

**Chapter
4****Maintaining Mathematical Proficiency**

Simplify the expression.

1. $-8x - 9x$

2. $25r - 5 + 7r - r$

3. $3 + 6(3x - 5) + x$

4. $3y - (2y - 5) + 11$

5. $-3(h + 7) - 7(10 - h)$

6. $5 - 8x^2 + 5x + 8x^2$

Find the volume or surface area of the solid.

7. volume of a right cylinder with radius 5 feet and height 15 feet

8. surface area of a rectangular prism with length 10 meters, width 20 meters, and height 4 meters

9. volume of a cube with side length 2.5 millimeters

10. surface area of a sphere with radius 1 foot

11. For what radius length can the value of the volume of a sphere equal the value of the surface area?

4.1

Graphing Polynomial Functions

For use with Exploration 4.1

Essential Question What are some common characteristics of the graphs of cubic and quartic polynomial functions?

1 EXPLORATION: Identifying Graphs of Polynomial Functions

Go to *BigIdeasMath.com* for an interactive tool to investigate this exploration.

Work with a partner. Match each polynomial function with its graph. Explain your reasoning. Use a graphing calculator to verify your answers.

a. $f(x) = x^3 - x$

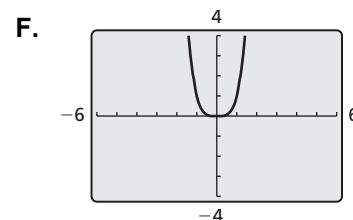
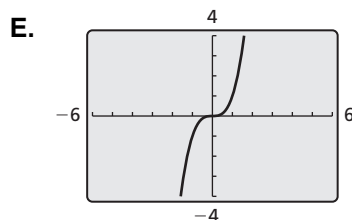
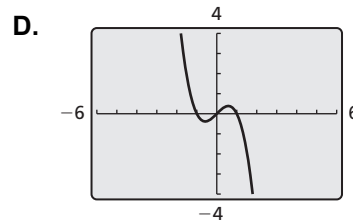
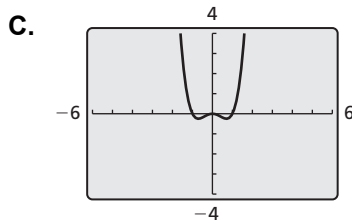
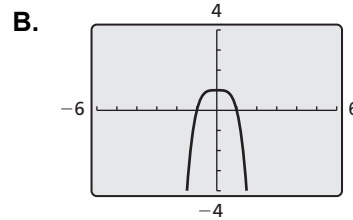
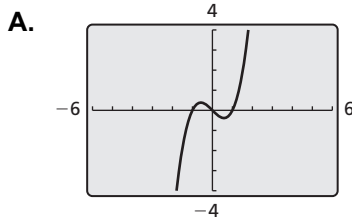
b. $f(x) = -x^3 + x$

c. $f(x) = -x^4 + 1$

d. $f(x) = x^4$

e. $f(x) = x^3$

f. $f(x) = x^4 - x^2$



4.1 Graphing Polynomial Functions (continued)**2** **EXPLORATION:** Identifying x -Intercepts of Polynomial Graphs

Work with a partner. Each of the polynomial graphs in Exploration 1 has x -intercept(s) of -1 , 0 , or 1 . Identify the x -intercept(s) of each graph. Explain how you can verify your answers.

Communicate Your Answer

3. What are some common characteristics of the graphs of cubic and quartic polynomial functions?

4. Determine whether each statement is *true* or *false*. Justify your answer.
 - a. When the graph of a cubic polynomial function rises to the left, it falls to the right.

 - b. When the graph of a quartic polynomial function falls to the left, it rises to the right.

4.1

Notetaking with Vocabulary
For use after Lesson 4.1

In your own words, write the meaning of each vocabulary term.

polynomial

polynomial function

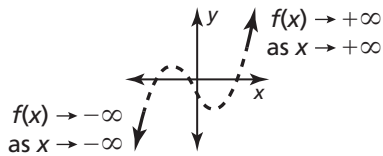
end behavior

Core Concepts

End Behavior of Polynomial Functions

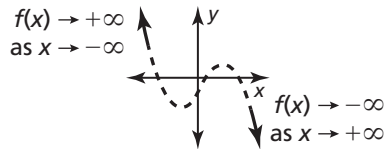
Degree: odd

Leading coefficient: positive



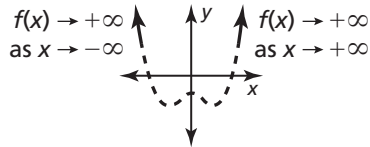
Degree: odd

Leading coefficient: negative



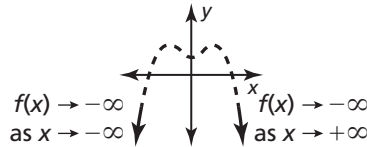
Degree: even

Leading coefficient: positive



Degree: even

Leading coefficient: negative



Notes:

4.1 Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1–4, decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

1. $f(x) = 2x^2 - 3x^4 + 6x + 1$

2. $m(x) = -\frac{3}{7}x^3 + \frac{7}{x} - 3$

3. $g(x) = \sqrt{15}x + \sqrt{5}$

4. $p(x) = -2\sqrt{3} + 3x - 2x^2$

In Exercises 5 and 6, evaluate the function for the given value of x .

5. $h(x) = -x^3 - 2x^2 - 3x + 4$; $x = 2$

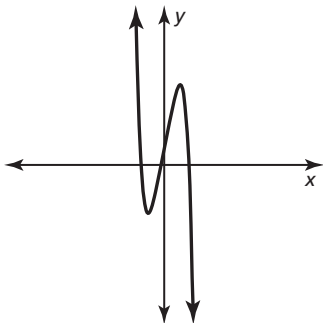
6. $g(x) = x^4 - 32x^2 + 256$; $x = -4$

In Exercises 7 and 8, describe the end behavior of the graph of the function.

7. $f(x) = -3x^6 + 4x^2 - 3x + 6$

8. $f(x) = \frac{4}{5}x + 6x + 3x^5 - 3x^3 - 2$

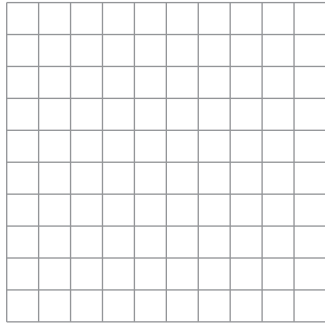
9. Describe the degree and leading coefficient of the polynomial function using the graph.



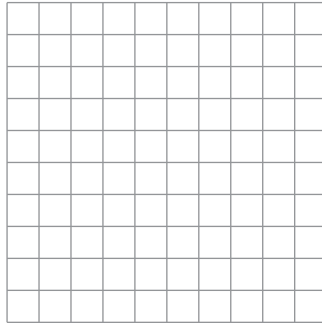
4.1 Notetaking with Vocabulary (continued)

In Exercises 10 and 11, graph the polynomial function.

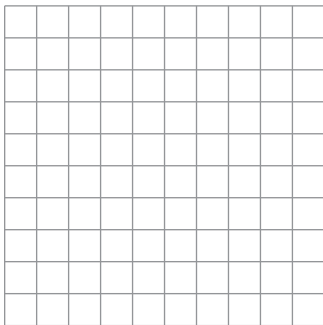
10. $p(x) = 16 - x^4$



11. $g(x) = x^2 + 3x^5 - x$



12. Sketch a graph of the polynomial function f if
 f is increasing when $x < -1$ and $0 < x < 1$,
 f is decreasing when $-1 < x < 0$ and $x > 1$,
and $f(x) < 0$ for all real numbers.



