and  $y = a \cos b(x - h) + k$** 

To graph  $y = a \sin b(x - h) + k$  or  $y = a \cos b(x - h) + k$  where  $a > 0$  and  $b > 0$ , follow these steps:

**Step 1** Identify the amplitude  $a$ , the period  $\frac{2\pi}{b}$ , the horizontal shift  $h$ , and the vertical shift  $k$  of the graph.

**Step 2** Draw the horizontal line  $y = k$ , called the **midline** of the graph.

**Step 3** Find the five key points by translating the key points of  $y = a \sin bx$  or  $y = a \cos bx$  horizontally  $h$  units and vertically  $k$  units.

**Step 4** Draw the graph through the five translated key points.

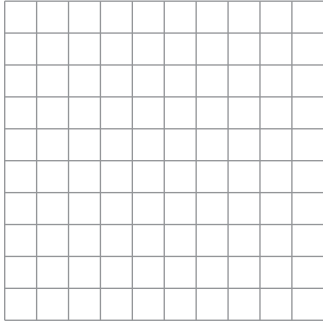
**Notes:**

**9.4** Notetaking with Vocabulary (continued)

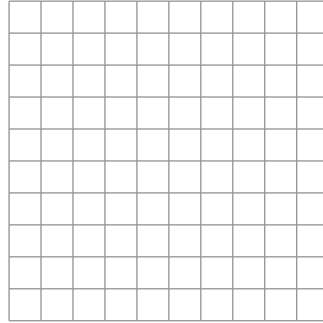
**Extra Practice**

In Exercises 1–4, identify the amplitude and period of the function. Then graph the function and describe the graph of  $g$  as a transformation of the graph of its parent function.

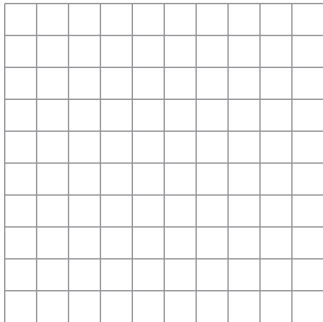
1.  $g(x) = \sin 2x$



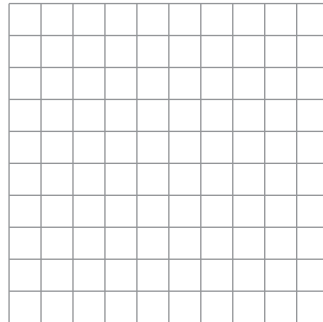
2.  $g(x) = \frac{1}{3} \cos 2x$



3.  $g(x) = 4 \sin 2\pi x$

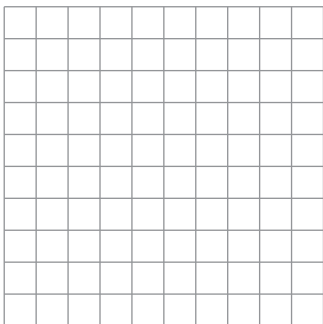


4.  $g(x) = \frac{1}{2} \cos 3\pi x$

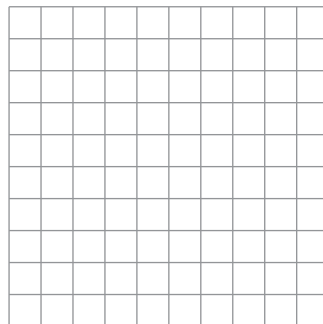


In Exercises 5 and 6, graph the function.

