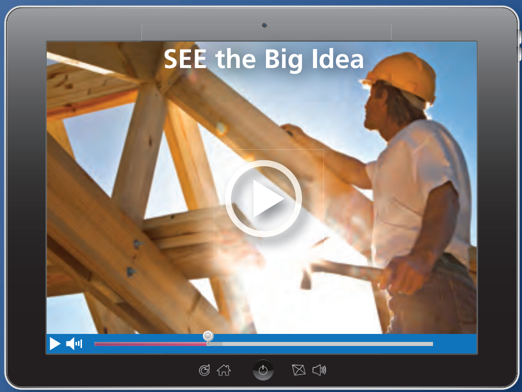
6 Relationships Within Triangles

Biking *(p. 314)*

Montana *(p. 309)*



Roof Truss *(p. 299)*

Windmill *(p. 288)*

Bridge *(p. 273)*

Maintaining Mathematical Proficiency

Writing an Equation of a Perpendicular Line

Example 1 **Write the equation of a line passing through the point (−2, 0) that is perpendicular to the line $y = 2x + 8$.**

Step 1 Find the slope m of the perpendicular line. The line $y = 2x + 8$ has a slope of 2. Use the Slopes of Perpendicular Lines Theorem (Theorem 3.14).

$2 \cdot m = -1$ — The product of the slopes of ⊥ lines is −1.

$m = -\frac{1}{2}$ — Divide each side by 2.

Step 2 Find the y-intercept b by using $m = -\frac{1}{2}$ and $(x, y) = (-2, 0)$.

$y = mx + b$ — Use the slope-intercept form.

$0 = -\frac{1}{2}(-2) + b$ — Substitute for m, x, and y.

$-1 = b$ — Solve for b.

▶ Because $m = -\frac{1}{2}$ and $b = -1$, an equation of the line is $y = -\frac{1}{2}x - 1$.

Write an equation of the line passing through point *P* that is perpendicular to the given line.

1. $P(3, 1)$, $y = \frac{1}{3}x - 5$

2. $P(4, -3)$, $y = -x - 5$

3. $P(-1, -2)$, $y = -4x + 13$

Writing Compound Inequalities

Example 2 **Write each sentence as an inequality.**

a. A number x is greater than or equal to −1 and less than 6.

A number x is greater than or equal to −1 and less than 6.

$x \geq -1$ *and* $x < 6$

▶ An inequality is $-1 \leq x < 6$.

b. A number y is at most 4 or at least 9.

A number y is at most 4 or at least 9.

$y \leq 4$ *or* $y \geq 9$

▶ An inequality is $y \leq 4$ *or* $y \geq 9$.

Write the sentence as an inequality.

4. A number w is at least −3 and no more than 8.

5. A number m is more than 0 and less than 11.

6. A number s is less than or equal to 5 or greater than 2.

7. A number d is fewer than 12 or no less than −7.

8. **ABSTRACT REASONING** Is it possible for the solution of a compound inequality to be all real numbers? Explain your reasoning.

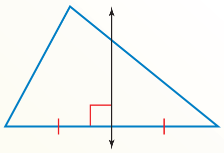
Mathematical Processes

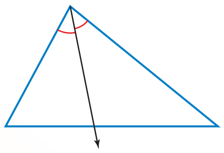
Mathematically proficient students use technological tools to explore concepts.

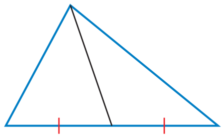
Lines, Rays, and Segments in Triangles

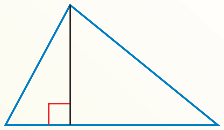
Core Concept

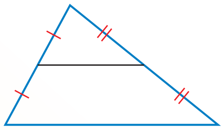
Lines, Rays, and Segments in Triangles


Perpendicular Bisector


Angle Bisector


Median

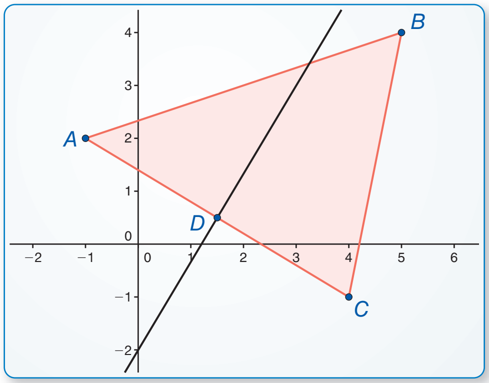

Altitude


Midsegment

EXAMPLE 1 **Drawing a Perpendicular Bisector**

Use dynamic geometry software to construct the perpendicular bisector of one of the sides of the triangle with vertices $A(-1, 2)$, $B(5, 4)$, and $C(4, -1)$. Find the lengths of the two segments of the bisected side.

SOLUTION



Sample

Points
$A(-1, 2)$
$B(5, 4)$
$C(4, -1)$
Line
$-5x + 3y = -6$
Segments
$AD = 2.92$
$CD = 2.92$

▶ The two segments of the bisected side have the same length, $AD = CD = 2.92$ units.

Monitoring Progress

Refer to the figures at the top of the page to describe each type of line, ray, or segment in a triangle.

1. perpendicular bisector **2.** angle bisector **3.** median

4. altitude **5.** midsegment

6.1 Perpendicular and Angle Bisectors

Essential Question What conjectures can you make about a point on the perpendicular bisector of a segment and a point on the bisector of an angle?

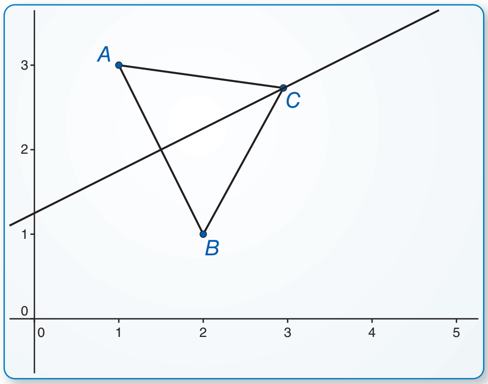
EXPLORATION 1 Points on a Perpendicular Bisector

Work with a partner. Use dynamic geometry software.

a. Draw any segment and label it $\overline{AB}$. Construct the perpendicular bisector of $\overline{AB}$.

b. Label a point C that is on the perpendicular bisector of $\overline{AB}$ but is not on $\overline{AB}$.

c. Draw $\overline{CA}$ and $\overline{CB}$ and find their lengths. Then move point C to other locations on the perpendicular bisector and note the lengths of $\overline{CA}$ and $\overline{CB}$.



Sample
Points
$A(1, 3)$
$B(2, 1)$
$C(2.95, 2.73)$
Segments
$AB = 2.24$
$CA = ?$
$CB = ?$
Line
$-x + 2y = 2.5$

d. Repeat parts (a)–(c) with other segments. Describe any relationship(s) you notice.

EXPLORATION 2 Points on an Angle Bisector

Work with a partner. Use dynamic geometry software.

a. Draw two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ to form $\angle BAC$. Construct the bisector of $\angle BAC$.

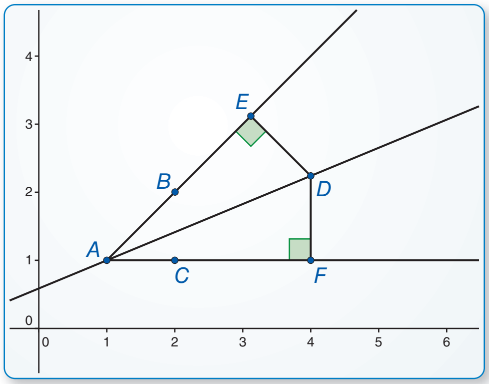
b. Label a point D on the bisector of $\angle BAC$.

c. Construct and find the lengths of the perpendicular segments from D to the sides of $\angle BAC$. Move point D along the angle bisector and note how the lengths change.

d. Repeat parts (a)–(c) with other angles. Describe any relationship(s) you notice.

USING TOOLS STRATEGICALLY

To be proficient in math, you need to visualize the results of varying assumptions, explore consequences, and compare predictions with data.



Sample
Points
$A(1, 1)$
$B(2, 2)$
$C(2, 1)$
$D(4, 2.24)$
Rays
$AB = -x + y = 0$
$AC = y = 1$
Line
$-0.38x + 0.92y = 0.54$

Communicate Your Answer

3. What conjectures can you make about a point on the perpendicular bisector of a segment and a point on the bisector of an angle?

4. In Exploration 2, what is the distance from point D to $\overrightarrow{AB}$ when the distance from D to $\overrightarrow{AC}$ is 5 units? Justify your answer.

6.1 Lesson

Core Vocabulary

equidistant, *p. 272*

Previous
perpendicular bisector
angle bisector

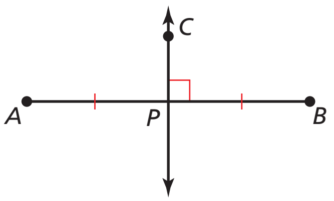
What You Will Learn

- Use perpendicular bisectors to find measures.
- Use angle bisectors to find measures and distance relationships.
- Write equations for perpendicular bisectors.

Using Perpendicular Bisectors

In Section 3.4, you learned that a *perpendicular bisector* of a line segment is the line that is perpendicular to the segment at its midpoint.

A point is **equidistant** from two figures when the point is the *same distance* from each figure.



$\overleftrightarrow{CP}$ is a $\perp$ bisector of $\overline{AB}$.

STUDY TIP

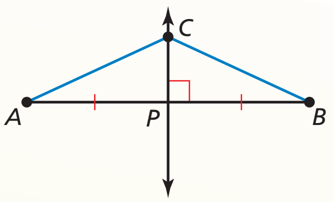
A perpendicular bisector can be a segment, a ray, a line, or a plane.

Theorems

Theorem 6.1 Perpendicular Bisector Theorem

In a plane, if a point lies on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

If $\overleftrightarrow{CP}$ is the $\perp$ bisector of $\overline{AB}$, then $CA = CB$.

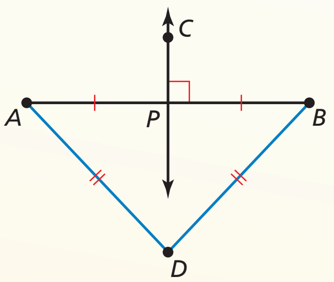


Proof p. 272

Theorem 6.2 Converse of the Perpendicular Bisector Theorem

In a plane, if a point is equidistant from the endpoints of a segment, then it lies on the perpendicular bisector of the segment.

If $DA = DB$, then point D lies on the $\perp$ bisector of $\overline{AB}$.

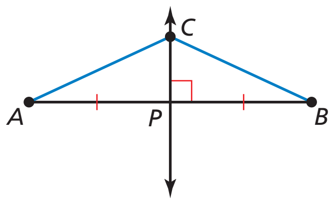


Proof Ex. 32, p. 278

PROOF Perpendicular Bisector Theorem

Given $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$.

