

**Chapter
8****Maintaining Mathematical Proficiency**

Complete the table to evaluate the function.

1. $y = 4 + 2^x$

x	y
1	
2	
3	

2. $y = 2x^3 - 5$

x	y
0	
1	
2	

3. $y = -2x + 10$

x	y
-4	
-2	
0	

Solve the equation. Check your solution(s).

4. $50 = 4 + 2x$

5. $\frac{1}{3} = 3\left(\frac{1}{3}\right)^x$

6. $45 = 5(2x - 1)$

7. $3^x + 12 = 93$

8. $\frac{2}{5}x + 8 = 2$

9. $\frac{32}{125} = 25\left(\frac{2}{5}\right)^x$

8.1**Defining and Using Sequences and Series**

For use with Exploration 8.1

Essential Question How can you write a rule for the n th term of a sequence?

A **sequence** is an ordered list of numbers. There can be a limited number or an infinite number of *terms* of a sequence.

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots \quad \text{Terms of a sequence}$$

Here is an example.

$$1, 4, 7, 10, \dots, 3n - 2, \dots$$

1 EXPLORATION: Writing Rules for Sequences

Work with a partner. Match each sequence with its graph on the next page. The horizontal axes represent n , the position of each term in the sequence. Then write a rule for the n th term of the sequence, and use the rule to find a_{10} .

a. $1, 2.5, 4, 5.5, 7, \dots$

b. $8, 6.5, 5, 3.5, 2, \dots$

c. $\frac{1}{4}, \frac{4}{4}, \frac{9}{4}, \frac{16}{4}, \frac{25}{4}, \dots$

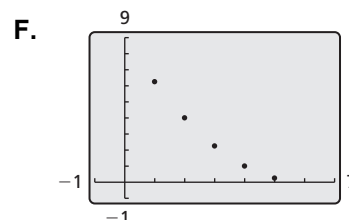
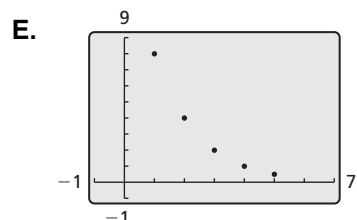
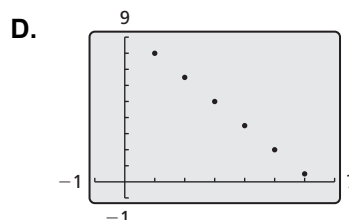
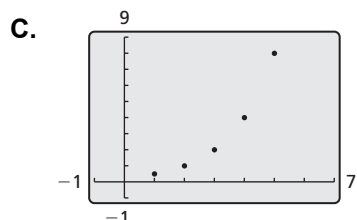
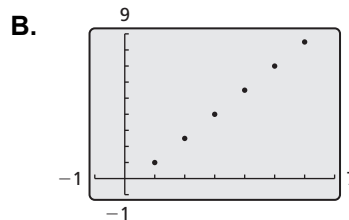
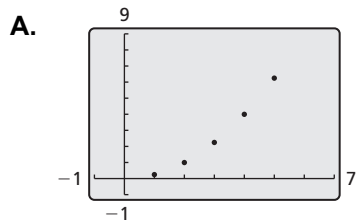
d. $\frac{25}{4}, \frac{16}{4}, \frac{9}{4}, \frac{4}{4}, \frac{1}{4}, \dots$

e. $\frac{1}{2}, 1, 2, 4, 8, \dots$

f. $8, 4, 2, 1, \frac{1}{2}, \dots$

8.1 Defining and Using Sequences and Series (continued)

1 **EXPLORATION:** Writing Rules for Sequences (continued)



Communicate Your Answer

2. How can you write a rule for the n th term of a sequence?

3. What do you notice about the relationship between the terms in (a) an arithmetic sequence and (b) a geometric sequence? Justify your answers.

8.1**Notetaking with Vocabulary**

For use after Lesson 8.1

In your own words, write the meaning of each vocabulary term.

sequence

terms of a sequence

series

summation notation

sigma notation

Core Concepts**Sequences**

A **sequence** is an ordered list of numbers. A *finite sequence* is a function that has a limited number of terms and whose domain is the finite set $\{1, 2, 3, \dots, n\}$. The values in the range are called the **terms** of the sequence.