Describe the degree and leading coefficient of the function f .

13. The number of students S (in thousands) who graduate in four years from a university can be modeled by the function $S(t) = -\frac{1}{4}t^3 + t^2 + 23$, where t is the number of years since 2010.

- a. Use a graphing calculator to graph the function for the interval $0 \leq t \leq 5$. Describe the behavior of the graph on this interval.
- b. What is the average rate of change in the number of four-year graduates from 2010 to 2015?
- c. Do you think this model can be used for years before 2010 or after 2015? Explain your reasoning.

4.2**Adding, Subtracting, and Multiplying Polynomials**

For use with Exploration 4.2

Essential Question How can you cube a binomial?**1 EXPLORATION:** Cubing Binomials**Work with a partner.** Find each product. Show your steps.

$$\begin{aligned} \text{a. } (x + 1)^3 &= (x + 1)(x + 1)^2 \\ &= (x + 1) \underline{\hspace{2cm}} \\ &= \underline{\hspace{2cm}} \\ &= \underline{\hspace{2cm}} \end{aligned}$$

Rewrite as a product of first and second powers.

Multiply second power.

Multiply binomial and trinomial.

Write in standard form, $ax^3 + bx^2 + cx + d$.

$$\begin{aligned} \text{b. } (a + b)^3 &= (a + b)(a + b)^2 \\ &= (a + b) \underline{\hspace{2cm}} \\ &= \underline{\hspace{2cm}} \\ &= \underline{\hspace{2cm}} \end{aligned}$$

Rewrite as a product of first and second powers.

Multiply second power.

Multiply binomial and trinomial.

Write in standard form.

$$\begin{aligned} \text{c. } (x - 1)^3 &= (x - 1)(x - 1)^2 \\ &= (x - 1) \underline{\hspace{2cm}} \\ &= \underline{\hspace{2cm}} \\ &= \underline{\hspace{2cm}} \end{aligned}$$

Rewrite as a product of first and second powers.

Multiply second power.

Multiply binomial and trinomial.

Write in standard form.

$$\begin{aligned} \text{d. } (a - b)^3 &= (a - b)(a - b)^2 \\ &= (a - b) \underline{\hspace{2cm}} \\ &= \underline{\hspace{2cm}} \\ &= \underline{\hspace{2cm}} \end{aligned}$$

Rewrite as a product of first and second powers.

Multiply second power.

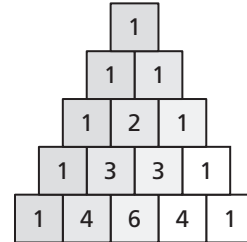
Multiply binomial and trinomial.

Write in standard form.

4.2 Adding, Subtracting, and Multiplying Polynomials (continued)**2 EXPLORATION:** Generalizing Patterns for Cubing a Binomial

Work with a partner.

- a. Use the results of Exploration 1 to describe a pattern for the coefficients of the terms when you expand the cube of a binomial. How is your pattern related to Pascal's Triangle, shown at the right?



- b. Use the results of Exploration 1 to describe a pattern for the exponents of the terms in the expansion of a cube of a binomial.
- c. Explain how you can use the patterns you described in parts (a) and (b) to find the product $(2x - 3)^3$. Then find this product.

Communicate Your Answer

3. How can you cube a binomial?

4. Find each product.

a. $(x + 2)^3$

b. $(x - 2)^3$

c. $(2x - 3)^3$

d. $(x - 3)^3$

e. $(-2x + 3)^3$

f. $(3x - 5)^3$

4.2**Notetaking with Vocabulary**

For use after Lesson 4.2

In your own words, write the meaning of each vocabulary term.

Pascal's Triangle

Core Concepts**Special Product Patterns****Sum and Difference**

$$(a + b)(a - b) = a^2 - b^2$$

Example

$$(x + 3)(x - 3) = x^2 - 9$$

Square of a Binomial

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

Example

$$(y + 4)^2 = y^2 + 8y + 16$$

$$(2t - 5)^2 = 4t^2 - 20t + 25$$

Cube of a Binomial

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

Example

$$(z + 3)^3 = z^3 + 9z^2 + 27z + 27$$

$$(m - 2)^3 = m^3 - 6m^2 + 12m - 8$$

Notes:

4.2 Notetaking with Vocabulary (continued)**Pascal's Triangle**

In Pascal's Triangle, the first and last numbers in each row are 1. Every number other than 1 is the sum of the closest two numbers in the row directly above it. The numbers in Pascal's Triangle are the same numbers that are the coefficients of binomial expansions, as shown in the first six rows.

	n	$(a + b)^n$	Binomial Expansion	Pascal's Triangle
0th row	0	$(a + b)^0 =$	1	1
1st row	1	$(a + b)^1 =$	$1a + 1b$	1 1
2nd row	2	$(a + b)^2 =$	$1a^2 + 2ab + 1b^2$	1 2 1
3rd row	3	$(a + b)^3 =$	$1a^3 + 3a^2b + 3ab^2 + 1b^3$	1 3 3 1
4th row	4	$(a + b)^4 =$	$1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$	1 4 6 4 1
5th row	5	$(a + b)^5 =$	$1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$	1 5 10 10 5 1

Notes:

4.2 Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1–3, find the sum or difference.

1. $(-4x^2 - 6x + 18) + (-x^2 + 7x + 8)$

2. $(6x^2 - 12x + 48) - (-x^2 + 24x - 63)$

3. $(-11x^4 - x^3 - 3x^2 + 10x - 2) - (-11x^4 + 5x^2 - 7x + 13)$

In Exercises 4–9, find the product.

4. $2x^2(2x^3 - x^2 + 3x - 5)$

5. $(x^4 - 10x^2 + 25)(3x^2 - 6x - 1)$

6. $(x + 1)(x - 2)(x + 6)$

7. $(2x - 3)(6 - x)(4 - 5x)$

8. $(3y - 8)(3y + 8)$

9. $(2v - 1)^3$

In Exercises 10 and 11, use Pascal's Triangle to expand the binomial.

10. $(4t - 2)^4$

11. $(g + 6)^5$

4.3

Dividing Polynomials

For use with Exploration 4.3

Essential Question How can you use the factors of a cubic polynomial to solve a division problem involving the polynomial?

1 EXPLORATION: Dividing Polynomials

Go to *BigIdeasMath.com* for an interactive tool to investigate this exploration.