Prove $CA = CB$

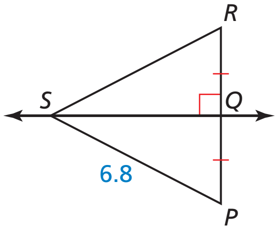


Paragraph Proof Because $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$, $\overleftrightarrow{CP}$ is perpendicular to $\overline{AB}$ and point P is the midpoint of $\overline{AB}$. By the definition of midpoint, $AP = BP$, and by the definition of perpendicular lines, $m\angle CPA = m\angle CPB = 90°$. Then by the definition of segment congruence, $\overline{AP} \cong \overline{BP}$, and by the definition of angle congruence, $\angle CPA \cong \angle CPB$. By the Reflexive Property of Congruence (Theorem 2.1), $\overline{CP} \cong \overline{CP}$. So, $\triangle CPA \cong \triangle CPB$ by the SAS Congruence Theorem (Theorem 5.5), and $\overline{CA} \cong \overline{CB}$ because corresponding parts of congruent triangles are congruent. So, $CA = CB$ by the definition of segment congruence.

EXAMPLE 1 Using the Perpendicular Bisector Theorems

Find each measure.

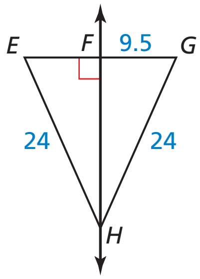
a. *RS*



From the figure, $\overleftrightarrow{SQ}$ is the perpendicular bisector of $\overline{PR}$. By the Perpendicular Bisector Theorem, $PS = RS$.

▶ So, $RS = PS = 6.8$.

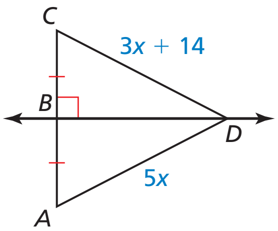
b. *EG*



Because $EH = GH$ and $\overleftrightarrow{HF} \perp \overline{EG}$, $\overleftrightarrow{HF}$ is the perpendicular bisector of $\overline{EG}$ by the Converse of the Perpendicular Bisector Theorem. By the definition of segment bisector, $EG = 2GF$.

▶ So, $EG = 2(9.5) = 19$.

c. *AD*



From the figure, $\overleftrightarrow{BD}$ is the perpendicular bisector of $\overline{AC}$.

$AD = CD$	Perpendicular Bisector Theorem
$5x = 3x + 14$	Substitute.
$x = 7$	Solve for *x*.

▶ So, $AD = 5x = 5(7) = 35$.

EXAMPLE 2 Solving a Real-Life Problem

Is there enough information in the diagram to conclude that point *N* lies on the perpendicular bisector of $\overline{KM}$?

SOLUTION

It is given that $\overline{KL} \cong \overline{ML}$. So, $\overline{LN}$ is a segment bisector of $\overline{KM}$. You do not know whether $\overline{LN}$ is perpendicular to $\overline{KM}$ because it is not indicated in the diagram.

▶ So, you cannot conclude that point *N* lies on the perpendicular bisector of $\overline{KM}$.

Monitoring Progress

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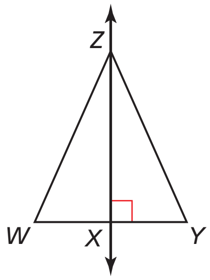
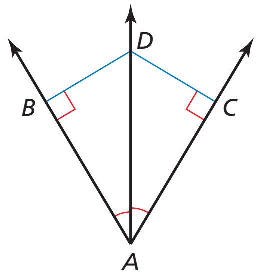
Use the diagram and the given information to find the indicated measure.



1. $\overleftrightarrow{ZX}$ is the perpendicular bisector of $\overline{WY}$, and $YZ = 13.75$. Find *WZ*.

2. $\overleftrightarrow{ZX}$ is the perpendicular bisector of $\overline{WY}$, $WZ = 4n - 13$, and $YZ = n + 17$. Find *YZ*.

3. Find *WX* when $WZ = 20.5$, $WY = 14.8$, and $YZ = 20.5$.



Using Angle Bisectors

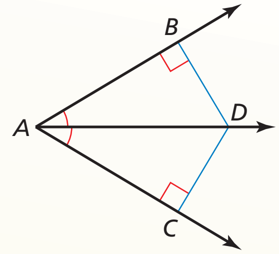
In Section 1.5, you learned that an *angle bisector* is a ray that divides an angle into two congruent adjacent angles. You also know that the *distance from a point to a line* is the length of the perpendicular segment from the point to the line. So, in the figure, $\overrightarrow{AD}$ is the bisector of $\angle BAC$, and the distance from point D to $\overrightarrow{AB}$ is DB, where $\overline{DB} \perp \overrightarrow{AB}$.

Theorems

Theorem 6.3 Angle Bisector Theorem

If a point lies on the bisector of an angle, then it is equidistant from the two sides of the angle.

If $\overrightarrow{AD}$ bisects $\angle BAC$ and $\overline{DB} \perp \overrightarrow{AB}$ and $\overline{DC} \perp \overrightarrow{AC}$, then $DB = DC$.

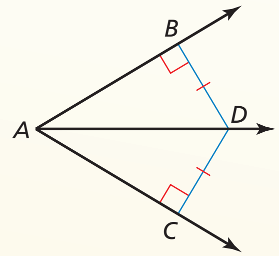


Proof Ex. 33(a), p. 278

Theorem 6.4 Converse of the Angle Bisector Theorem

If a point is in the interior of an angle and is equidistant from the two sides of the angle, then it lies on the bisector of the angle.

If $\overline{DB} \perp \overrightarrow{AB}$ and $\overline{DC} \perp \overrightarrow{AC}$ and $DB = DC$, then $\overrightarrow{AD}$ bisects $\angle BAC$.

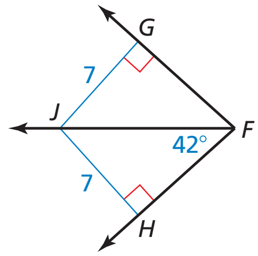


Proof Ex. 33(b), p. 278

EXAMPLE 3 Using the Angle Bisector Theorems

Find each measure.

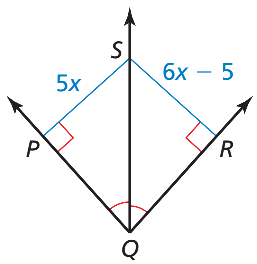
a. $m\angle GFJ$



Because $\overline{JG} \perp \overrightarrow{FG}$ and $\overline{JH} \perp \overrightarrow{FH}$ and $JG = JH = 7$, $\overrightarrow{FJ}$ bisects $\angle GFH$ by the Converse of the Angle Bisector Theorem.

▶ So, $m\angle GFJ = m\angle HFJ = 42°$.

b. RS



$PS = RS$	Angle Bisector Theorem
$5x = 6x - 5$	Substitute.
$5 = x$	Solve for x.

▶ So, $RS = 6x - 5 = 6(5) - 5 = 25$.

Monitoring Progress

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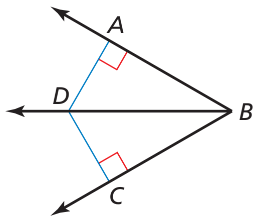
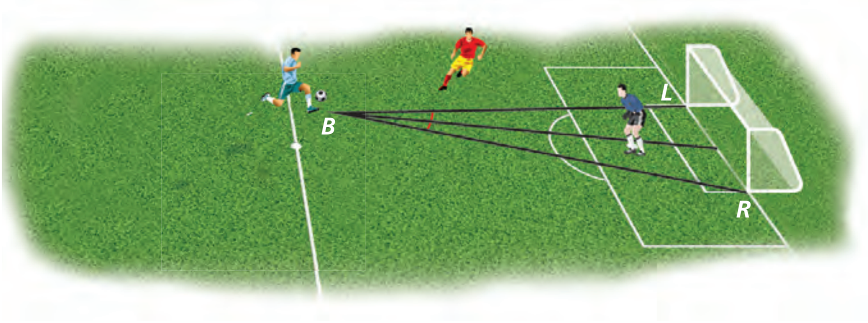
Use the diagram and the given information to find the indicated measure.



4. $\overrightarrow{BD}$ bisects $\angle ABC$, and $DC = 6.9$. Find DA.
5. $\overrightarrow{BD}$ bisects $\angle ABC$, $AD = 3z + 7$, and $CD = 2z + 11$. Find CD.
6. Find $m\angle ABC$ when $AD = 3.2$, $CD = 3.2$, and $m\angle DBC = 39°$.

EXAMPLE 4 Solving a Real-Life Problem

A soccer goalie's position relative to the ball and goalposts forms congruent angles, as shown. Will the goalie have to move farther to block a shot toward the right goalpost R or the left goalpost L?



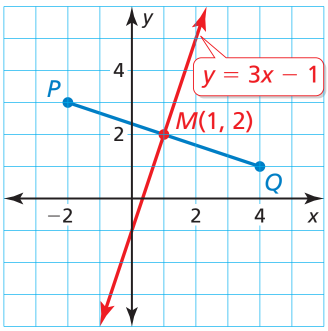
SOLUTION

The congruent angles tell you that the goalie is on the bisector of $\angle LBR$. By the Angle Bisector Theorem, the goalie is equidistant from $\overrightarrow{BR}$ and $\overrightarrow{BL}$.

▶ So, the goalie must move the same distance to block either shot.

Writing Equations for Perpendicular Bisectors

EXAMPLE 5 Writing an Equation for a Bisector



Write an equation of the perpendicular bisector of the segment with endpoints $P(-2, 3)$ and $Q(4, 1)$.

SOLUTION

Step 1 Graph $\overline{PQ}$. By definition, the perpendicular bisector of $\overline{PQ}$ is perpendicular to $\overline{PQ}$ at its midpoint.

Step 2 Find the midpoint M of $\overline{PQ}$.

$$M\left(\frac{-2+4}{2}, \frac{3+1}{2}\right) = M\left(\frac{2}{2}, \frac{4}{2}\right) = M(1, 2)$$

Step 3 Find the slope of the perpendicular bisector.

$$\text{slope of } \overline{PQ} = \frac{1-3}{4-(-2)} = \frac{-2}{6} = -\frac{1}{3}$$

Because the slopes of perpendicular lines are negative reciprocals, the slope of the perpendicular bisector is 3.

Step 4 Write an equation. The bisector of $\overline{PQ}$ has slope 3 and passes through (1, 2).

$y = mx + b$ — Use slope-intercept form.

$2 = 3(1) + b$ — Substitute for m, x, and y.

$-1 = b$ — Solve for b.

▶ So, an equation of the perpendicular bisector of $\overline{PQ}$ is $y = 3x - 1$.