5.  $g(x) = \sin \frac{1}{2}(x - \pi) + 1$



6.  $g(x) = \cos \left( x + \frac{\pi}{2} \right) - 3$



# 9.5

## Graphing Other Trigonometric Functions

For use with Exploration 9.5

**Essential Question** What are the characteristics of the graph of the tangent function?

### 1 EXPLORATION: Graphing the Tangent Function

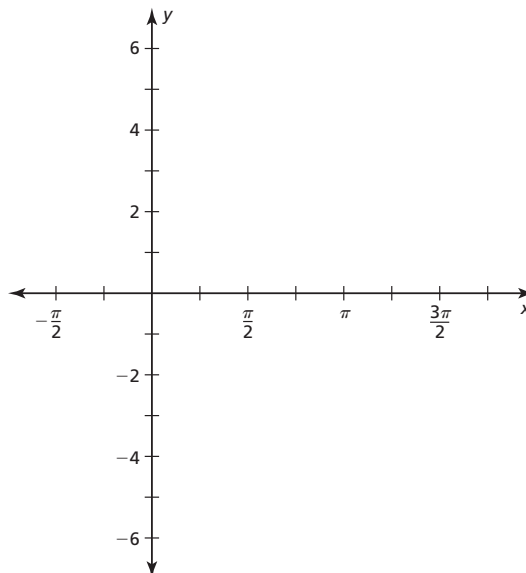
Go to *BigIdeasMath.com* for an interactive tool to investigate this exploration.

Work with a partner.

- a. Complete the table for  $y = \tan x$ , where  $x$  is an angle measure in radians.

$x$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y = \tan x$									
$x$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\pi$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$
$y = \tan x$									

- b. The graph of  $y = \tan x$  has vertical asymptotes at  $x$ -values where  $\tan x$  is undefined. Plot the points  $(x, y)$  from part (a). Then use the asymptotes to sketch the graph of  $y = \tan x$ .



**9.5** Graphing Other Trigonometric Functions (continued)**1** **EXPLORATION:** Graphing the Tangent Function (continued)

- c. For the graph of  $y = \tan x$ , identify the asymptotes, the  $x$ -intercepts, and the intervals for which the function is increasing or decreasing over  $-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$ . Is the tangent function *even*, *odd*, or *neither*?

**Communicate Your Answer**

2. What are the characteristics of the graph of the tangent function?

3. Describe the asymptotes of the graph of  $y = \cot x$  on the interval  $-\frac{\pi}{2} < x < \frac{3\pi}{2}$ .

**9.5****Notetaking with Vocabulary**

For use after Lesson 9.5

In your own words, write the meaning of each vocabulary term.

asymptote

period

amplitude

 $x$ -intercept

transformations

**Core Concepts****Characteristics of  $y = \tan x$  and  $y = \cot x$** 

The functions  $y = \tan x$  and  $y = \cot x$  have the following characteristics.

- The domain of  $y = \tan x$  is all real numbers except odd multiples of  $\frac{\pi}{2}$ . At these  $x$ -values, the graph has vertical asymptotes.
- The domain of  $y = \cot x$  is all real numbers except multiples of  $\pi$ . At these  $x$ -values, the graph has vertical asymptotes.
- The range of each function is all real numbers. So, the functions do not have maximum or minimum values, and the graphs do not have an amplitude.
- The period of each graph is  $\pi$ .
- The  $x$ -intercepts for  $y = \tan x$  occur when  $x = 0, \pm\pi, \pm2\pi, \pm3\pi, \dots$
- The  $x$ -intercepts for  $y = \cot x$  occur when  $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \dots$

**Notes:**



**9.5** Notetaking with Vocabulary (continued)**Period and Vertical Asymptotes of  $y = a \tan bx$  and  $y = a \cot bx$** 

The period and vertical asymptotes of the graphs of  $y = a \tan bx$  and  $y = a \cot bx$ , where  $a$  and  $b$  are nonzero real numbers, are as follows.

- The period of the graph of each function is  $\frac{\pi}{|b|}$ .
- The vertical asymptotes for  $y = a \tan bx$  are at odd multiples of  $\frac{\pi}{2|b|}$ .
- The vertical asymptotes for  $y = a \cot bx$  are at multiples of  $\frac{\pi}{|b|}$ .

**Notes:**

**Characteristics of  $y = \sec x$  and  $y = \csc x$** 

The functions  $y = \sec x$  and  $y = \csc x$  have the following characteristics.

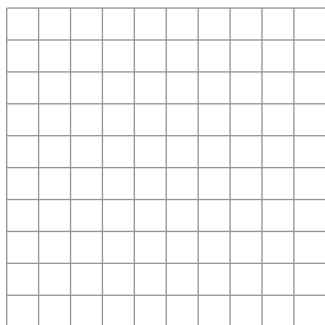
- The domain of  $y = \sec x$  is all real numbers except odd multiples of  $\frac{\pi}{2}$ . At these  $x$ -values, the graph has vertical asymptotes.
- The domain of  $y = \csc x$  is all real numbers except multiples of  $\pi$ . At these  $x$ -values, the graph has vertical asymptotes.
- The range of each function is  $y \leq -1$  and  $y \geq 1$ . So, the graphs do not have an amplitude.
- The period of each graph is  $2\pi$ .