Work with a partner. Match each division statement with the graph of the related cubic polynomial $f(x)$. Explain your reasoning. Use a graphing calculator to verify your answers.

a. $\frac{f(x)}{x} = (x - 1)(x + 2)$

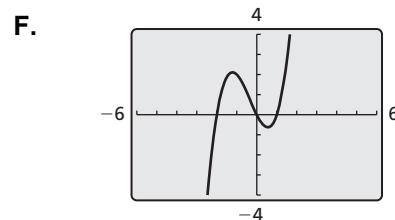
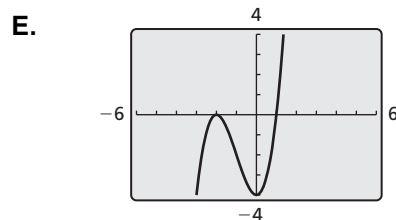
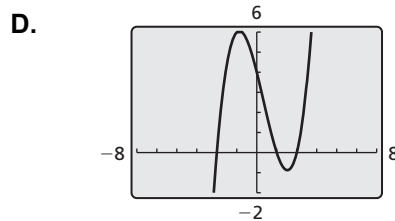
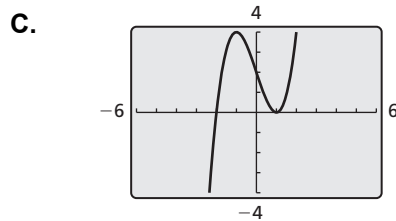
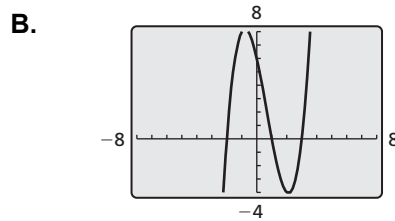
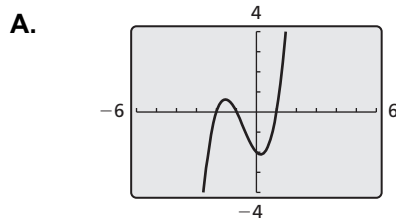
b. $\frac{f(x)}{x - 1} = (x - 1)(x + 2)$

c. $\frac{f(x)}{x + 1} = (x - 1)(x + 2)$

d. $\frac{f(x)}{x - 2} = (x - 1)(x + 2)$

e. $\frac{f(x)}{x + 2} = (x - 1)(x + 2)$

f. $\frac{f(x)}{x - 3} = (x - 1)(x + 2)$



4.3 Dividing Polynomials (continued)**2** **EXPLORATION:** Dividing Polynomials

Work with a partner. Use the results of Exploration 1 to find each quotient. Write your answers in standard form. Check your answers by multiplying.

a. $(x^3 + x^2 - 2x) \div x$

b. $(x^3 - 3x + 2) \div (x - 1)$

c. $(x^3 + 2x^2 - x - 2) \div (x + 1)$

d. $(x^3 - x^2 - 4x + 4) \div (x - 2)$

e. $(x^3 + 3x^2 - 4) \div (x + 2)$

f. $(x^3 - 2x^2 - 5x + 6) \div (x - 3)$

Communicate Your Answer

3. How can you use the factors of a cubic polynomial to solve a division problem involving the polynomial?

4.3

Notetaking with Vocabulary

For use after Lesson 4.3

In your own words, write the meaning of each vocabulary term.

polynomial long division

synthetic division

Core Concepts

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

Notes:

4.3 Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1–4, divide using polynomial long division.

1. $(x^2 + 6x + 12) \div (x - 3)$

2. $(x^3 - 4x^2) \div (x^2 - 16)$

3. $(4x^3 + 13x^2 + 27x + 6) \div (4x + 1)$

4. $(x^4 + 2x^3 + 5x^2 + 3x) \div (x^2 - x)$

In Exercises 5–8, divide using synthetic division.

5. $(x^2 - 10x + 2) \div (x - 2)$

6. $(x^3 + 4x^2 + 6x + 4) \div (x + 2)$

7. $(2x^3 - 54) \div (x + 3)$

8. $(2x^4 - 11x^3 + 11x^2 + 4x + 4) \div (x - 4)$

4.3 Notetaking with Vocabulary (continued)

In Exercises 9–12, match the equivalent expressions. Justify your answers.

9. $(x^2 - x - 8) \div (x - 4)$

A. $x + 3 + \frac{4}{x - 4}$

10. $(x^2 - x + 8) \div (x - 4)$

B. $x + 5 + \frac{12}{x - 4}$

11. $(x^2 + x - 8) \div (x - 4)$

C. $x + 5 + \frac{28}{x - 4}$

12. $(x^2 + x + 8) \div (x - 4)$

D. $x + 3 + \frac{20}{x - 4}$

In Exercises 13–16, use synthetic division to evaluate the function for the indicated value of x .

13. $f(x) = -3x^3 + 4x^2 - 17x - 6; x = 2$

14. $f(x) = -x^4 + x^2 + 4; x = -1$

15. $f(x) = x^3 - 10x^2 + 31x - 30; x = -5$

16. $f(x) = x^3 + 8x + 27; x = 3$

17. What is the value of k such that $(-x^4 + 5x^2 + kx - 8) \div (x - 4)$ has a remainder of 0?

4.4

Factoring Polynomials

For use with Exploration 4.4

Essential Question How can you factor a polynomial?

1 EXPLORATION: Factoring Polynomials

Work with a partner. Match each polynomial equation with the graph of its related polynomial function. Use the x -intercepts of the graph to write each polynomial in factored form. Explain your reasoning.

a. $x^2 + 5x + 4 = 0$

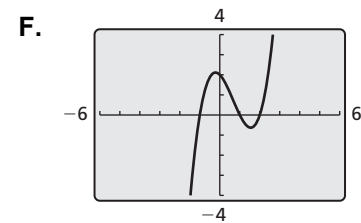
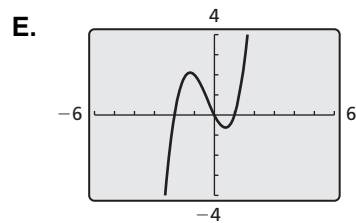
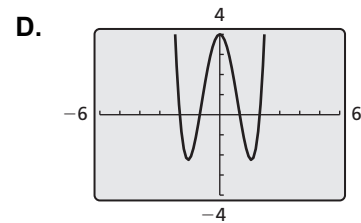
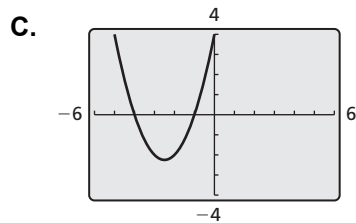
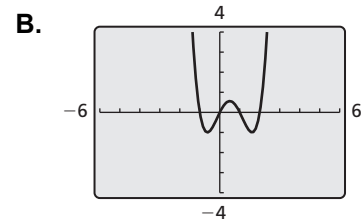
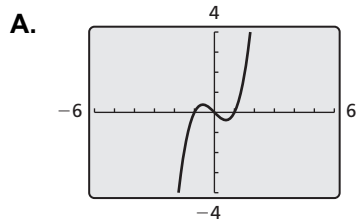
b. $x^3 - 2x^2 - x + 2 = 0$

c. $x^3 + x^2 - 2x = 0$

d. $x^3 - x = 0$

e. $x^4 - 5x^2 + 4 = 0$

f. $x^4 - 2x^3 - x^2 + 2x = 0$



4.4 Factoring Polynomials (continued)**2 EXPLORATION: Factoring Polynomials**

Work with a partner. Use the x -intercepts of the graph of the polynomial function to write each polynomial in factored form. Explain your reasoning. Check your answers by multiplying.

a. $f(x) = x^2 - x - 2$

b. $f(x) = x^3 - x^2 - 2x$

c. $f(x) = x^3 - 2x^2 - 3x$

d. $f(x) = x^3 - 3x^2 - x + 3$

e. $f(x) = x^4 + 2x^3 - x^2 - 2x$

f. $f(x) = x^4 - 10x^2 + 9$

Communicate Your Answer

- How can you factor a polynomial?
- What information can you obtain about the graph of a polynomial function written in factored form?