Monitoring Progress

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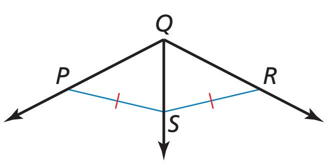


7. Do you have enough information to conclude that $\overrightarrow{QS}$ bisects $\angle PQR$? Explain.

8. Write an equation of the perpendicular bisector of the segment with endpoints $(-1, -5)$ and $(3, -1)$.

6.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** Point C is in the interior of $\angle DEF$. If $\angle DEC$ and $\angle CEF$ are congruent, then $\overrightarrow{EC}$ is the ________ of $\angle DEF$.

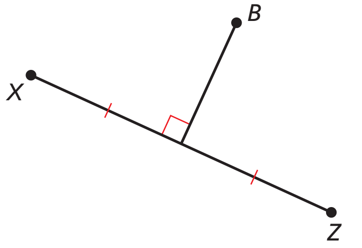
2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

Is point B the same distance from both X and Z?

Is point B equidistant from X and Z?

Is point B collinear with X and Z?

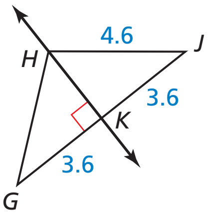
Is point B on the perpendicular bisector of $\overline{XZ}$?



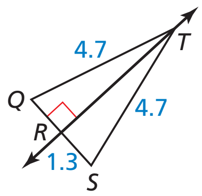
Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, find the indicated measure. Explain your reasoning. *(See Example 1.)*

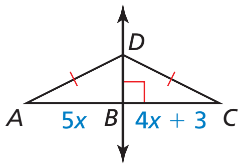
3. GH



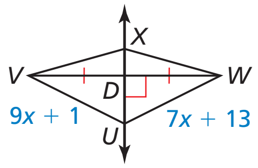
4. QR



5. AB

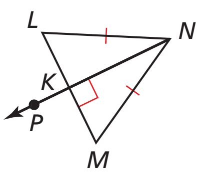


6. UW

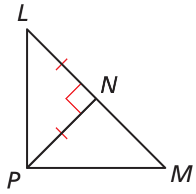


In Exercises 7–10, tell whether the information in the diagram allows you to conclude that point P lies on the perpendicular bisector of $\overline{LM}$. Explain your reasoning. *(See Example 2.)*

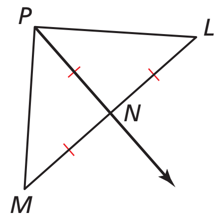
7.



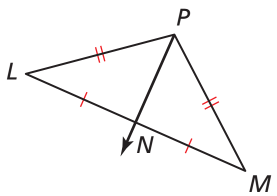
8.



9.

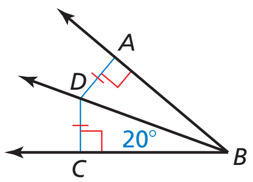


10.

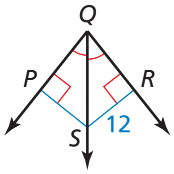


In Exercises 11–14, find the indicated measure. Explain your reasoning. *(See Example 3.)*

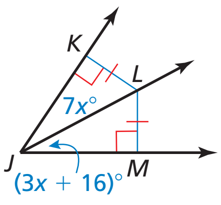
11. $m\angle ABD$



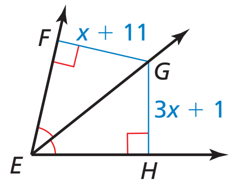
12. PS



13. $m\angle KJL$

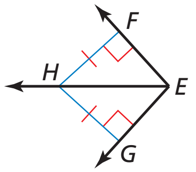


14. FG

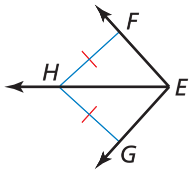


In Exercises 15 and 16, tell whether the information in the diagram allows you to conclude that $\overrightarrow{EH}$ bisects $\angle FEG$. Explain your reasoning. *(See Example 4.)*

15.

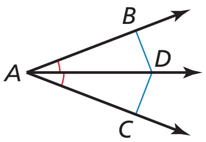


16.

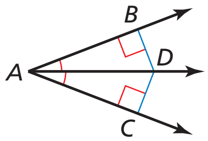


In Exercises 17 and 18, tell whether the information in the diagram allows you to conclude that $DB = DC$. Explain your reasoning.

17.



18.



In Exercises 19–22, write an equation of the perpendicular bisector of the segment with the given endpoints. *(See Example 5.)*

19. $M(1, 5)$, $N(7, -1)$

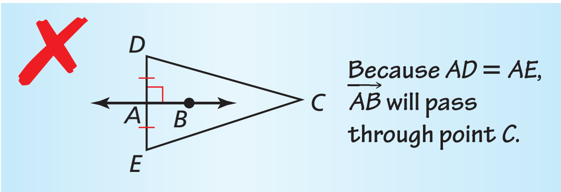
20. $Q(-2, 0)$, $R(6, 12)$

21. $U(-3, 4)$, $V(9, 8)$

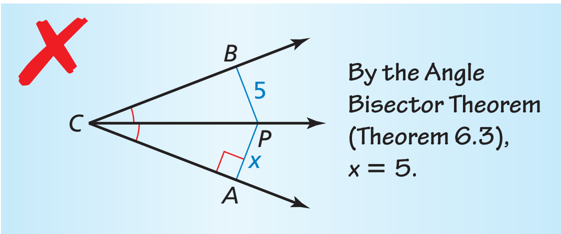
22. $Y(10, -7)$, $Z(-4, 1)$

ERROR ANALYSIS **In Exercises 23 and 24, describe and correct the error in the student's reasoning.**

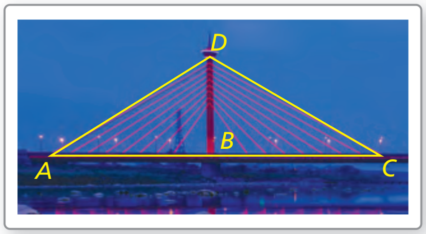
23.



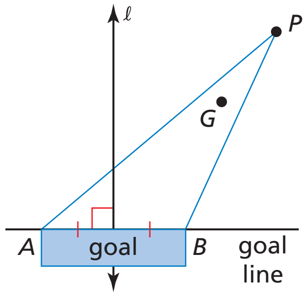
24.



25. **MODELING MATHEMATICS** In the photo, the road is perpendicular to the support beam and $\overline{AB} \cong \overline{CB}$. Which theorem allows you to conclude that $\overline{AD} \cong \overline{CD}$?



26. **MODELING WITH MATHEMATICS** The diagram shows the position of the goalie and the puck during a hockey game. The goalie is at point G, and the puck is at point P.



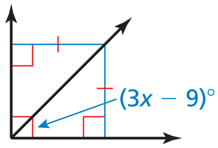
a. What should be the relationship between $\overrightarrow{PG}$ and $\angle APB$ to give the goalie equal distances to travel on each side of $\overrightarrow{PG}$?

b. How does $m\angle APB$ change as the puck gets closer to the goal? Does this change make it easier or more difficult for the goalie to defend the goal? Explain your reasoning.

27. **CONSTRUCTION** Use a compass and straightedge to construct a copy of $\overline{XY}$. Construct a perpendicular bisector and plot a point Z on the bisector so that the distance between point Z and $\overline{XY}$ is 3 centimeters. Measure $\overline{XZ}$ and $\overline{YZ}$. Which theorem does this construction demonstrate?

28. **WRITING** Explain how the Converse of the Perpendicular Bisector Theorem (Theorem 6.2) is related to the construction of a perpendicular bisector.

29. **REASONING** What is the value of x in the diagram?



Ⓐ 13

Ⓑ 18

Ⓒ 33

Ⓓ not enough information

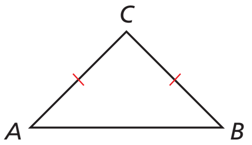
30. **REASONING** Which point lies on the perpendicular bisector of the segment with endpoints $M(7, 5)$ and $N(-1, 5)$?

Ⓐ $(2, 0)$ Ⓑ $(3, 9)$

Ⓒ $(4, 1)$ Ⓓ $(1, 3)$

31. **MAKING AN ARGUMENT** Your friend says it is impossible for an angle bisector of a triangle to be the same line as the perpendicular bisector of the opposite side. Is your friend correct? Explain your reasoning.

32. PROVING A THEOREM Prove the Converse of the Perpendicular Bisector Theorem (Thm. 6.2). (*Hint:* Construct a line through point C perpendicular to $\overline{AB}$ at point P.)



Given $CA = CB$

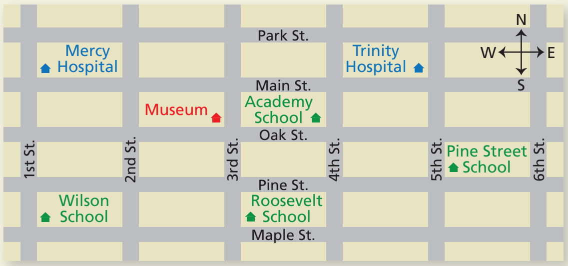
Prove Point C lies on the perpendicular bisector of $\overline{AB}$.

33. PROVING A THEOREM Use a congruence theorem to prove each theorem.

a. Angle Bisector Theorem (Thm. 6.3)

b. Converse of the Angle Bisector Theorem (Thm. 6.4)

34. HOW DO YOU SEE IT? The figure shows a map of a city. The city is arranged so each block north to south is the same length and each block east to west is the same length.



a. Which school is approximately equidistant from both hospitals? Explain your reasoning.

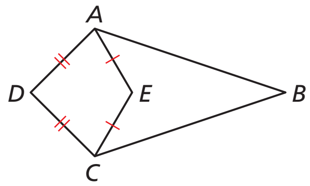
b. Is the museum approximately equidistant from Wilson School and Roosevelt School? Explain your reasoning.

35. MATHEMATICAL CONNECTIONS Write an equation whose graph consists of all the points in the given quadrants that are equidistant from the x- and y-axes.

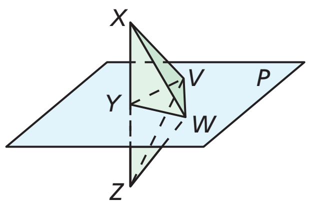
a. I and III **b.** II and IV **c.** I and II

36. THOUGHT PROVOKING The postulates and theorems in this book represent Euclidean geometry. In spherical geometry, all points are on the surface of a sphere. A line is a circle on the sphere whose diameter is equal to the diameter of the sphere. In spherical geometry, is it possible for two lines to be perpendicular but not bisect each other? Explain your reasoning.

37. PROOF Use the information in the diagram to prove that $\overline{AB} \cong \overline{CB}$ if and only if points D, E, and B are collinear.