Domain:	1	2	3	4	...	n	Relative position of each term
	↓	↓	↓	↓		↓	
Range:	a_1	a_2	a_3	a_4	...	a_n	Terms of the sequence

An *infinite sequence* is a function that continues without stopping and whose domain is the set of positive integers. Here are examples of a finite sequence and an infinite sequence.

Finite sequence: 2, 4, 6, 8 **Infinite sequence:** 2, 4, 6, 8, ...

A sequence can be specified by an equation, or *rule*. For example, both sequences above can be described by the rule $a_n = 2n$ or $f(n) = 2n$.

Notes:

8.1 Notetaking with Vocabulary (continued)**Series and Summation Notation**

When the terms of a sequence are added together, the resulting expression is a **series**. A series can be finite or infinite.

Finite series: $2 + 4 + 6 + 8$

Infinite series: $2 + 4 + 6 + 8 + \dots$

You can use **summation notation** to write a series. For example, the two series above can be written in summation notation as follows:

Finite series: $2 + 4 + 6 + 8 = \sum_{i=1}^4 2i$

Infinite series: $2 + 4 + 6 + 8 + \dots = \sum_{i=1}^{\infty} 2i$

For both series, the *index of summation* is i and the *lower limit of summation* is 1. The *upper limit of summation* is 4 for the finite series and ∞ (infinity) for the infinite series. Summation notation is also called **sigma notation** because it uses the uppercase Greek letter *sigma*, written Σ .

Notes:

Formulas for Special Series

Sum of n terms of 1: $\sum_{i=1}^n 1 = n$

Sum of first n positive integers: $\sum_{i=1}^n i = \frac{n(n+1)}{2}$

Sum of squares of first n positive integers: $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

Notes:

8.1 Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1 and 2, write the first six terms of the sequence.

1. $a_n = n^3 - 1$

2. $f(n) = (-2)^{n-1}$

In Exercises 3 and 4, describe the pattern, write the next term, and write a rule for the n th term of the sequence.

3. $-3, -1, 1, 3, \dots$

4. $\frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \frac{8}{5}, \dots$

5. Write the series $-1 + 4 - 9 + 16 - 25 + \dots$ using summation notation.

In Exercises 6 and 7, find the sum.

6. $\sum_{n=2}^5 \frac{n}{n-1}$

7. $\sum_{i=1}^{18} i^2$

8.2

Analyzing Arithmetic Sequences and Series

For use with Exploration 8.2

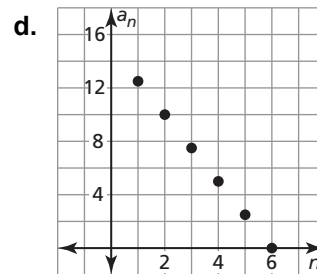
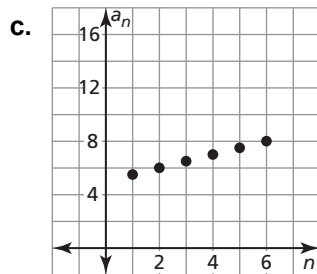
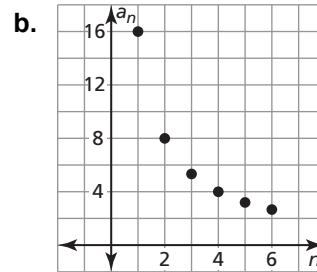
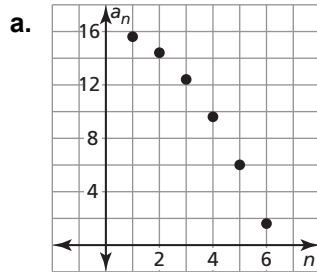
Essential Question How can you recognize an arithmetic sequence from its graph?

In an **arithmetic sequence**, the difference of consecutive terms, called the *common difference*, is constant. For example, in the arithmetic sequence 1, 4, 7, 10, . . . , the common difference is 3.

1 EXPLORATION: Recognizing Graphs of Arithmetic Sequences

Go to *BigIdeasMath.com* for an interactive tool to investigate this exploration.