4.4**Notetaking with Vocabulary**

For use after Lesson 4.4

In your own words, write the meaning of each vocabulary term.

factored completely

factor by grouping

quadratic form

Core Concepts**Special Factoring Patterns****Sum of Two Cubes**

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Example

$$\begin{aligned} 64x^3 + 1 &= (4x)^3 + 1^3 \\ &= (4x + 1)(16x^2 - 4x + 1) \end{aligned}$$

Difference of Two Cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Example

$$\begin{aligned} 27x^3 - 8 &= (3x)^3 - 2^3 \\ &= (3x - 2)(9x^2 + 6x + 4) \end{aligned}$$

Notes:

4.4 Notetaking with Vocabulary (continued)**The Factor Theorem**

A polynomial $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

Notes:

Extra Practice

In Exercises 1–14, factor the polynomial completely.

1. $20x^3 - 220x^2 + 600x$

2. $m^5 - 81m$

3. $27a^3 + 8b^3$

4. $5t^6 + 2t^5 - 5t^4 - 2t^3$

5. $y^4 - 13y^2 - 48$

6. $5p^3 + 5p - 5p^2 - 5$

7. $810k^4 - 160$

8. $a^5 + a^3 - a^2 - 1$

4.4 Notetaking with Vocabulary (continued)

9. $2x^6 - 8x^5 - 42x^4$

10. $5z^3 + 5z^2 - 6z - 6$

11. $12x^2 - 22x - 20$

12. $3m^2 - 48m^6$

13. $4x^3 - 4x^2 + x$

14. $5m^4 - 70m^3 + 245m^2$

In Exercises 15–17, show that the binomial is a factor of the polynomial. Then factor the function completely.

15. $f(x) = x^3 - 13x - 12; x + 1$

16. $f(x) = 6x^3 + 8x^2 - 34x - 12; x - 2$

17. $f(x) = 2x^4 - 12x^3 + 6x^2 + 20x; x - 5$

4.5

Solving Polynomial Equations

For use with Exploration 4.5

Essential Question How can you determine whether a polynomial equation has a repeated solution?

1 EXPLORATION: Cubic Equations and Repeated Solutions

Work with a partner. Some cubic equations have three distinct solutions. Others have repeated solutions. Match each cubic polynomial equation with the graph of its related polynomial function. Then solve each equation. For those equations that have repeated solutions, describe the behavior of the related function near the repeated zero using the graph or a table of values.

a. $x^3 - 6x^2 + 12x - 8 = 0$

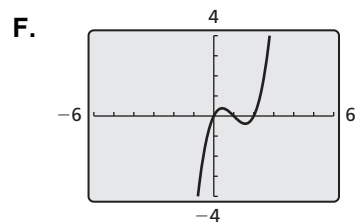
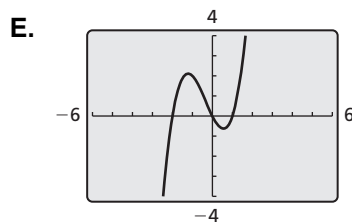
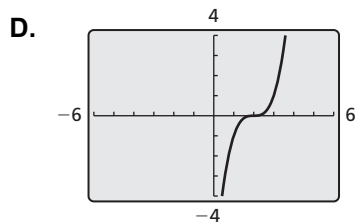
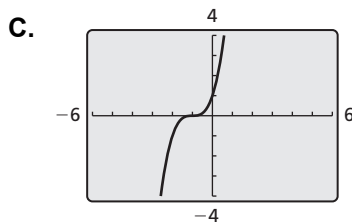
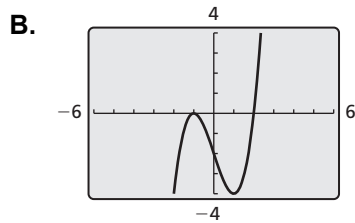
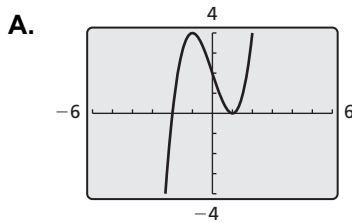
b. $x^3 + 3x^2 + 3x + 1 = 0$

c. $x^3 - 3x + 2 = 0$

d. $x^3 + x^2 - 2x = 0$

e. $x^3 - 3x - 2 = 0$

f. $x^3 - 3x^2 + 2x = 0$



4.5 Solving Polynomial Equations (continued)**2 EXPLORATION:** Quartic Equations and Repeated Solutions

Go to *BigIdeasMath.com* for an interactive tool to investigate this exploration.

Work with a partner. Determine whether each quartic equation has repeated solutions using the graph of the related quartic function or a table of values. Explain your reasoning. Then solve each equation.

a. $x^4 - 4x^3 + 5x^2 - 2x = 0$

b. $x^4 - 2x^3 - x^2 + 2x = 0$

c. $x^4 - 4x^3 + 4x^2 = 0$

d. $x^4 + 3x^3 = 0$

Communicate Your Answer

- How can you determine whether a polynomial equation has a repeated solution?
- Write a cubic or a quartic polynomial equation that is different from the equations in Explorations 1 and 2 and has a repeated solution.

4.5**Notetaking with Vocabulary**

For use after Lesson 4.5

In your own words, write the meaning of each vocabulary term.

repeated solution

Core Concepts**The Rational Root Theorem**

If $f(x) = a_nx^n + \cdots + a_1x + a_0$ has *integer* coefficients, then every rational solution of $f(x) = 0$ has the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

Notes:

The Irrational Conjugates Theorem

Let f be a polynomial function with rational coefficients, and let a and b be rational numbers such that \sqrt{b} is irrational. If $a + \sqrt{b}$ is a zero of f , then $a - \sqrt{b}$ is also a zero of f .

Notes:

4.5 Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1–6, solve the equation.

1. $36r^3 - r = 0$

2. $20x^3 + 80x^2 = -60x$

3. $3m^2 = 75m^4$

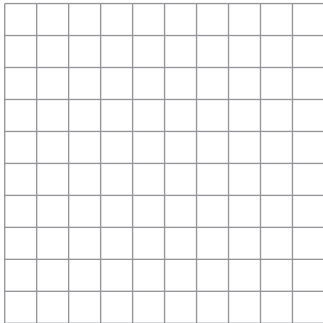
4. $-13y^2 + 36 = -y^4$

5. $2x^3 - x^2 - 2x = -1$

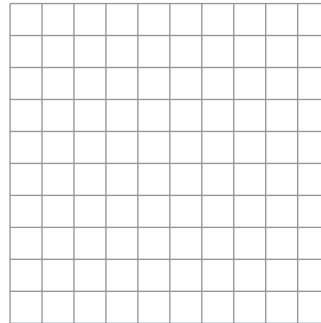
6. $-20c^2 + 50c = 8c^3 - 125$

In Exercises 7–10, find the zeros of the function. Then sketch a graph of the function.