38. PROOF Prove the statements in parts (a)–(c).



Given Plane P is a perpendicular bisector of $\overline{XZ}$ at point Y.

Prove **a.** $\overline{XW} \cong \overline{ZW}$

b. $\overline{XV} \cong \overline{ZV}$

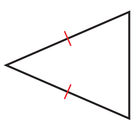
c. $\angle VXW \cong \angle VZW$

Maintaining Mathematical Proficiency

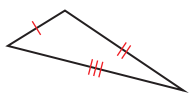
Reviewing what you learned in previous grades and lessons

Classify the triangle by its sides.

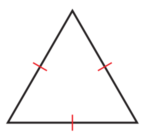
39.



40.

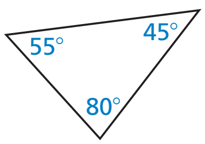


41.

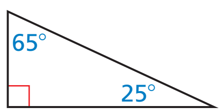


Classify the triangle by its angles.

42.



43.



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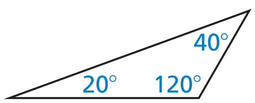


6.2 Bisectors of Triangles

Essential Question

What conjectures can you make about the perpendicular bisectors and the angle bisectors of a triangle?

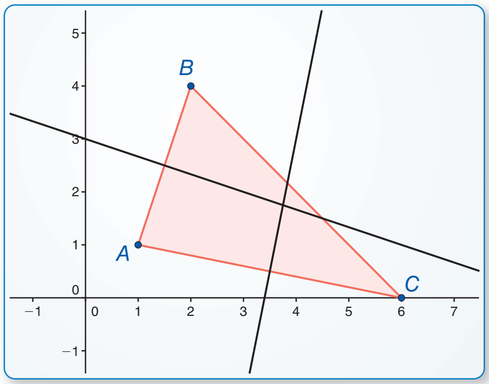
EXPLORATION 1 Properties of the Perpendicular Bisectors of a Triangle

Work with a partner. Use dynamic geometry software. Draw any $\triangle ABC$.

a. Construct the perpendicular bisectors of all three sides of $\triangle ABC$. Then drag the vertices to change $\triangle ABC$. What do you notice about the perpendicular bisectors?

b. Label a point D at the intersection of the perpendicular bisectors.

c. Draw the circle with center D through vertex A of $\triangle ABC$. Then drag the vertices to change $\triangle ABC$. What do you notice?



Sample

Points
$A(1, 1)$
$B(2, 4)$
$C(6, 0)$
Segments
$BC = 5.66$
$AC = 5.10$
$AB = 3.16$
Lines
$x + 3y = 9$
$-5x + y = -17$

LOOKING FOR STRUCTURE

To be proficient in math, you need to see complicated things as single objects or as being composed of several objects.

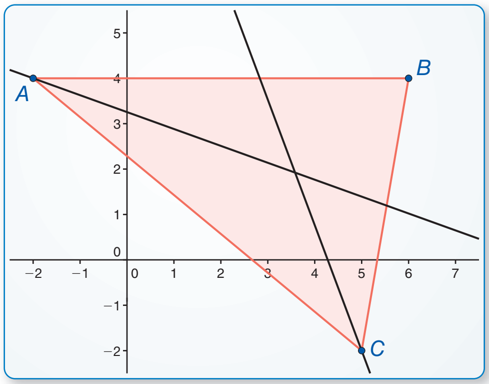
EXPLORATION 2 Properties of the Angle Bisectors of a Triangle

Work with a partner. Use dynamic geometry software. Draw any $\triangle ABC$.

a. Construct the angle bisectors of all three angles of $\triangle ABC$. Then drag the vertices to change $\triangle ABC$. What do you notice about the angle bisectors?

b. Label a point D at the intersection of the angle bisectors.

c. Find the distance between D and $\overline{AB}$. Draw the circle with center D and this distance as a radius. Then drag the vertices to change $\triangle ABC$. What do you notice?



Sample

Points
$A(-2, 4)$
$B(6, 4)$
$C(5, -2)$
Segments
$BC = 6.08$
$AC = 9.22$
$AB = 8$
Lines
$0.35x + 0.94y = 3.06$
$-0.94x - 0.34y = -4.02$

Communicate Your Answer

3. What conjectures can you make about the perpendicular bisectors and the angle bisectors of a triangle?

6.2 Lesson

What You Will Learn

- Use and find the circumcenter of a triangle.
- Use and find the incenter of a triangle.

Core Vocabulary

concurrent, *p. 280*
point of concurrency, *p. 280*
circumcenter, *p. 280*
incenter, *p. 283*

Previous
perpendicular bisector
angle bisector

Using the Circumcenter of a Triangle

When three or more lines, rays, or segments intersect in the same point, they are called **concurrent** lines, rays, or segments. The point of intersection of the lines, rays, or segments is called the **point of concurrency**.

In a triangle, the three perpendicular bisectors are concurrent. The point of concurrency is the **circumcenter** of the triangle.

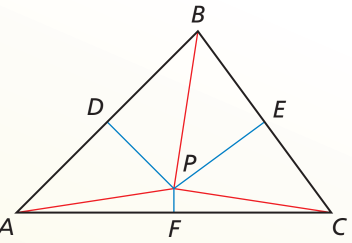
Theorems

Theorem 6.5 Circumcenter Theorem

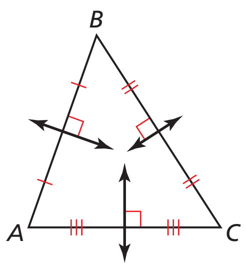
The circumcenter of a triangle is equidistant from the vertices of the triangle.

If $\overline{PD}$, $\overline{PE}$, and $\overline{PF}$ are perpendicular bisectors, then $PA = PB = PC$.

Proof p. 280



PROOF **Circumcenter Theorem**



Given $\triangle ABC$; the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$

Prove The perpendicular bisectors intersect in a point; that point is equidistant from A, B, and C.

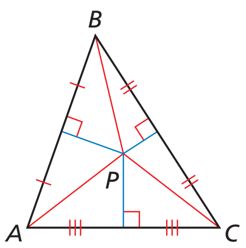
Plan for Proof Show that P, the point of intersection of the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$, also lies on the perpendicular bisector of $\overline{AC}$. Then show that point P is equidistant from the vertices of the triangle.

Plan in Action

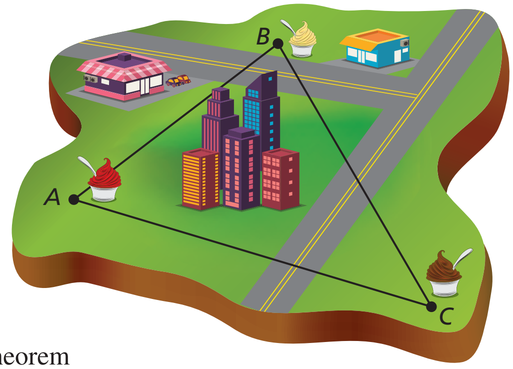
STATEMENTS	REASONS
1. $\triangle ABC$; the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$	**1.** Given
2. The perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ intersect at some point P.	**2.** Because the sides of a triangle cannot be parallel, these perpendicular bisectors must intersect in some point. Call it P.
3. Draw $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$.	**3.** Two Point Postulate (Post. 2.1)
4. $PA = PB$, $PB = PC$	**4.** Perpendicular Bisector Theorem (Thm. 6.1)
5. $PA = PC$	**5.** Transitive Property of Equality
6. P is on the perpendicular bisector of $\overline{AC}$.	**6.** Converse of the Perpendicular Bisector Theorem (Thm. 6.2)
7. $PA = PB = PC$. So, P is equidistant from the vertices of the triangle.	**7.** From the results of Steps 4 and 5 and the definition of equidistant

STUDY TIP

Use diagrams like the one below to help visualize your proof.



EXAMPLE 1 Solving a Real-Life Problem



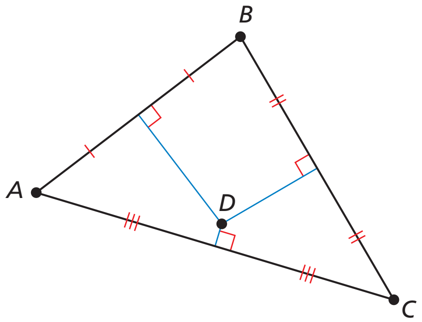
Three snack carts sell frozen yogurt from points A, B, and C outside a city. Each of the three carts is the same distance from the frozen yogurt distributor.

Find the location of the distributor.

SOLUTION

The distributor is equidistant from the three snack carts. The Circumcenter Theorem shows that you can find a point equidistant from three points by using the perpendicular bisectors of the triangle formed by those points.

Copy the positions of points A, B, and C and connect the points to draw $\triangle ABC$. Then use a ruler and protractor to draw the three perpendicular bisectors of $\triangle ABC$. The circumcenter D is the location of the distributor.



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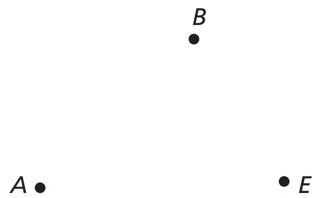
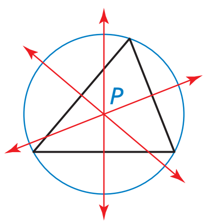
1. Three snack carts sell hot pretzels from points A, B, and E. What is the location of the pretzel distributor if it is equidistant from the three carts? Sketch the triangle and show the location.



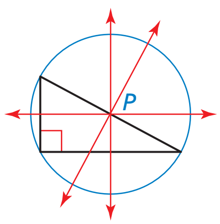
READING

The prefix *circum-* means "around" or "about," as in *circumference* (distance around a circle).

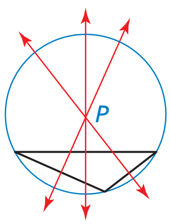
The circumcenter P is equidistant from the three vertices, so P is the center of a circle that passes through all three vertices. As shown below, the location of P depends on the type of triangle. The circle with center P is said to be *circumscribed* about the triangle.



Acute triangle
P is inside triangle.



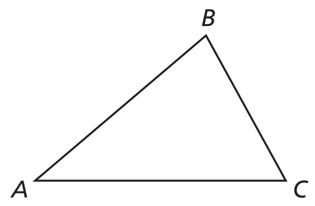
Right triangle
P is on triangle.



Obtuse triangle
P is outside triangle.

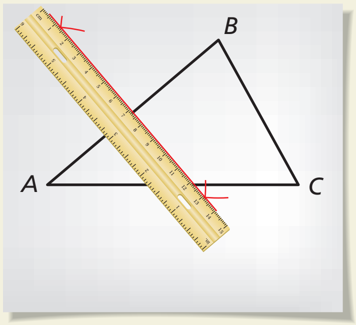
CONSTRUCTION Circumscribing a Circle About a Triangle

Use a compass and straightedge to construct a circle that is circumscribed about $\triangle ABC$.