**Notes:**

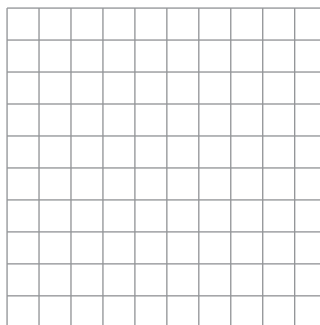
**9.5** Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1–6, graph one period of the function. Describe the graph of  $g$  as a transformation of the graph of its parent function.

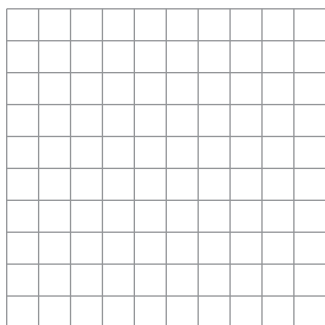
1.  $g(x) = \tan 2x$



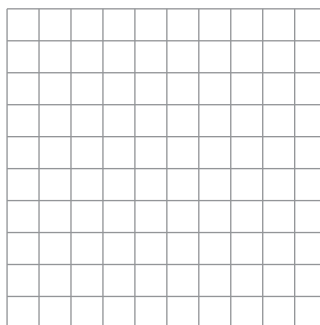
2.  $g(x) = 2 \cot \frac{1}{2}x$



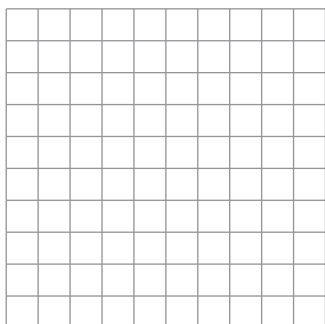
3.  $g(x) = \frac{1}{4} \tan \frac{\pi}{4}x$



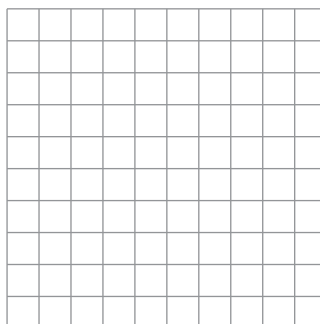
4.  $g(x) = \frac{1}{2} \cot 3x$



5.  $g(x) = 2 \sec 2x$



6.  $g(x) = \csc 2\pi x$



# 9.6

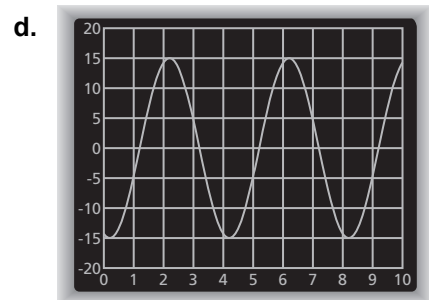
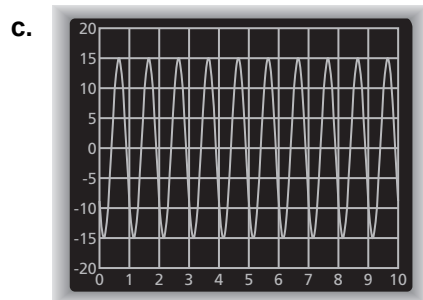
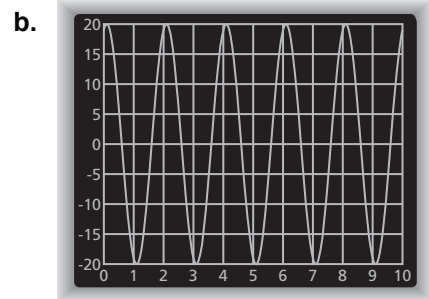
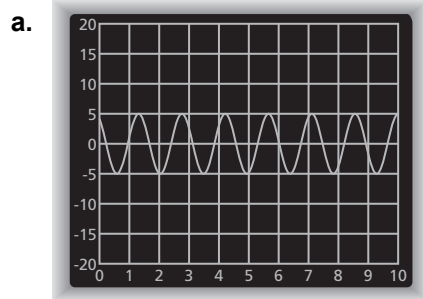
## Modeling with Trigonometric Functions

For use with Exploration 9.6

**Essential Question** What are the characteristics of the real-life problems that can be modeled by trigonometric functions?