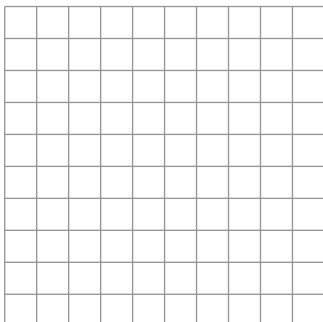
7. $f(x) = x^4 - x^3 - 12x^2$



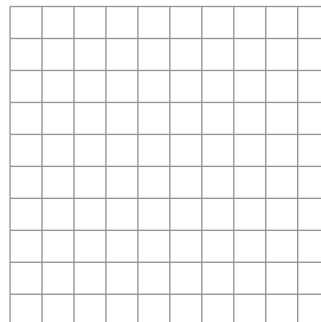
8. $f(x) = -4x^3 + 12x^2 - 9x$



9. $f(x) = x^3 + 4x^2 - 6x - 24$



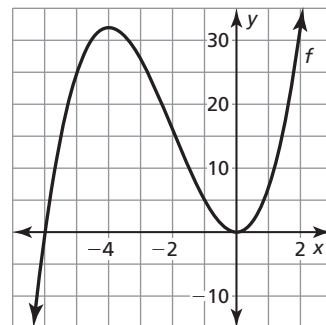
10. $f(x) = x^4 - 18x^2 + 81$



4.5 Notetaking with Vocabulary (continued)

11. According to the Rational Root Theorem, which is *not* a possible solution of the equation $2x^4 + 3x^3 - 6x + 7 = 0$?
- A. 3.5 B. 0.5 C. 7 D. 2
12. Find all the real zeros of the function $f(x) = 3x^4 + 11x^3 - 40x^2 - 132x + 48$.
13. Write a polynomial function g of least degree that has rational coefficients, a leading coefficient of 1, and the zeros -5 and $4 + \sqrt{2}$.
14. Use the information in the graph to answer the questions.

a. What are the real zeros of the function f ?



b. Write an equation of the cubic function in factored form.

4.6

The Fundamental Theorem of Algebra

For use with Exploration 4.6

Essential Question How can you determine whether a polynomial equation has imaginary solutions?

1 EXPLORATION: Cubic Equations and Imaginary Solutions

Work with a partner. Match each cubic polynomial equation with the graph of its related polynomial function. Then find *all* solutions. Make a conjecture about how you can use a graph or table of values to determine the number and types of solutions of a cubic polynomial equation.

a. $x^3 - 3x^2 + x + 5 = 0$

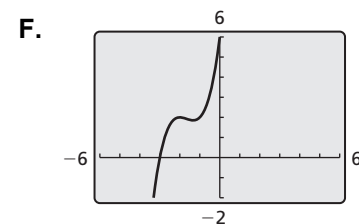
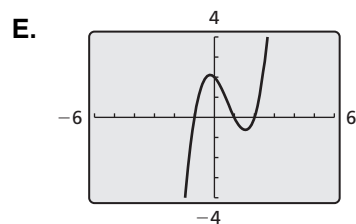
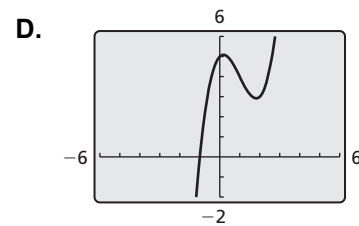
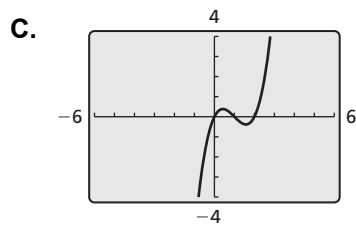
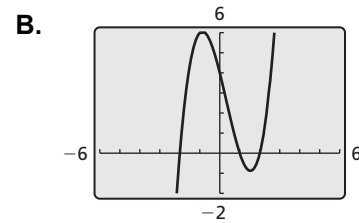
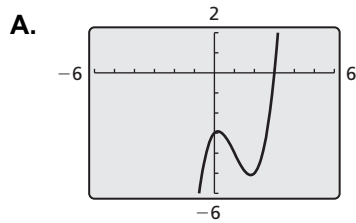
b. $x^3 - 2x^2 - x + 2 = 0$

c. $x^3 - x^2 - 4x + 4 = 0$

d. $x^3 + 5x^2 + 8x + 6 = 0$

e. $x^3 - 3x^2 + x - 3 = 0$

f. $x^3 - 3x^2 + 2x = 0$



4.6 The Fundamental Theorem of Algebra (continued)**2** **EXPLORATION:** Quartic Equations and Imaginary Solutions

Go to *BigIdeasMath.com* for an interactive tool to investigate this exploration.

Work with a partner. Use the graph of the related quartic function, or a table of values, to determine whether each quartic equation has imaginary solutions. Explain your reasoning. Then find *all* solutions.

a. $x^4 - 2x^3 - x^2 + 2x = 0$

b. $x^4 - 1 = 0$

c. $x^4 + x^3 - x - 1 = 0$

d. $x^4 - 3x^3 + x^2 + 3x - 2 = 0$

Communicate Your Answer

- How can you determine whether a polynomial equation has imaginary solutions?
- Is it possible for a cubic equation to have three imaginary solutions? Explain your reasoning.

4.6**Notetaking with Vocabulary**

For use after Lesson 4.6

In your own words, write the meaning of each vocabulary term.

complex conjugates

Core Concepts**The Fundamental Theorem of Algebra**

Theorem If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has at least one solution in the set of complex numbers.

Corollary If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has exactly n solutions provided each solution repeated twice is counted as two solutions, each solution repeated three times is counted as three solutions, and so on.

Notes:

4.6 Notetaking with Vocabulary (continued)**The Complex Conjugates Theorem**

If f is a polynomial function with real coefficients, and $a + bi$ is an imaginary zero of f , then $a - bi$ is also a zero of f .

Notes:

Descartes's Rule of Signs

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial function with real coefficients.

- The number of *positive real zeros* of f is equal to the number of changes in sign of the coefficients of $f(x)$ or is less than this by an even number.
- The number of *negative real zeros* of f is equal to the number of changes in sign of the coefficients of $f(-x)$ or is less than this by an even number.

Notes:

4.6 Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1–4, find all zeros of the polynomial function.

1. $h(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$

2. $f(x) = x^3 - 3x^2 - 15x + 125$

3. $g(x) = x^4 - 48x^2 - 49$

4. $h(x) = -5x^3 + 9x^2 - 18x - 4$

In Exercises 5–8, write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

5. $-4, 1, 7$

6. $10, -\sqrt{5}$

7. $8, 3 - i$

8. $0, 2 - \sqrt{2}, 2 + 3i$

4.7

Transformations of Polynomial Functions

For use with Exploration 4.7

Essential Question How can you transform the graph of a polynomial function?

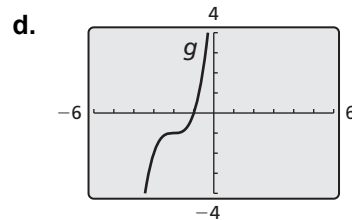
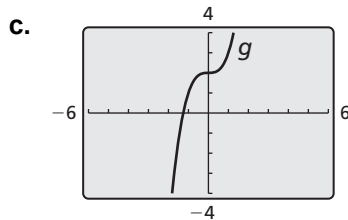
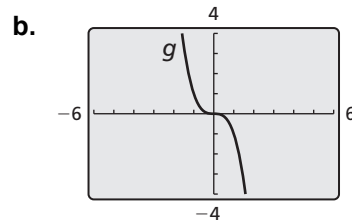
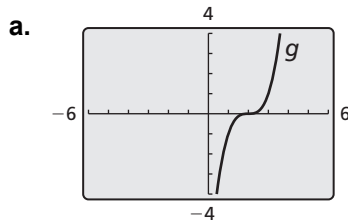
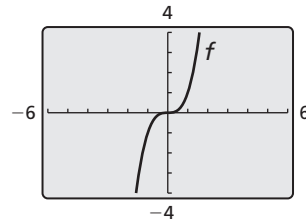
1 EXPLORATION: Transforming the Graph of the Cubic Function

Go to *BigIdeasMath.com* for an interactive tool to investigate this exploration.