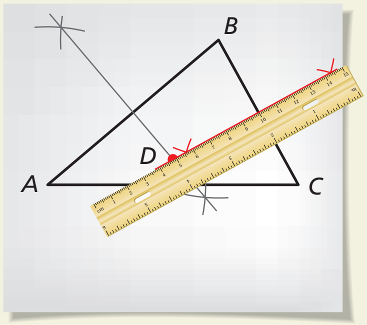
SOLUTION

Step 1



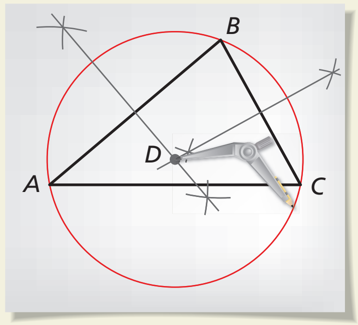
Draw a bisector Draw the perpendicular bisector of $\overline{AB}$.

Step 2



Draw a bisector Draw the perpendicular bisector of $\overline{BC}$. Label the intersection of the bisectors D. This is the circumcenter.

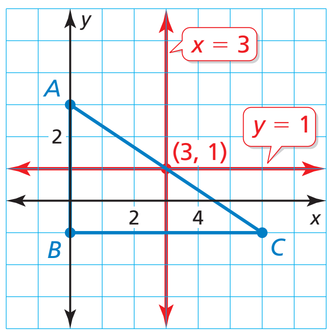
Step 3



Draw a circle Place the compass at D. Set the width by using any vertex of the triangle. This is the radius of the *circumcircle*. Draw the circle. It should pass through all three vertices A, B, and C.

EXAMPLE 2 Finding the Circumcenter of a Triangle

Find the coordinates of the circumcenter of $\triangle ABC$ with vertices $A(0, 3)$, $B(0, -1)$, and $C(6, -1)$.



SOLUTION

Step 1 Graph $\triangle ABC$.

Step 2 Find equations for two perpendicular bisectors. Use the Slopes of Perpendicular Lines Theorem (Theorem 3.14), which states that horizontal lines are perpendicular to vertical lines.

The midpoint of $\overline{AB}$ is $(0, 1)$. The line through $(0, 1)$ that is perpendicular to $\overline{AB}$ is $y = 1$.

The midpoint of $\overline{BC}$ is $(3, -1)$. The line through $(3, -1)$ that is perpendicular to $\overline{BC}$ is $x = 3$.

Step 3 Find the point where $x = 3$ and $y = 1$ intersect. They intersect at $(3, 1)$.

▶ So, the coordinates of the circumcenter are $(3, 1)$.

STUDY TIP

Note that you only need to find the equations for *two* perpendicular bisectors. You can use the perpendicular bisector of the third side to verify your result.

MAKING SENSE OF PROBLEMS

Because $\triangle ABC$ is a right triangle, the circumcenter lies on the triangle.

Monitoring Progress

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Find the coordinates of the circumcenter of the triangle with the given vertices.

2. $R(-2, 5)$, $S(-6, 5)$, $T(-2, -1)$

3. $W(-1, 4)$, $X(1, 4)$, $Y(1, -6)$

Using the Incenter of a Triangle

Just as a triangle has three perpendicular bisectors, it also has three angle bisectors. The angle bisectors of a triangle are also concurrent. This point of concurrency is the **incenter** of the triangle. For any triangle, the incenter always lies inside the triangle.

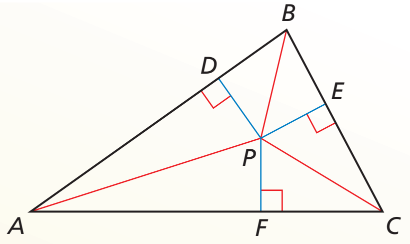
Theorem

Theorem 6.6 Incenter Theorem

The incenter of a triangle is equidistant from the sides of the triangle.

If $\overline{AP}$, $\overline{BP}$, and $\overline{CP}$ are angle bisectors of $\triangle ABC$, then $PD = PE = PF$.

Proof Ex. 38, p. 287

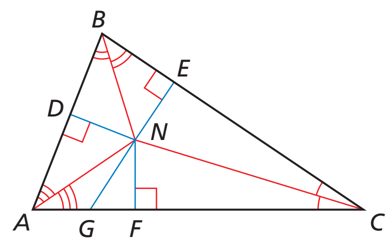


EXAMPLE 3 Using the Incenter of a Triangle

In the figure shown, $ND = 5x - 1$ and $NE = 2x + 11$.

a. Find NF.

b. Can NG be equal to 18? Explain your reasoning.



SOLUTION

a. N is the incenter of $\triangle ABC$ because it is the point of concurrency of the three angle bisectors. So, by the Incenter Theorem, $ND = NE = NF$.

Step 1 Solve for x.

$ND = NE$ — Incenter Theorem

$5x - 1 = 2x + 11$ — Substitute.

$x = 4$ — Solve for x.

Step 2 Find ND (or NE).

$$ND = 5x - 1 = 5(4) - 1 = 19$$

▶ So, because $ND = NF$, $NF = 19$.

b. Recall that the shortest distance between a point and a line is a perpendicular segment. In this case, the perpendicular segment is $\overline{NF}$, which has a length of 19. Because $18 < 19$, NG cannot be equal to 18.

Monitoring Progress

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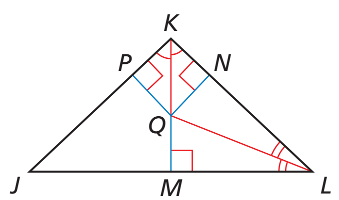
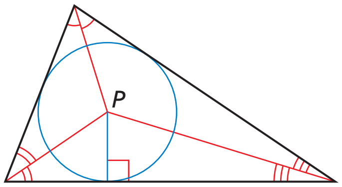
4. In the figure shown, $QM = 3x + 8$ and $QN = 7x + 2$. Find QP.

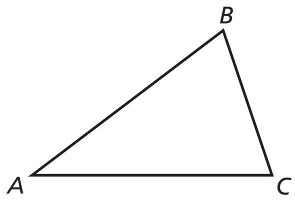




Because the incenter P is equidistant from the three sides of the triangle, a circle drawn using P as the center and the distance to one side of the triangle as the radius will just touch the other two sides of the triangle. The circle is said to be *inscribed* within the triangle.

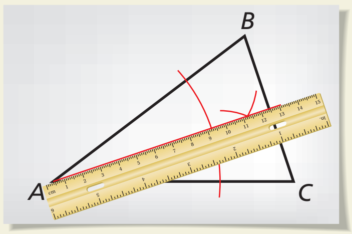
CONSTRUCTION Inscribing a Circle Within a Triangle

Use a compass and straightedge to construct a circle that is inscribed within $\triangle ABC$.



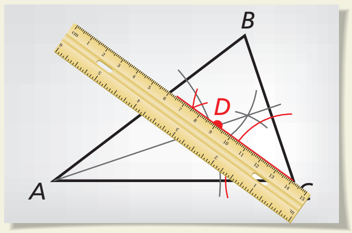
SOLUTION

Step 1



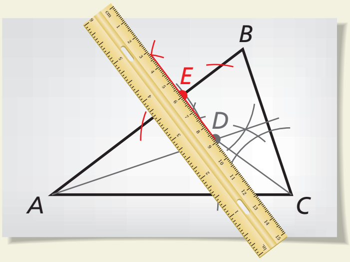
Draw a bisector Draw the angle bisector of $\angle A$.

Step 2



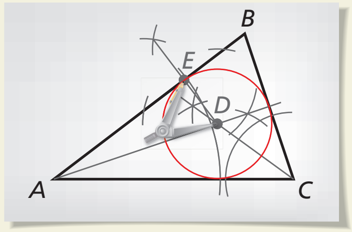
Draw a bisector Draw the angle bisector of $\angle C$. Label the intersection of the bisectors D. This is the incenter.

Step 3



Draw a perpendicular line Draw the perpendicular line from D to $\overline{AB}$. Label the point where it intersects $\overline{AB}$ as E.

Step 4



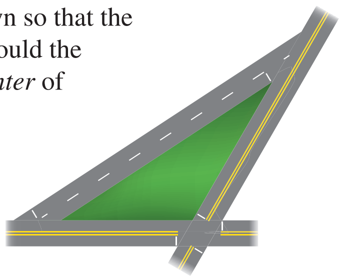
Draw a circle Place the compass at D. Set the width to E. This is the radius of the *incircle*. Draw the circle. It should touch each side of the triangle.

EXAMPLE 4 Solving a Real-Life Problem

ATTENDING TO PRECISION

Pay close attention to how a problem is stated. The city wants the lamppost to be the *same distance* from the three streets, not from where the streets intersect.

A city wants to place a lamppost on the boulevard shown so that the lamppost is the same distance from all three streets. Should the location of the lamppost be at the *circumcenter* or *incenter* of the triangular boulevard? Explain.



SOLUTION

Because the shape of the boulevard is an obtuse triangle, its circumcenter lies outside the triangle. So, the location of the lamppost cannot be at the circumcenter. The city wants the lamppost to be the same distance from all three streets. By the Incenter Theorem, the incenter of a triangle is equidistant from the sides of a triangle.

▶ So, the location of the lamppost should be at the incenter of the boulevard.

Monitoring Progress

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5. Draw a sketch to show the location L of the lamppost in Example 4.

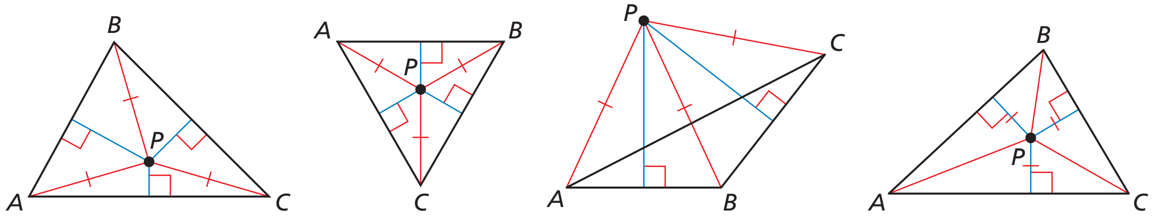
6.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** When three or more lines, rays, or segments intersect in the same point, they are called ___________ lines, rays, or segments.

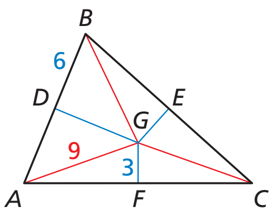
2. **WHICH ONE DOESN'T BELONG?** Which triangle does *not* belong with the other three? Explain your reasoning.