### 1 EXPLORATION: Modeling Electric Currents

**Work with a partner.** Find a sine function that models the electric current shown in each oscilloscope screen. State the amplitude and period of the graph.





**9.6**

**Notetaking with Vocabulary**

For use after Lesson 9.6

In your own words, write the meaning of each vocabulary term.

frequency

sinusoid

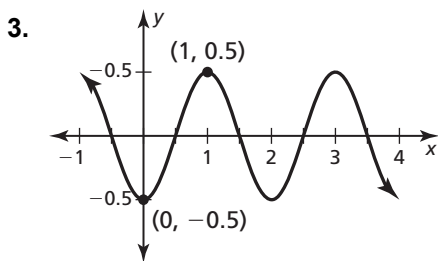
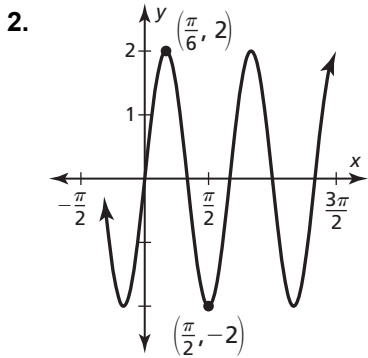
**Notes:**

**9.6** Notetaking with Vocabulary (continued)

**Extra Practice**

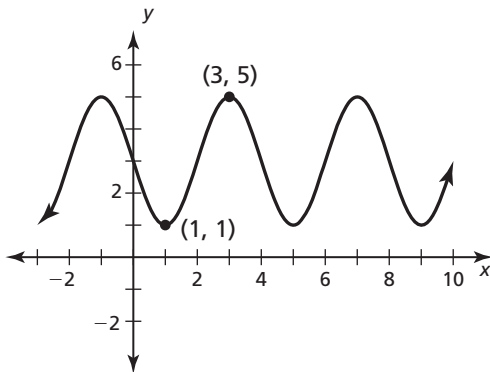
1. An alternating current generator (AC generator) converts motion to electricity by generating sinusoidal voltage. Assuming that there is no vertical offset and phase shift, the voltage oscillates between  $-170$  volts and  $+170$  volts with a frequency of 60 hertz. Write and graph a sine model that gives the voltage  $V$  as a function of the time  $t$  (in seconds).

In Exercises 2–5, write a function for the sinusoid.

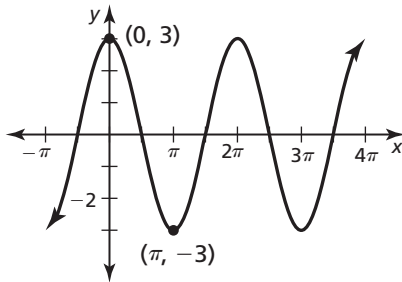


**9.6** Notetaking with Vocabulary (continued)

4.



5.



6. The pedal of a bicycle wheel is 7 inches long. The lowest point of the pedal is 4 inches above the ground. A cyclist pedals 3 revolutions per second. Write a model for the height  $h$  (in inches) of the pedal as a function of the time  $t$  (in seconds) given that the pedal is at its lowest point when  $t = 0$ .

7. The London Eye, the tallest Ferris wheel in Europe, has a diameter of 120 meters and the whole structure is 135 meters tall. The Ferris wheel completes one revolution in about 30 minutes. Write a model for the height  $h$  (in meters) of a passenger capsule as a function of the time  $t$  (in seconds) given that the capsule is at its highest point when  $t = 0$ .

# 9.7

## Using Trigonometric Identities

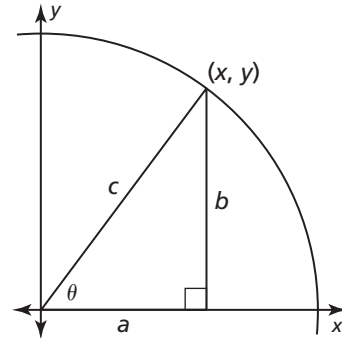
For use with Exploration 9.7

**Essential Question** How can you verify a trigonometric identity?

**1 EXPLORATION:** Writing a Trigonometric Identity

Go to *BigIdeasMath.com* for an interactive tool to investigate this exploration.