Work with a partner. Determine whether each graph shows an arithmetic sequence. If it does, then write a rule for the n th term of the sequence, and use a spreadsheet to find the sum of the first 20 terms. What do you notice about the graph of an arithmetic sequence?



8.2**Notetaking with Vocabulary**

For use after Lesson 8.2

In your own words, write the meaning of each vocabulary term.

arithmetic sequence

common difference

arithmetic series

Core Concepts**Rule for an Arithmetic Sequence**

Algebra The n th term of an arithmetic sequence with first term a_1 and common difference d is given by:

$$a_n = a_1 + (n - 1)d$$

Example The n th term of an arithmetic sequence with a first term of 3 and a common difference of 2 is given by:

$$a_n = 3 + (n - 1)2, \text{ or } a_n = 2n + 1$$

Notes:**The Sum of a Finite Arithmetic Series**

The sum of the first n terms of an arithmetic series is $S_n = n\left(\frac{a_1 + a_n}{2}\right)$.

In words, S_n is the mean of the first and n th terms, multiplied by the number of terms.

Notes:

8.2 Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1–4, tell whether the sequence is arithmetic. Explain your reasoning.

1. 1, 4, 7, 12, 17, ...

2. 26, 23, 20, 17, 14, ...

3. 0.3, 0.5, 0.7, 0.9, 1.1, ...

4. $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \dots$

In Exercises 5–8, write a rule for the n th term of the sequence. Then find a_{20} .

5. 3, 9, 15, 21, ...

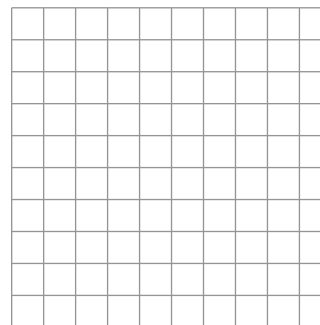
6. 8, 3, -2, -7, ...

7. $-1, -\frac{1}{2}, 0, \frac{1}{2}, \dots$

8. 0.7, 0.2, -0.3, -0.8, ...

8.2 Notetaking with Vocabulary (continued)

9. Write a rule for the n th term of the sequence where $a_{12} = -13$ and $d = -2$. Then graph the first six terms of the sequence.



In Exercises 10 and 11, write a rule for the n th term of the sequence.

10. $a_8 = 59, a_{13} = 99$

11. $a_{18} = -5, a_{27} = -8$

12. Find the sum $\sum_{i=1}^{22} (5 - 2i)$.

8.3

Analyzing Geometric Sequences and Series

For use with Exploration 8.3

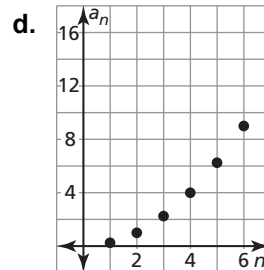
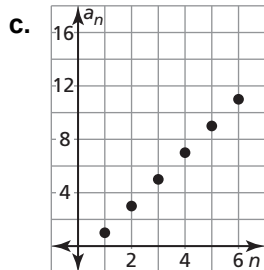
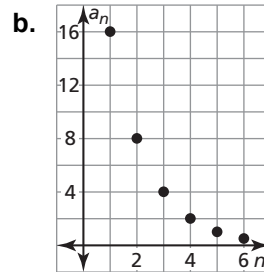
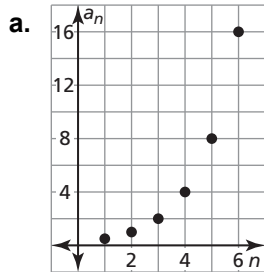
Essential Question How can you recognize a geometric sequence from its graph?

In a **geometric sequence**, the ratio of any term to the previous term, called the *common ratio*, is constant. For example, in the geometric sequence 1, 2, 4, 8, . . . , the common ratio is 2.

1 EXPLORATION: Recognizing Graphs of Geometric Sequences

Go to BigIdeasMath.com for an interactive tool to investigate this exploration.

Work with a partner. Determine whether each graph shows a geometric sequence. If it does, then write a rule for the n th term of the sequence and use a spreadsheet to find the sum of the first 20 terms. What do you notice about the graph of a geometric sequence?