Work with a partner. The graph of the cubic function

$$f(x) = x^3$$

is shown. The graph of each cubic function g represents a transformation of the graph of f . Write a rule for g . Use a graphing calculator to verify your answers.



4.7 Transformations of Polynomial Functions (continued)

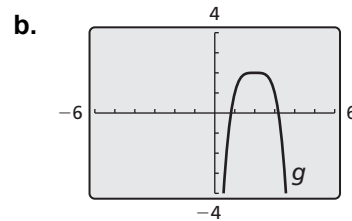
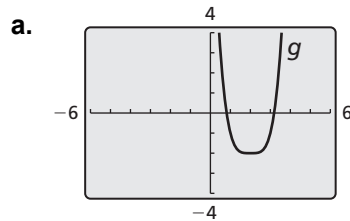
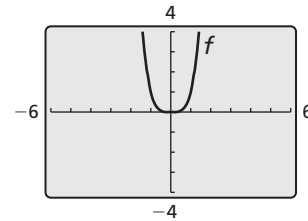
2 EXPLORATION: Transforming the Graph of the Quartic Function

Go to *BigIdeasMath.com* for an interactive tool to investigate this exploration.

Work with a partner. The graph of the quartic function

$$f(x) = x^4$$

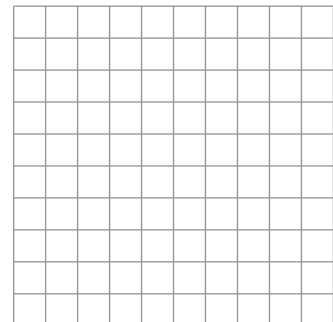
is shown. The graph of each quartic function g represents a transformation of the graph of f . Write a rule for g . Use a graphing calculator to verify your answers.



Communicate Your Answer

3. How can you transform the graph of a polynomial function?

4. Describe the transformation of $f(x) = x^4$ represented by $g(x) = (x + 1)^4 + 3$. Then graph $g(x)$.



4.7**Notetaking with Vocabulary**

For use after Lesson 4.7

In your own words, write the meaning of each vocabulary term.

polynomial function

transformations

Core Concepts

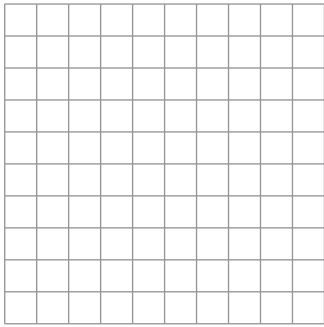
Transformation	$f(x)$ Notation	Examples
Horizontal Translation Graph shifts left or right.	$f(x - h)$	$g(x) = (x - 5)^4$ 5 units right $g(x) = (x + 2)^4$ 2 units left
Vertical Translation Graph shifts up or down.	$f(x) + k$	$g(x) = x^4 + 1$ 1 unit up $g(x) = x^4 - 4$ 4 units down
Reflection Graph flips over x - or y -axis.	$f(-x)$ $-f(x)$	$g(x) = (-x)^4 = x^4$ over y -axis $g(x) = -x^4$ over x -axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y -axis	$f(ax)$	$g(x) = (2x)^4$ shrink by $\frac{1}{2}$ $g(x) = \left(\frac{1}{2}x\right)^4$ stretch by 2
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x -axis.	$a \cdot f(x)$	$g(x) = 8x^4$ stretch by 8 $g(x) = \frac{1}{4}x^4$ shrink by $\frac{1}{4}$

Notes:

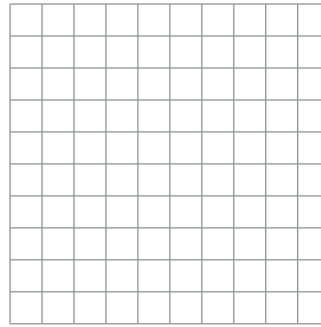
4.7 Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1–6, describe the transformation of f represented by g . Then graph each function.

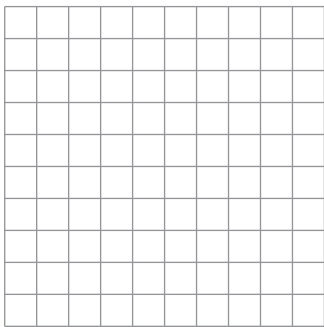
1. $f(x) = x^4$; $g(x) = x^4 - 9$



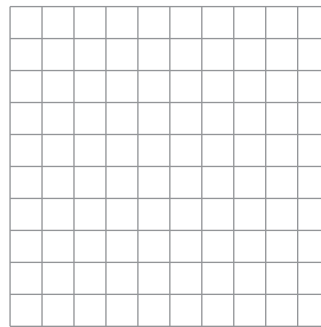
2. $f(x) = x^5$; $g(x) = (x + 1)^5 + 2$



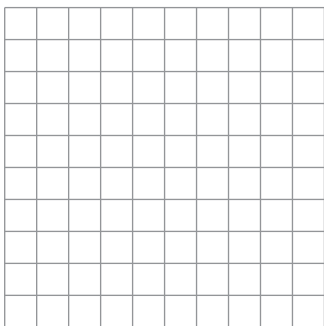
3. $f(x) = x^6$; $g(x) = -5(x - 2)^6$



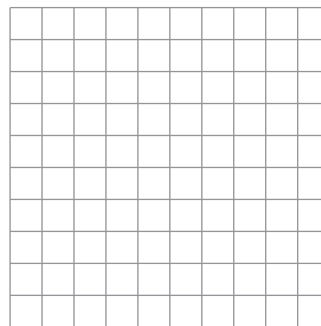
4. $f(x) = x^3$; $g(x) = \left(\frac{1}{2}x\right)^3 - 4$



5. $f(x) = x^4$; $g(x) = \frac{1}{8}(-x)^4$

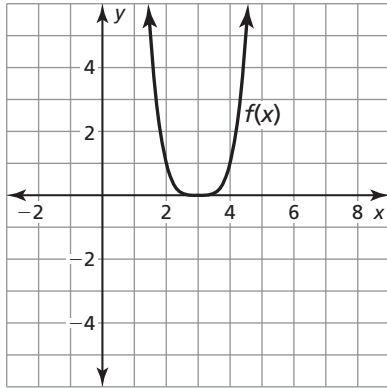


6. $f(x) = x^5$; $g(x) = (x - 10)^5 + 1$



4.7 Notetaking with Vocabulary (continued)

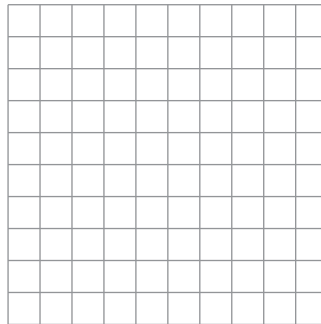
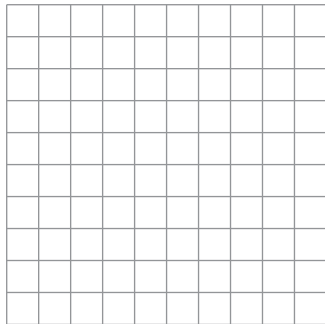
7. Graph the function $g(x) = -f(x - 3)$ on the same coordinate plane as $f(x)$.