**Work with a partner.** In the figure, the point  $(x, y)$  is on a circle of radius  $c$  with center at the origin.



- a. Write an equation that relates  $a$ ,  $b$ , and  $c$ .
- b. Write expressions for the sine and cosine ratios of angle  $\theta$ .
- c. Use the results from parts (a) and (b) to find the sum of  $\sin^2 \theta$  and  $\cos^2 \theta$ . What do you observe?
- d. Complete the table to verify that the identity you wrote in part (c) is valid for angles (of your choice) in each of the four quadrants.

	$\theta$	$\sin^2 \theta$	$\cos^2 \theta$	$\sin^2 \theta + \cos^2 \theta$
QI				
QII				
QIII				
QIV				



**9.7 Using Trigonometric Identities (continued)****2 EXPLORATION:** Writing Other Trigonometric Identities

**Work with a partner.** The trigonometric identity you derived in Exploration 1 is called a Pythagorean identity. There are two other Pythagorean identities. To derive them, recall the four relationships:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta} \qquad \csc \theta = \frac{1}{\sin \theta}$$

- a. Divide each side of the Pythagorean identity you derived in Exploration 1 by  $\cos^2 \theta$  and simplify. What do you observe?
  
  
  
  
  
  
  
  
  
  
- b. Divide each side of the Pythagorean identity you derived in Exploration 1 by  $\sin^2 \theta$  and simplify. What do you observe?

**Communicate Your Answer**

3. How can you verify a trigonometric identity?
  
  
  
  
  
  
  
  
  
  
4. Is  $\sin \theta = \cos \theta$  a trigonometric identity? Explain your reasoning.
  
  
  
  
  
  
  
  
  
  
5. Give some examples of trigonometric identities that are different than those in Explorations 1 and 2.

**9.7****Notetaking with Vocabulary**

For use after Lesson 9.7

In your own words, write the meaning of each vocabulary term.

trigonometric identity

**Core Concepts****Fundamental Trigonometric Identities****Reciprocal Identities**

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

**Tangent and Cotangent Identities**

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

**Pythagorean Identities**

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

**Cofunction Identities**

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$$

**Negative Angle Identities**

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = -\tan \theta$$

**Notes:**

**9.7** Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1–4, find the values of the other five trigonometric functions of  $\theta$ .

1.  $\sin \theta = \frac{1}{5}, \frac{\pi}{2} < \theta < \pi$

2.  $\cos \theta = -\frac{4}{5}, \pi < \theta < \frac{3\pi}{2}$

3.  $\cot \theta = \frac{4}{7}, 0 < \theta < \frac{\pi}{2}$

4.  $\sec \theta = \frac{\sqrt{13}}{3}, \frac{3\pi}{2} < \theta < 2\pi$

In Exercises 5–8, simplify the expression.

5.  $-\frac{\tan \theta}{\sec \theta}$

6.  $\cos\left(\frac{\pi}{2} - \theta\right)(1 - \cos^2 \theta)$

**9.7** Notetaking with Vocabulary (continued)

7. 
$$\frac{2 \sec^2 x - 2 \tan^2 x}{\tan(-x) \cos(-x)}$$

8. 
$$\frac{-\sin\left(\frac{\pi}{2} - \theta\right)}{\sec(-\theta)} - \sin^2 \theta$$

In Exercises 9 and 10, verify the identity.

9. 
$$\frac{1 - \cos^2 \theta}{\sec^2 \theta} - \sin^2 \theta = -\sin^4 \theta$$

10. 
$$\csc x + \cot x = \frac{\sin x}{1 - \cos x}$$

**9.8****Using Sum and Difference Formulas**

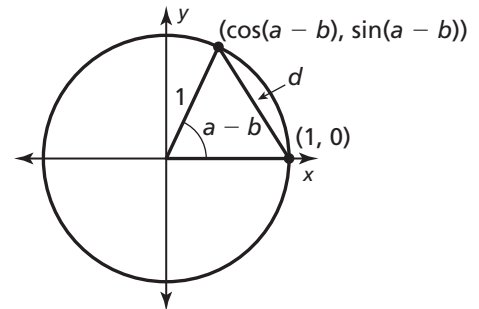
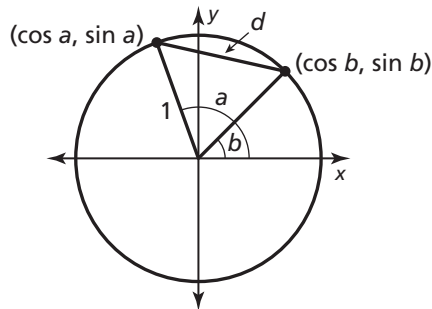
For use with Exploration 9.8