8.3 Analyzing Geometric Sequences and Series (continued)**2 EXPLORATION: Finding the Sum of a Geometric Sequence**

Work with a partner. You can write the n th term of a geometric sequence with first term a_1 and common ratio r as

$$a_n = a_1 r^{n-1}.$$

So, you can write the sum S_n of the first n terms of a geometric sequence as

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}.$$

Rewrite this formula by finding the difference $S_n - rS_n$ and solve for S_n . Then verify your rewritten formula by finding the sums of the first 20 terms of the geometric sequences in Exploration 1. Compare your answers to those you obtained using a spreadsheet.

Communicate Your Answer

3. How can you recognize a geometric sequence from its graph?

4. Find the sum of the terms of each geometric sequence.
 - a. 1, 2, 4, 8, ..., 8192

 - b. 0.1, 0.01, 0.001, 0.0001, ..., 10^{-10}

8.3**Notetaking with Vocabulary**

For use after Lesson 8.3

In your own words, write the meaning of each vocabulary term.

geometric sequence

common ratio

geometric series

Core Concepts**Rule for a Geometric Sequence**

Algebra The n th term of a geometric sequence with first term a_1 and common ratio r is given by:

$$a_n = a_1 r^{n-1}$$

Example The n th term of a geometric sequence with a first term of 2 and a common ratio of 3 is given by:

$$a_n = 2(3)^{n-1}$$

Notes:

The Sum of a Finite Geometric Series

The sum of the first n terms of a geometric series with common ratio $r \neq 1$ is

$$S_n = a_1 \left(\frac{1 - r^n}{1 - r} \right).$$

Notes:

8.3 Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1–4, tell whether the sequence is geometric. Explain your reasoning.

1. 4, 12, 36, 108, 324, ...

2. 45, 40, 35, 30, 25, ...

3. 1.3, 7.8, 46.8, 280.8, 1684.8, ...

4. $\frac{3}{2}, -\frac{3}{4}, \frac{3}{8}, -\frac{3}{16}, \frac{3}{32}, \dots$

In Exercises 5–8, write a rule for the n th term of the sequence. Then find a_6 .

5. 6, 18, 54, 162, ...

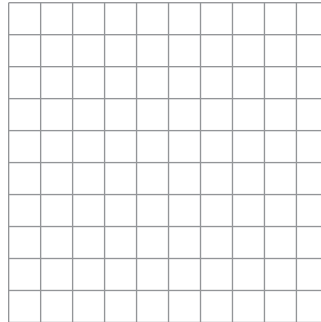
6. 3, -6, 12, -24, ...

7. $1, \frac{5}{2}, \frac{25}{4}, \frac{125}{8}, \dots$

8. -2.4, -16.8, -117.6, -823.2, ...

8.3 Notetaking with Vocabulary (continued)

9. Write a rule for the n th term where $a_8 = 384$ and $r = 2$.
Then graph the first six terms of the sequence.



In Exercises 10 and 11, write a rule for the n th term of the geometric sequence.

10. $a_3 = 54, a_6 = 1458$

11. $a_2 = -2, a_5 = \frac{2}{125}$

12. Find the sum $\sum_{i=0}^{10} 3\left(\frac{3}{2}\right)^{i-1}$.

8.4**Finding Sums of Infinite Geometric Series**

For use with Exploration 8.4

Essential Question How can you find the sum of an infinite geometric series?

1 EXPLORATION: Finding Sums of Infinite Geometric Series

Go to *BigIdeasMath.com* for an interactive tool to investigate this exploration.

Work with a partner. Enter each geometric series in a spreadsheet. Then use the spreadsheet to determine whether the infinite geometric series has a finite sum. If it does, find the sum. Explain your reasoning. (The figure shows a partially completed spreadsheet for part (a).)

a. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

b. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$

c. $1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16} + \dots$

d. $1 + \frac{5}{4} + \frac{25}{16} + \frac{125}{64} + \frac{625}{256} + \dots$

e. $1 + \frac{4}{5} + \frac{16}{25} + \frac{64}{125} + \frac{256}{625} + \dots$

f. $1 + \frac{9}{10} + \frac{81}{100} + \frac{729}{1000} + \frac{6561}{10,000} + \dots$

	A	B
1	1	1
2	2	0.5
3	3	0.25
4	4	0.125
5	5	0.0625
6	6	0.03125
7	7	
8	8	
9	9	
10	10	
11	11	
12	12	
13	13	
14	14	
15	15	
16	Sum	

8.4 Finding Sums of Infinite Geometric Series (continued)**2 EXPLORATION: Writing a Conjecture**

Work with a partner. Look back at the infinite geometric series in Exploration 1. Write a conjecture about how you can determine whether the infinite geometric series

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \dots$$

has a finite sum.