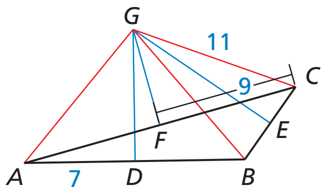
Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, the perpendicular bisectors of △*ABC* intersect at point *G* and are shown in blue. Find the indicated measure.

3. Find *BG*.

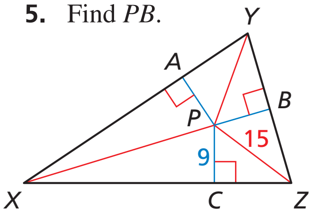


4. Find *GA*.

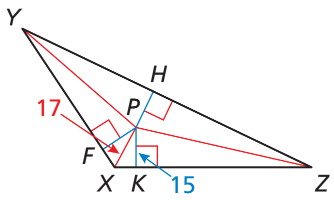


In Exercises 5 and 6, the angle bisectors of △*XYZ* intersect at point *P* and are shown in red. Find the indicated measure.

5. Find *PB*.



6. Find *HP*.



In Exercises 7–10, find the coordinates of the circumcenter of the triangle with the given vertices. *(See Example 2.)*

7. $A(2, 6)$, $B(8, 6)$, $C(8, 10)$

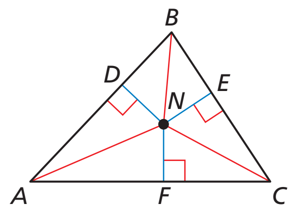
8. $D(-7, -1)$, $E(-1, -1)$, $F(-7, -9)$

9. $H(-10, 7)$, $J(-6, 3)$, $K(-2, 3)$

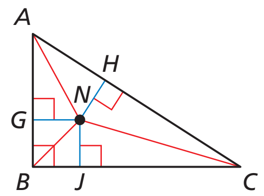
10. $L(3, -6)$, $M(5, -3)$, $N(8, -6)$

In Exercises 11–14, *N* is the incenter of △*ABC*. Use the given information to find the indicated measure. *(See Example 3.)*

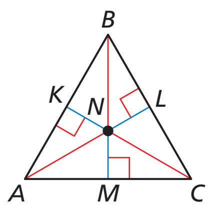
11. $ND = 6x - 2$
    $NE = 3x + 7$
    Find *NF*.



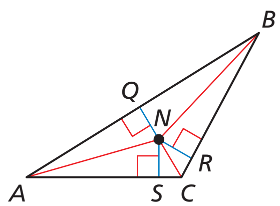
12. $NG = x + 3$
    $NH = 2x - 3$
    Find *NJ*.



13. $NK = 2x - 2$
    $NL = -x + 10$
    Find *NM*.



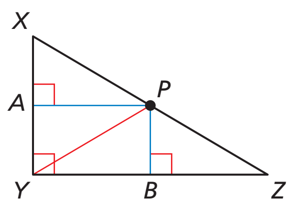
14. $NQ = 2x$
    $NR = 3x - 2$
    Find *NS*.



15. *P* is the circumcenter of △*XYZ*. Use the given information to find *PZ*.

    $PX = 3x + 2$

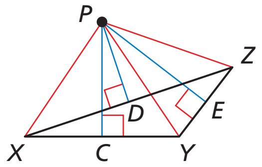
    $PY = 4x - 8$



16. P is the circumcenter of $\triangle XYZ$. Use the given information to find PY.

$PX = 4x + 3$

$PZ = 6x - 11$



CONSTRUCTION **In Exercises 17–20, draw a triangle of the given type. Find the circumcenter. Then construct the circumscribed circle.**

17. right

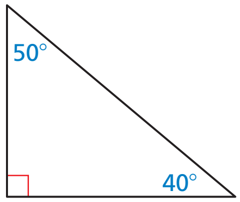
18. obtuse

19. acute isosceles

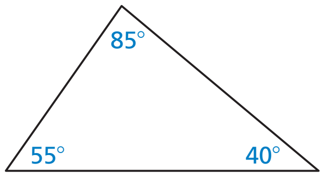
20. equilateral

CONSTRUCTION **In Exercises 21–24, copy the triangle with the given angle measures. Find the incenter. Then construct the inscribed circle.**

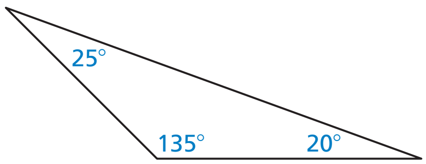
21.



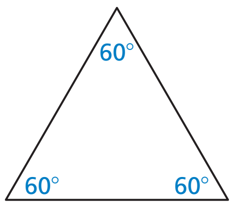
22.



23.

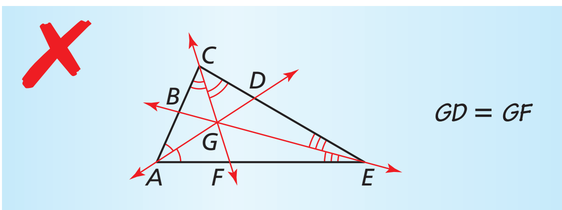


24.

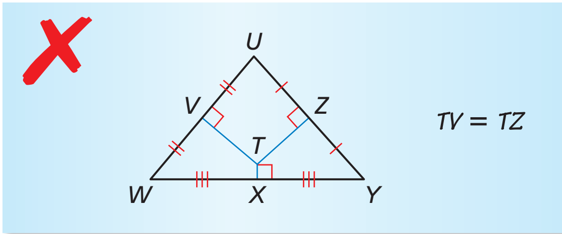


ERROR ANALYSIS **In Exercises 25 and 26, describe and correct the error in identifying equal distances inside the triangle.**

25.

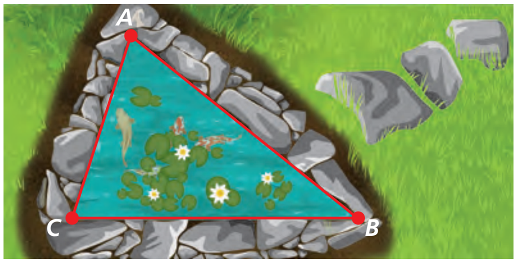


26.



27. **MODELING WITH MATHEMATICS** You and two friends plan to meet to walk your dogs together. You want the meeting place to be the same distance from each person's house. Explain how you can use the diagram to locate the meeting place. *(See Example 1.)*

28. **MODELING WITH MATHEMATICS** You are placing a fountain in a triangular koi pond. You want the fountain to be the same distance from each edge of the pond. Where should you place the fountain? Explain your reasoning. Use a sketch to support your answer. *(See Example 4.)*



CRITICAL THINKING **In Exercises 29–32, complete the statement with *always*, *sometimes*, or *never*. Explain your reasoning.**

29. The circumcenter of a scalene triangle is __________ inside the triangle.

30. If the perpendicular bisector of one side of a triangle intersects the opposite vertex, then the triangle is __________ isosceles.

31. The perpendicular bisectors of a triangle intersect at a point that is __________ equidistant from the midpoints of the sides of the triangle.

32. The angle bisectors of a triangle intersect at a point that is __________ equidistant from the sides of the triangle.

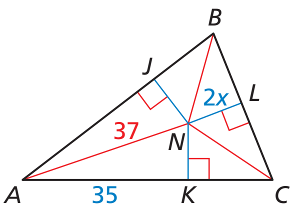
CRITICAL THINKING **In Exercises 33 and 34, find the coordinates of the circumcenter of the triangle with the given vertices.**

33. $A(2, 5)$, $B(6, 6)$, $C(12, 3)$

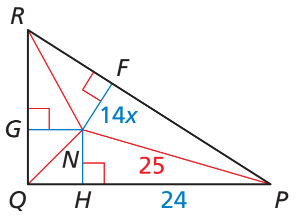
34. $D(-9, -5)$, $E(-5, -9)$, $F(-2, -2)$

MATHEMATICAL CONNECTIONS **In Exercises 35 and 36, find the value of *x* that makes *N* the incenter of the triangle.**

35.



36.

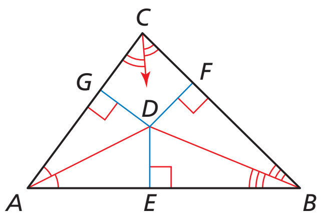


37. **PROOF** Where is the circumcenter located in any right triangle? Write a coordinate proof of this result.

38. **PROVING A THEOREM** Write a proof of the Incenter Theorem (Theorem 6.6).

Given $\triangle ABC$, $\overline{AD}$ bisects $\angle CAB$, $\overline{BD}$ bisects $\angle CBA$, $\overline{DE} \perp \overline{AB}$, $\overline{DF} \perp \overline{BC}$, and $\overline{DG} \perp \overline{CA}$

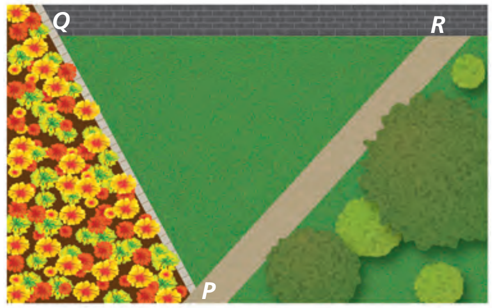
Prove The angle bisectors intersect at D, which is equidistant from $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.



39. **WRITING** Explain the difference between the circumcenter and the incenter of a triangle.

40. **REASONING** Is the incenter of a triangle ever located outside the triangle? Explain your reasoning.

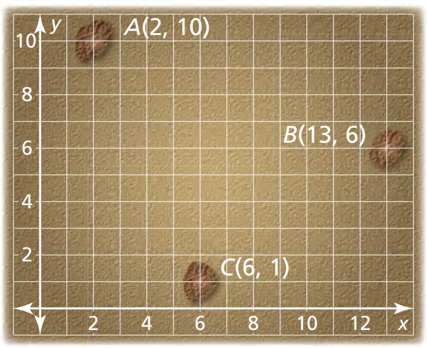
41. **MODELING WITH MATHEMATICS** You are installing a circular pool in the triangular courtyard shown. You want to have the largest pool possible on the site without extending into the walkway.



a. Copy the triangle and show how to install the pool so that it just touches each edge. Then explain how you can be sure that you could not fit a larger pool on the site.

b. You want to have the largest pool possible while leaving at least 1 foot of space around the pool. Would the center of the pool be in the same position as in part (a)? Justify your answer.

42. **MODELING WITH MATHEMATICS** Archaeologists find three stones. They believe that the stones were once part of a circle of stones with a community fire pit at its center. They mark the locations of stones A, B, and C on a graph, where distances are measured in feet.



a. Explain how archaeologists can use a sketch to estimate the center of the circle of stones.

b. Copy the diagram and find the approximate coordinates of the point at which the archaeologists should look for the fire pit.

43. **REASONING** Point P is inside $\triangle ABC$ and is equidistant from points A and B. On which of the following segments must P be located?