**Essential Question** How can you evaluate trigonometric functions of the sum or difference of two angles?

**1 EXPLORATION: Deriving a Difference Formula**

**Work with a partner.**

- a. Explain why the two triangles shown are congruent.



- b. Use the Distance Formula to write an expression for  $d$  in the first unit circle.
- c. Use the Distance Formula to write an expression for  $d$  in the second unit circle.
- d. Write an equation that relates the expressions in parts (b) and (c). Then simplify this equation to obtain a formula for  $\cos(a - b)$ .

**2 EXPLORATION: Deriving a Sum Formula**

**Work with a partner.** Use the difference formula you derived in Exploration 1 to write a formula for  $\cos(a + b)$  in terms of sine and cosine of  $a$  and  $b$ . *Hint:* Use the fact that

$$\cos(a + b) = \cos[a - (-b)].$$

**9.8 Using Sum and Difference Formulas (continued)****3 EXPLORATION:** Deriving Difference and Sum Formulas

**Work with a partner.** Use the formulas you derived in Explorations 1 and 2 to write formulas for  $\sin(a - b)$  and  $\sin(a + b)$  in terms of sine and cosine of  $a$  and  $b$ . *Hint:* Use the cofunction identities

$$\sin\left(\frac{\pi}{2} - a\right) = \cos a \text{ and } \cos\left(\frac{\pi}{2} - a\right) = \sin a$$

and the fact that

$$\cos\left[\left(\frac{\pi}{2} - a\right) + b\right] = \sin(a - b) \text{ and } \sin(a + b) = \sin[a - (-b)].$$

**Communicate Your Answer**

4. How can you evaluate trigonometric functions of the sum or difference of two angles?
  
  
  
  
  
  
  
  
  
  
5.
  - a. Find the exact values of  $\sin 75^\circ$  and  $\cos 75^\circ$  using sum formulas. Explain your reasoning.
  
  
  
  
  
  
  
  
  
  
  - b. Find the exact values of  $\sin 75^\circ$  and  $\cos 75^\circ$  using difference formulas. Compare your answers to those in part (a).

**9.8****Notetaking with Vocabulary**

For use after Lesson 9.8

In your own words, write the meaning of each vocabulary term.

ratio

**Core Concepts****Sum and Difference Formulas****Sum Formulas**

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

**Difference Formulas**

$$\sin(a - b) = \sin a \cos b - \cos a \sin b$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

**Notes:**

**9.8** Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1–4, find the exact value of the expression.

1.  $\sin(-75^\circ)$

2.  $\tan 120^\circ$

3.  $\cos\left(\frac{7\pi}{12}\right)$

4.  $\cos\frac{35\pi}{12}$

In Exercises 5–8, evaluate the expression given that  $\sin a = -\frac{4}{5}$  with $\pi < a < \frac{3\pi}{2}$  and  $\cos b = \frac{5}{13}$  with  $0 < b < \frac{\pi}{2}$ .

5.  $\cos(a - b)$

6.  $\sin(a + b)$



**9.8** Notetaking with Vocabulary (continued)

7.  $\tan(a + b)$

8.  $\tan(a - b)$

In Exercises 9–12, simplify the expression.

9.  $\sin\left(x + \frac{\pi}{2}\right)$

10.  $\tan(x - 2\pi)$

11.  $\cos(x - 2\pi)$

12.  $\cos\left(x + \frac{5\pi}{2}\right)$

In Exercises 13 and 14, solve the equation for  $0 \leq x \leq 2\pi$ .

13.  $\sin\left(x + \frac{3\pi}{2}\right) = 1$

14.  $\sin\left(x - \frac{\pi}{2}\right) + \cos\left(x - \frac{\pi}{2}\right) = 0$