3 EXPLORATION: Writing a Formula

Work with a partner. In Lesson 8.3, you learned that the sum of the first n terms of a geometric series with first term a_1 and common ratio $r \neq 1$ is

$$S_n = a_1 \left(\frac{1 - r^n}{1 - r} \right).$$

When an infinite geometric series has a finite sum, what happens to r^n as n increases? Explain your reasoning. Write a formula to find the sum of an infinite geometric series. Then verify your formula by checking the sums you obtained in Exploration 1.

Communicate Your Answer

4. How can you find the sum of an infinite geometric series?

5. Find the sum of each infinite geometric series, if it exists.

a. $1 + 0.1 + 0.01 + 0.001 + 0.0001 + \dots$ b. $2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \frac{32}{81} + \dots$

8.4**Notetaking with Vocabulary**

For use after Lesson 8.4

In your own words, write the meaning of each vocabulary term.

partial sum

Core Concepts**The Sum of an Infinite Geometric Series**

The sum of an infinite geometric series with first term a_1 and common ratio r is given by

$$S = \frac{a_1}{1 - r}$$

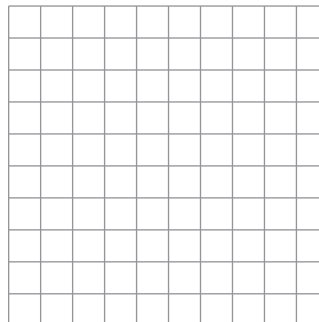
provided $|r| < 1$. If $|r| \geq 1$, then the series has no sum.

Notes:

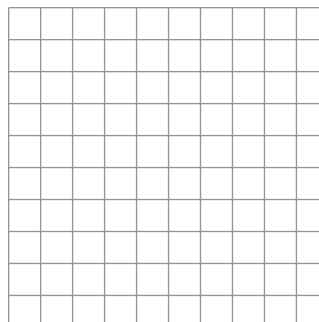
8.4 Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1 and 2, consider the infinite geometric series. Find and graph the partial sums S_n for $n = 1, 2, 3, 4,$ and 5 . Then describe what happens to S_n as n increases.

1. $\frac{1}{4} + \frac{1}{6} + \frac{1}{9} + \frac{2}{27} + \frac{4}{81} + \dots$



2. $3 + \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \frac{3}{625} + \dots$



8.4 Notetaking with Vocabulary (continued)

In Exercises 3–6, find the sum of the infinite geometric series, if it exists.

3.
$$\sum_{n=1}^{\infty} 6\left(\frac{3}{5}\right)^{n-1}$$

4.
$$\sum_{i=1}^{\infty} \frac{10}{3}\left(\frac{5}{2}\right)^{i-1}$$

5.
$$5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \dots$$

6.
$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots$$

7. A child pushes a tumbler toy and lets it swing freely. On the first swing, the toy travels 30 centimeters. On each successive swing, the toy travels 75% of the distance of the previous swing. What is the total distance the toy swings?

8. Write 0.121212... as a fraction in simplest form.

8.5**Using Recursive Rules with Sequences**

For use with Exploration 8.5

Essential Question How can you define a sequence recursively?

A **recursive rule** gives the beginning term(s) of a sequence and a *recursive equation* that tells how a_n is related to one or more preceding terms.

1 EXPLORATION: Evaluating a Recursive Rule

Go to *BigIdeasMath.com* for an interactive tool to investigate this exploration.

Work with a partner. Use each recursive rule and a spreadsheet to write the first six terms of the sequence. Classify the sequence as arithmetic, geometric, or neither. Explain your reasoning. (The figure shows a partially completed spreadsheet for part (a).)