Ⓐ $\overline{AB}$

Ⓑ the perpendicular bisector of $\overline{AB}$

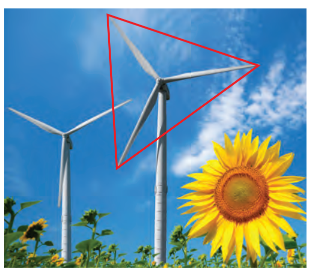
Ⓒ $\overline{AC}$

Ⓓ the perpendicular bisector of $\overline{AC}$

44. CRITICAL THINKING A high school is being built for the four towns shown on the map. Each town agrees that the school should be an equal distance from each of the four towns. Is there a single point where they could agree to build the school? If so, find it. If not, explain why not. Justify your answer with a diagram.

45. MAKING AN ARGUMENT Your friend says that the circumcenter of an equilateral triangle is also the incenter of the triangle. Is your friend correct? Explain your reasoning.

46. HOW DO YOU SEE IT? The arms of the windmill are the angle bisectors of the red triangle. What point of concurrency is the point that connects the three arms?

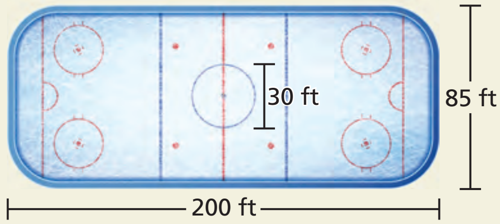


47. ABSTRACT REASONING You are asked to draw a triangle and all its perpendicular bisectors and angle bisectors.

a. For which type of triangle would you need the fewest segments? What is the minimum number of segments you would need? Explain.

b. For which type of triangle would you need the most segments? What is the maximum number of segments you would need? Explain.

48. THOUGHT PROVOKING The diagram shows an official hockey rink used by the National Hockey League. Create a triangle using hockey players as vertices in which the center circle is inscribed in the triangle. The center dot should be the incenter of your triangle. Sketch a drawing of the locations of your hockey players. Then label the actual lengths of the sides and the angle measures in your triangle.

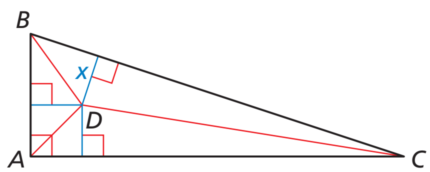


COMPARING METHODS In Exercises 49 and 50, state whether you would use *perpendicular bisectors* or *angle bisectors*. Then solve the problem.

49. You need to cut the largest circle possible from an isosceles triangle made of paper whose sides are 8 inches, 12 inches, and 12 inches. Find the radius of the circle.

50. On a map of a camp, you need to create a circular walking path that connects the pool at (10, 20), the nature center at (16, 2), and the tennis court at (2, 4). Find the coordinates of the center of the circle and the radius of the circle.

51. CRITICAL THINKING Point D is the incenter of $\triangle ABC$. Write an expression for the length x in terms of the three side lengths AB, AC, and BC.



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

The endpoints of $\overline{AB}$ are given. Find the coordinates of the midpoint M. Then find AB.

52. $A(-3, 5), B(3, 5)$

53. $A(2, -1), B(10, 7)$

54. $A(-5, 1), B(4, -5)$

55. $A(-7, 5), B(5, 9)$

Write an equation of the line passing through point P that is perpendicular to the given line. Graph the equations of the lines to check that they are perpendicular.

56. $P(2, 8), y = 2x + 1$

57. $P(6, -3), y = -5$

58. $P(-8, -6),\ 2x + 3y = 18$

59. $P(-4, 1), y + 3 = -4(x + 3)$

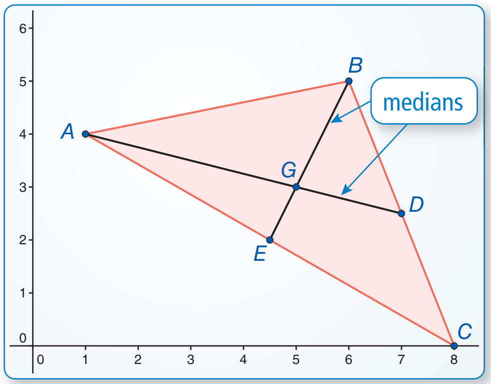
6.3 Medians and Altitudes of Triangles

Essential Question What conjectures can you make about the medians and altitudes of a triangle?

EXPLORATION 1 Finding Properties of the Medians of a Triangle

Work with a partner. Use dynamic geometry software. Draw any △*ABC*.

a. Plot the midpoint of $\overline{BC}$ and label it *D*. Draw $\overline{AD}$, which is a *median* of △*ABC*. Construct the medians to the other two sides of △*ABC*.



Sample
Points
A(1, 4)
B(6, 5)
C(8, 0)
D(7, 2.5)
E(4.5, 2)
G(5, 3)

b. What do you notice about the medians? Drag the vertices to change △*ABC*. Use your observations to write a conjecture about the medians of a triangle.

c. In the figure above, point *G* divides each median into a shorter segment and a longer segment. Find the ratio of the length of each longer segment to the length of the whole median. Is this ratio always the same? Justify your answer.

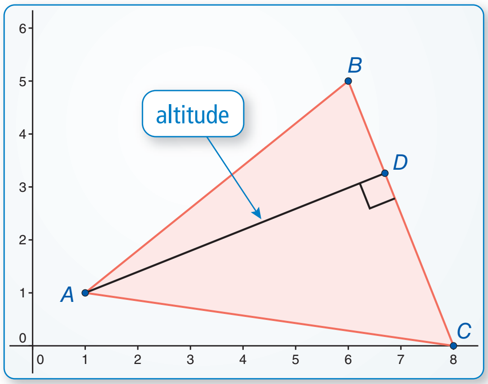
EXPLORATION 2 Finding Properties of the Altitudes of a Triangle

Work with a partner. Use dynamic geometry software. Draw any △*ABC*.

a. Construct the perpendicular segment from vertex *A* to $\overline{BC}$. Label the endpoint *D*. $\overline{AD}$ is an *altitude* of △*ABC*.

b. Construct the altitudes to the other two sides of △*ABC*. What do you notice?

c. Write a conjecture about the altitudes of a triangle. Test your conjecture by dragging the vertices to change △*ABC*.



LOOKING FOR STRUCTURE

To be proficient in math, you need to look closely to discern a pattern or structure.

Communicate Your Answer

3. What conjectures can you make about the medians and altitudes of a triangle?

4. The length of median $\overline{RU}$ in △*RST* is 3 inches. The point of concurrency of the three medians of △*RST* divides $\overline{RU}$ into two segments. What are the lengths of these two segments?

6.3 Lesson

Core Vocabulary

median of a triangle, *p. 290*
centroid, *p. 290*
altitude of a triangle, *p. 291*
orthocenter, *p. 291*

Previous
midpoint
concurrent
point of concurrency

What You Will Learn

- Use medians and find the centroids of triangles.
- Use altitudes and find the orthocenters of triangles.

Using the Median of a Triangle

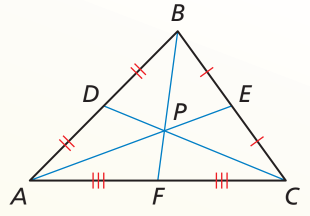
A **median of a triangle** is a segment from a vertex to the midpoint of the opposite side. The three medians of a triangle are concurrent. The point of concurrency, called the **centroid**, is inside the triangle.

Theorem

Theorem 6.7 Centroid Theorem

The centroid of a triangle is two-thirds of the distance from each vertex to the midpoint of the opposite side.

The medians of $\triangle ABC$ meet at point P, and $AP = \frac{2}{3}AE$, $BP = \frac{2}{3}BF$, and $CP = \frac{2}{3}CD$.



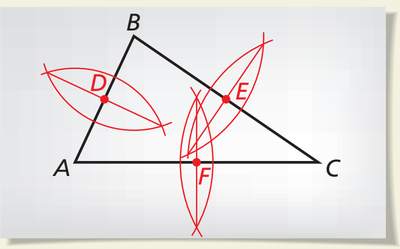
Proof *BigIdeasMath.com*

CONSTRUCTION Finding the Centroid of a Triangle

Use a compass and straightedge to construct the medians of $\triangle ABC$.

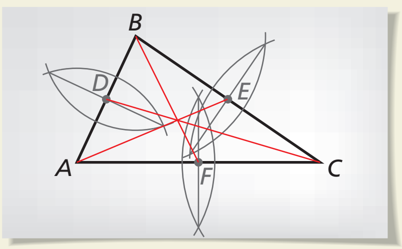
SOLUTION

Step 1



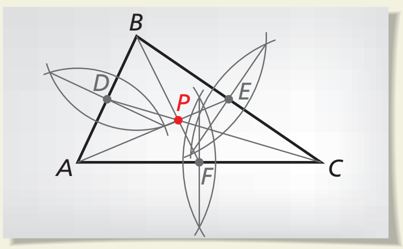
Find midpoints Draw $\triangle ABC$. Find the midpoints of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. Label the midpoints of the sides D, E, and F, respectively.

Step 2



Draw medians Draw $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$. These are the three medians of $\triangle ABC$.

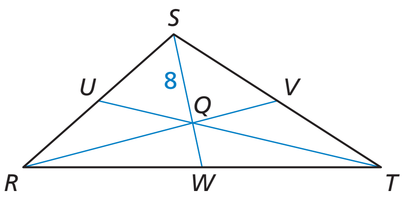
Step 3



Label a point Label the point where $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$ intersect as P. This is the centroid.

EXAMPLE 1 Using the Centroid of a Triangle

In $\triangle RST$, point Q is the centroid, and $SQ = 8$. Find QW and SW.



SOLUTION

$SQ = \frac{2}{3}SW$	Centroid Theorem
$8 = \frac{2}{3}SW$	Substitute 8 for SQ.
$12 = SW$	Multiply each side by the reciprocal, $\frac{3}{2}$.

Then $QW = SW - SQ = 12 - 8 = 4$.

- So, $QW = 4$ and $SW = 12$.

> **FINDING AN ENTRY POINT**
>
> The median $\overline{SV}$ is chosen in Example 2 because it is easier to find a distance on a vertical segment.

EXAMPLE 2 **Finding the Centroid of a Triangle**

Find the coordinates of the centroid of $\triangle RST$ with vertices $R(2, 1)$, $S(5, 8)$, and $T(8, 3)$.

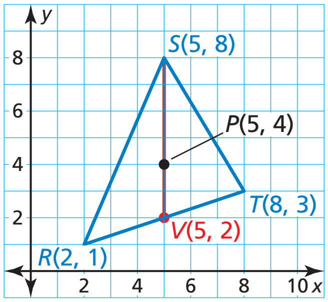
SOLUTION

Step 1 Graph $\triangle RST$.