In Exercises 8 and 9, write a rule for g and then graph each function. Describe the graph of g as a transformation of the graph of f .

8. $f(x) = x^3 + 8; g(x) = f(-x) - 9$

9. $f(x) = 2x^5 - x^3 + 1; g(x) = 5f(x)$



In Exercises 10 and 11, write a rule for g that represents the indicated transformations of the graph of f .

10. $f(x) = x^3 - 6x^2 + 5$; translation 1 unit left, followed by a reflection in the x -axis and a vertical stretch by a factor of 2

11. $f(x) = 3x^4 + x^3 + 3x^2 + 12$; horizontal shrink by a factor of $\frac{1}{3}$ and a translation 8 units down, followed by a reflection in the y -axis

4.8

Analyzing Graphs of Polynomial Functions

For use with Exploration 4.8

Essential Question How many turning points can the graph of a polynomial function have?

1 EXPLORATION: Approximating Turning Points

Go to *BigIdeasMath.com* for an interactive tool to investigate this exploration.

Work with a partner. Match each polynomial function with its graph. Explain your reasoning. Then use a graphing calculator to approximate the coordinates of the turning points of the graph of the function. Round your answers to the nearest hundredth.

a. $f(x) = 2x^2 + 3x - 4$

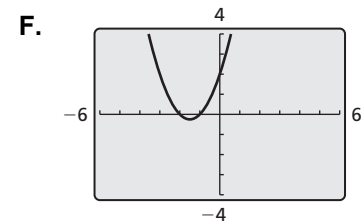
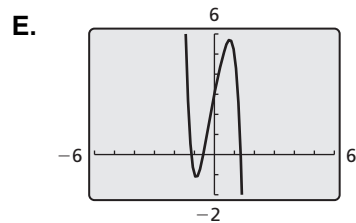
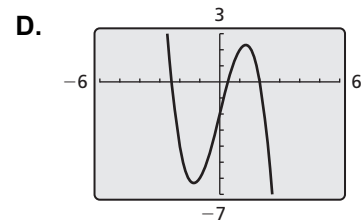
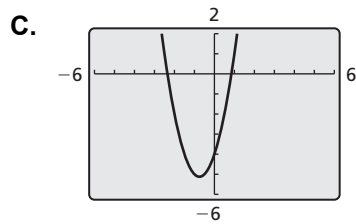
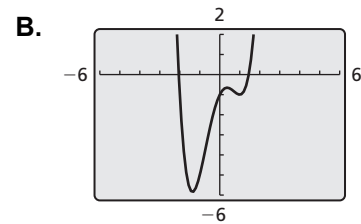
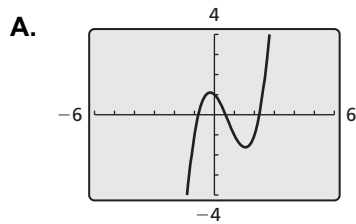
b. $f(x) = x^2 + 3x + 2$

c. $f(x) = x^3 - 2x^2 - x + 1$

d. $f(x) = -x^3 + 5x - 2$

e. $f(x) = x^4 - 3x^2 + 2x - 1$

f. $f(x) = -2x^5 - x^2 + 5x + 3$



4.8**Notetaking with Vocabulary**

For use after Lesson 4.8

In your own words, write the meaning of each vocabulary term.

local maximum

local minimum

even function

odd function

Core Concepts**Zeros, Factors, Solutions, and Intercepts**

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial function. The following statements are equivalent.

Zero: k is a zero of the polynomial function f .

Factor: $x - k$ is a factor of the polynomial $f(x)$.

Solution: k is a solution (or root) of the polynomial equation $f(x) = 0$.

x-Intercept: If k is a real number, then k is an x -intercept of the graph of the polynomial function f . The graph of f passes through $(k, 0)$.

Notes:

4.8 Notetaking with Vocabulary (continued)**The Location Principle**

If f is a polynomial function, and a and b are two real numbers such that $f(a) < 0$ and $f(b) > 0$, then f has at least one real zero between a and b .

Notes:

Turning Points of Polynomial Functions

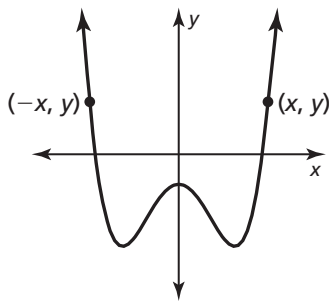
1. The graph of every polynomial function of degree n has *at most* $n - 1$ turning points.
2. If a polynomial function has n distinct real zeros, then its graph has *exactly* $n - 1$ turning points.

Notes:

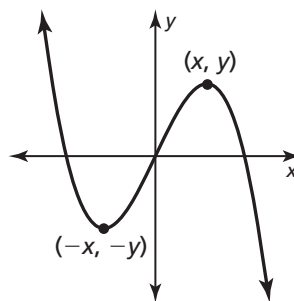
4.8 Notetaking with Vocabulary (continued)**Even and Odd Functions**

A function f is an **even function** when $f(-x) = f(x)$ for all x in its domain. The graph of an even function is *symmetric about the y-axis*.

A function f is an **odd function** when $f(-x) = -f(x)$ for all x in its domain. The graph of an odd function is *symmetric about the origin*. One way to recognize a graph that is symmetric about the origin is that it looks the same after a 180° rotation about the origin.

Even Function

For an even function, if (x, y) is on the graph, then $(-x, y)$ is also on the graph.

Odd Function

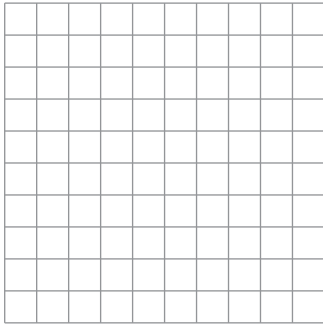
For an odd function, if (x, y) is on the graph, then $(-x, -y)$ is also on the graph.

Notes:

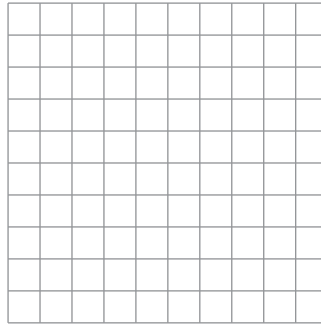
4.8 Notetaking with Vocabulary (continued)**Extra Practice**

In Exercises 1–6, graph the function. Identify the x -intercepts, and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing. Determine whether the function is *even*, *odd*, or *neither*.