	A	B
1	n	nth Term
2	1	7
3	2	10
4	3	
5	4	
6	5	
7	6	

B2+3

a. $a_1 = 7, a_n = a_{n-1} + 3$

b. $a_1 = 5, a_n = a_{n-1} - 2$

c. $a_1 = 1, a_n = 2a_{n-1}$

d. $a_1 = 1, a_n = \frac{1}{2}(a_{n-1})^2$

e. $a_1 = 3, a_n = a_{n-1} + 1$

f. $a_1 = 4, a_n = \frac{1}{2}a_{n-1} - 1$

g. $a_1 = 4, a_n = \frac{1}{2}a_{n-1}$

h. $a_1 = 4, a_2 = 5, a_n = a_{n-1} + a_{n-2}$

8.5 Using Recursive Rules with Sequences (continued)

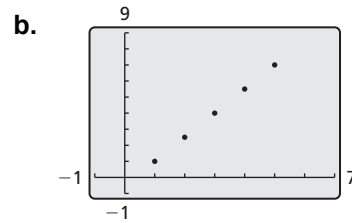
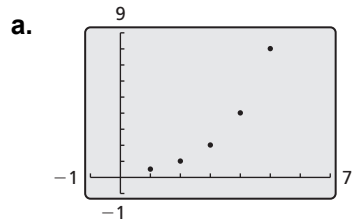
2 EXPLORATION: Writing a Recursive Rule

Work with a partner. Write a recursive rule for the sequence. Explain your reasoning.

- a. 3, 6, 9, 12, 15, 18, ...
- b. 18, 14, 10, 6, 2, -2, ...
- c. 3, 6, 12, 24, 48, 96, ...
- d. 128, 64, 32, 16, 8, 4, ...
- e. 5, 5, 5, 5, 5, 5, ...
- f. 1, 1, 2, 3, 5, 8, ...

3 EXPLORATION: Writing a Recursive Rule

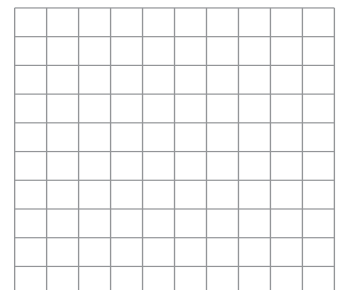
Work with a partner. Write a recursive rule for the sequence whose graph is shown.



Communicate Your Answer

4. How can you define a sequence recursively?

5. Write a recursive rule that is different from those in Explorations 1–3. Write the first six terms of the sequence. Then graph the sequence and classify it as arithmetic, geometric, or neither.



8.5**Notetaking with Vocabulary**

For use after Lesson 8.5

In your own words, write the meaning of each vocabulary term.

explicit rule

recursive rule

Core Concepts**Recursive Equations for Arithmetic and Geometric Sequences****Arithmetic Sequence**

$$a_n = a_{n-1} + d, \text{ where } d \text{ is the common difference}$$

Geometric Sequence

$$a_n = r \cdot a_{n-1}, \text{ where } r \text{ is the common ratio}$$

Notes:

8.5 Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1 and 2, write the first six terms of the sequence.

1. $a_1 = 2$

$$a_n = a_{n-1} + 5$$

2. $f(0) = 1$

$$f(n) = 2f(n - 1)$$

In Exercises 3–6, write a recursive rule for the sequence.

3. 9, 12, 15, 18, 21, ...

4. 50, 20, 8, $\frac{16}{5}$, $\frac{32}{25}$, ...

5. 3, 4, 1, -3, -4, ...

6. 1, 1, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{15}$, ...

8.5 Notetaking with Vocabulary (continued)

In Exercises 7–10, write a recursive rule for the sequence.

7. $a_n = 5 - 3n$

8. $a_n = 10(-2)^{n-1}$

9. $a_n = -1 + 8n$

10. $a_n = -3\left(\frac{3}{4}\right)^{n-1}$

In Exercises 11–14, write an explicit rule for each sequence.

11. $a_1 = -1, a_n = a_{n-1} + 7$

12. $a_1 = 24, a_n = 0.2a_{n-1}$

13. $a_1 = 1, a_n = a_{n-1} - 0.3$

14. $a_1 = -2, a_n = -5a_{n-1}$