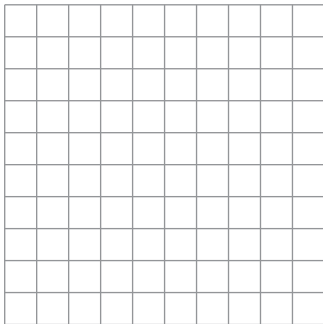
1. $f(x) = 4x^3 - 12x^2 - x + 15$



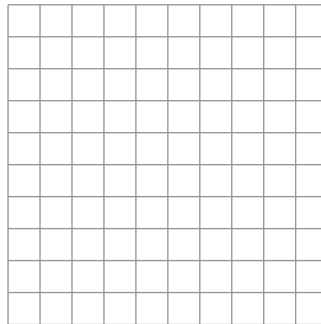
2. $g(x) = 2x^4 + 5x^3 - 21x^2 - 10x$



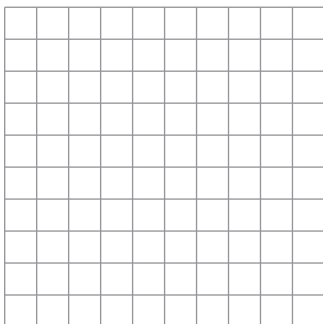
3. $h(x) = x^3 - x^2 - 13x - 3$



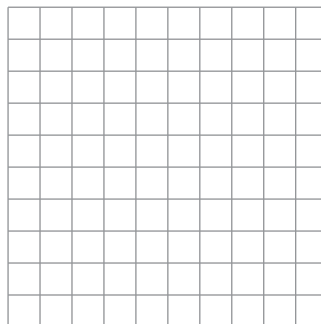
4. $k(x) = x^3 - 2x$



5. $f(x) = x^4 - 29x^2 + 100$



6. $g(x) = -\frac{1}{3}x^3 + x^2 - \frac{4}{3}$



4.9

Modeling with Polynomial Functions

For use with Exploration 4.9

Essential Question How can you find a polynomial model for real-life data?

1 EXPLORATION: Modeling Real-Life Data

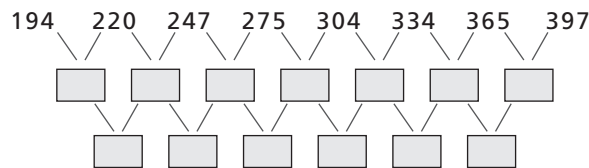
Go to *BigIdeasMath.com* for an interactive tool to investigate this exploration.

Work with a partner. The distance a baseball travels after it is hit depends on the angle at which it was hit and the initial speed. The table shows the distances a baseball hit at an angle of 35° travels at various initial speeds.

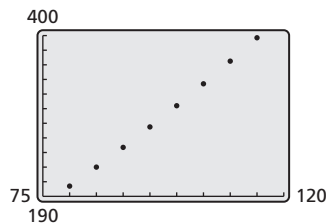
Initial speed, x (miles per hour)	80	85	90	95	100	105	110	115
Distance, y (feet)	194	220	247	275	304	334	365	397

- a. Recall that when data have equally-spaced x -values, you can analyze patterns in the differences of the y -values to determine what type of function can be used to model the data. If the first differences are constant, then the set of data fits a linear model. If the second differences are constant, then the set of data fits a quadratic model.

Find the first and second differences of the data. Are the data linear or quadratic? Explain your reasoning.



- b. Use a graphing calculator to draw a scatter plot of the data. Do the data appear linear or quadratic? Use the *regression* feature of the graphing calculator to find a linear or quadratic model that best fits the data.



4.9 Modeling with Polynomial Functions (continued)**1 EXPLORATION: Modeling Real-Life Data (continued)**

- c. Use the model you found in part (b) to find the distance a baseball travels when it is hit at an angle of 35° and travels at an initial speed of 120 miles per hour.
- d. According to the *Baseball Almanac*, “Any drive over 400 feet is noteworthy. A blow of 450 feet shows exceptional power, as the majority of major league players are unable to hit a ball that far. Anything in the 500-foot range is genuinely historic.” Estimate the initial speed of a baseball that travels a distance of 500 feet.

Communicate Your Answer

2. How can you find a polynomial model for real-life data?
3. How well does the model you found in Exploration 1(b) fit the data? Do you think the model is valid for any initial speed? Explain your reasoning.

4.9**Notetaking with Vocabulary**

For use after Lesson 4.9

In your own words, write the meaning of each vocabulary term.

finite differences

Core Concepts**Properties of Finite Differences**

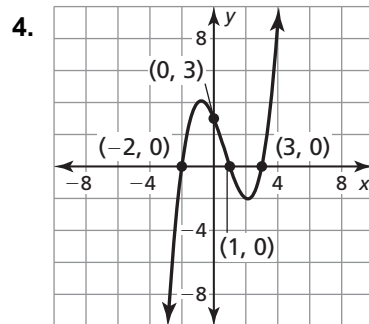
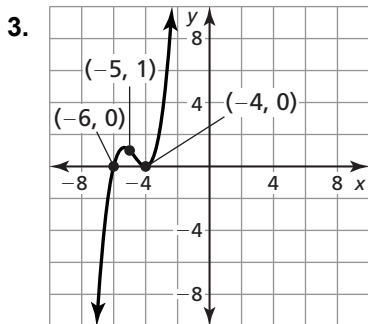
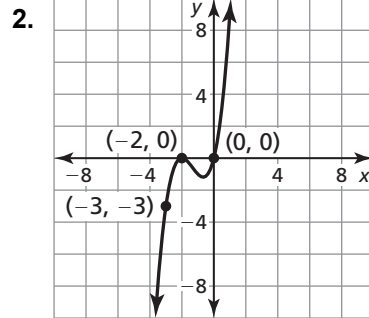
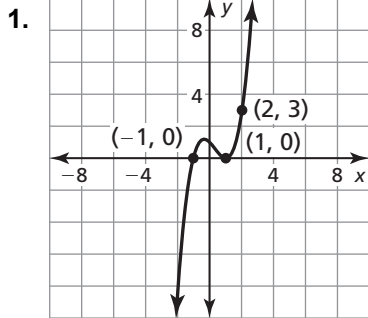
1. If a polynomial function $f(x)$ has degree n , then the n th differences of function values for equally-spaced x -values are nonzero and constant.
2. Conversely, if the n th differences of equally-spaced data are nonzero and constant, then the data can be represented by a polynomial function of degree n .

Notes:

4.9 Notetaking with Vocabulary (continued)

Extra Practice

In Exercises 1–4, write a cubic function whose graph passes through the given points.



In Exercises 5–8, use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.

5.

x	-2	-1	0	1	2	3
$f(x)$	-14	-6.5	0	5.5	10	13.5

6.

x	-2	-1	0	1	2
$f(x)$	30	4	0	0	-14

7. $(0, 0), (2, 0), (4, 40), (6, 168), (8, 432), (10, 880)$

8. $(0, 10), (1, 10), (2, 18), (3, 64), (4, 202), (5, 510)$

4.9 Notetaking with Vocabulary (continued)

9. The table shows the population y of bacteria after x hours. Find a polynomial model for the data for the first 4.5 hours. Use the model to estimate the population level of the bacteria (in thousands) after 1 day.

Number of Hours, x	0.5	1	2.5	3	4	4.5
Number of Bacteria, y	5.125	6	20.625	32	69	96.125

10. The table shows the value y (in thousands of dollars) of a signed autograph of a MVP football player, where x represents the number of years since 2000. Find a polynomial model for the data for the first 5 years. Use the model to estimate the year that the autograph will be valued at \$5,000,000?

Number of Years, x	1	2	3	4	5
Value of autograph, y	6	34	162	510	1246