Step 2 Use the Midpoint Formula to find the midpoint V of $\overline{RT}$ and sketch median $\overline{SV}$.

$$V\left(\frac{2+8}{2}, \frac{1+3}{2}\right) = (5, 2)$$

Step 3 Find the centroid. It is two-thirds of the distance from each vertex to the midpoint of the opposite side.



The distance from vertex $S(5, 8)$ to $V(5, 2)$ is $8 - 2 = 6$ units.
So, the centroid is $\frac{2}{3}(6) = 4$ units down from vertex S on $\overline{SV}$.

▶ So, the coordinates of the centroid P are $(5, 8 - 4)$, or $(5, 4)$.

> **JUSTIFYING CONCLUSIONS**
>
> You can check your result by using a different median to find the centroid.

Monitoring Progress

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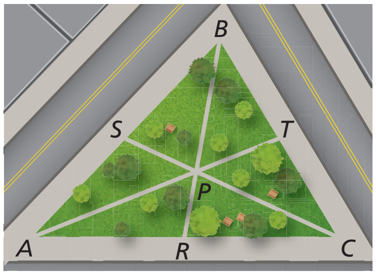
There are three paths through a triangular park. Each path goes from the midpoint of one edge to the opposite corner. The paths meet at point *P*.



1. Find PS and PC when $SC = 2100$ feet.
2. Find TC and BC when $BT = 1000$ feet.
3. Find PA and TA when $PT = 800$ feet.

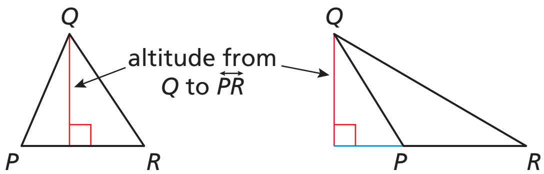
Find the coordinates of the centroid of the triangle with the given vertices.

4. $F(2, 5)$, $G(4, 9)$, $H(6, 1)$
5. $X(-3, 3)$, $Y(1, 5)$, $Z(-1, -2)$

> **READING**
>
> In the area formula for a triangle, $A = \frac{1}{2}bh$, you can use the length of any side for the base b. The height h is the length of the altitude to that side from the opposite vertex.

Using the Altitude of a Triangle

An **altitude of a triangle** is the perpendicular segment from a vertex to the opposite side or to the line that contains the opposite side.

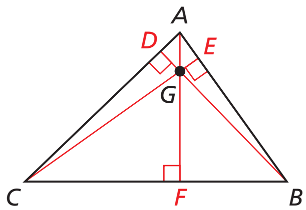


Core Concept

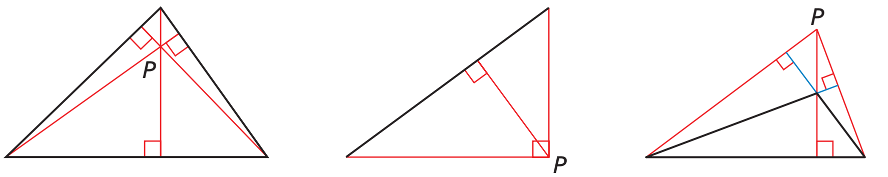
Orthocenter

The lines containing the altitudes of a triangle are concurrent. This point of concurrency is the **orthocenter** of the triangle.

The lines containing $\overline{AF}$, $\overline{BD}$, and $\overline{CE}$ meet at the orthocenter G of $\triangle ABC$.



As shown below, the location of the orthocenter P of a triangle depends on the type of triangle.



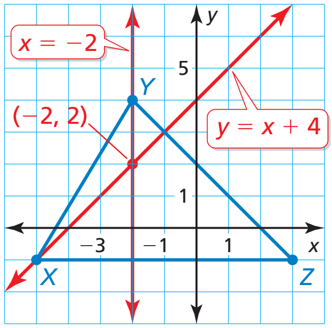
READING

The altitudes are shown in red. Notice that in the right triangle, the legs are also altitudes. The altitudes of the obtuse triangle are extended to find the orthocenter.

EXAMPLE 3 Finding the Orthocenter of a Triangle

Find the coordinates of the orthocenter of $\triangle XYZ$ with vertices $X(-5, -1)$, $Y(-2, 4)$, and $Z(3, -1)$.

SOLUTION



Step 1 Graph $\triangle XYZ$.

Step 2 Find an equation of the line that contains the altitude from Y to $\overline{XZ}$. Because $\overline{XZ}$ is horizontal, the altitude is vertical. The line that contains the altitude passes through $Y(-2, 4)$. So, the equation of the line is $x = -2$.

Step 3 Find an equation of the line that contains the altitude from X to $\overline{YZ}$.

$$\text{slope of } \overleftrightarrow{YZ} = \frac{-1 - 4}{3 - (-2)} = -1$$

Because the product of the slopes of two perpendicular lines is -1, the slope of a line perpendicular to $\overleftrightarrow{YZ}$ is 1. The line passes through $X(-5, -1)$.

$y = mx + b$ — Use slope-intercept form.

$-1 = 1(-5) + b$ — Substitute -1 for y, 1 for m, and -5 for x.

$4 = b$ — Solve for b.

So, the equation of the line is $y = x + 4$.

Step 4 Find the point of intersection of the graphs of the equations $x = -2$ and $y = x + 4$.

Substitute -2 for x in the equation $y = x + 4$. Then solve for y.

$y = x + 4$ — Write equation.

$y = -2 + 4$ — Substitute -2 for x.

$y = 2$ — Solve for y.

▶ So, the coordinates of the orthocenter are $(-2, 2)$.

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Tell whether the orthocenter of the triangle with the given vertices is *inside*, *on*, or *outside* the triangle. Then find the coordinates of the orthocenter.

6. $A(0, 3)$, $B(0, -2)$, $C(6, -3)$

7. $J(-3, -4)$, $K(-3, 4)$, $L(5, 4)$

In an isosceles triangle, the perpendicular bisector, angle bisector, median, and altitude from the vertex angle to the base are all the same segment. In an equilateral triangle, this is true for any vertex.

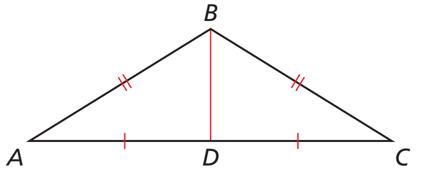
EXAMPLE 4 Proving a Property of Isosceles Triangles

Prove that the median from the vertex angle to the base of an isosceles triangle is an altitude.

SOLUTION

Given $\triangle ABC$ is isosceles, with base $\overline{AC}$.
$\overline{BD}$ is the median to base $\overline{AC}$.

Prove $\overline{BD}$ is an altitude of $\triangle ABC$.



Paragraph Proof Legs $\overline{AB}$ and $\overline{BC}$ of isosceles $\triangle ABC$ are congruent. $\overline{CD} \cong \overline{AD}$ because $\overline{BD}$ is the median to $\overline{AC}$. Also, $\overline{BD} \cong \overline{BD}$ by the Reflexive Property of Congruence (Thm. 2.1). So, $\triangle ABD \cong \triangle CBD$ by the SSS Congruence Theorem (Thm. 5.8). $\angle ADB \cong \angle CDB$ because corresponding parts of congruent triangles are congruent. Also, $\angle ADB$ and $\angle CDB$ are a linear pair. $\overline{BD}$ and $\overline{AC}$ intersect to form a linear pair of congruent angles, so $\overline{BD} \perp \overline{AC}$ and $\overline{BD}$ is an altitude of $\triangle ABC$.

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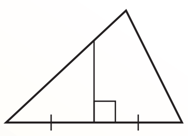
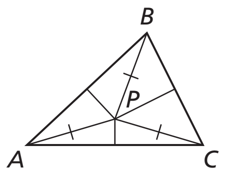
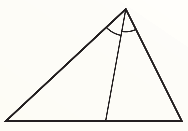
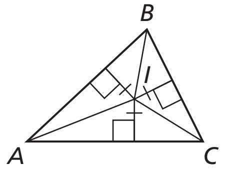
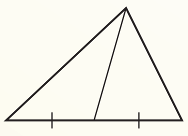
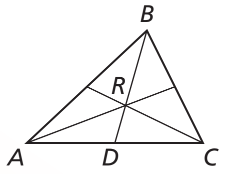
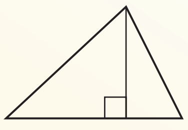
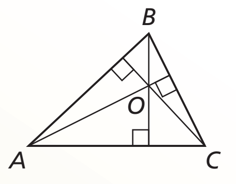
8. **WHAT IF?** In Example 4, you want to show that median $\overline{BD}$ is also an angle bisector. How would your proof be different?

Concept Summary

Segments, Lines, Rays, and Points in Triangles

	Example	Point of Concurrency	Property	Example
perpendicular bisector		circumcenter	The circumcenter P of a triangle is equidistant from the vertices of the triangle.	
angle bisector		incenter	The incenter I of a triangle is equidistant from the sides of the triangle.	
median		centroid	The centroid R of a triangle is two thirds of the distance from each vertex to the midpoint of the opposite side.	
altitude		orthocenter	The lines containing the altitudes of a triangle are concurrent at the orthocenter O.	

6.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

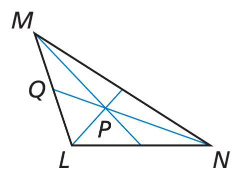
1. **VOCABULARY** Name the four types of points of concurrency. Which lines intersect to form each of the points?

2. **COMPLETE THE SENTENCE** The length of a segment from a vertex to the centroid is ____________ the length of the median from that vertex.

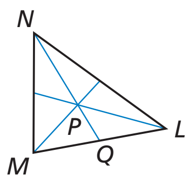
Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, point P is the centroid of $\triangle LMN$. Find PN and QP. *(See Example 1.)*

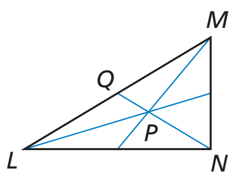
3. $QN = 9$



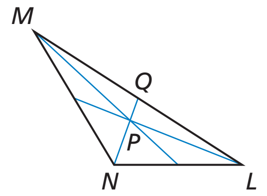
4. $QN = 21$



5. $QN = 30$

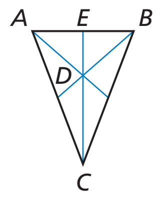


6. $QN = 42$

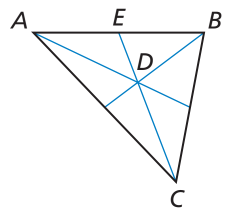


In Exercises 7–10, point D is the centroid of $\triangle ABC$. Find CD and CE.

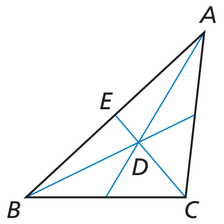
7. $DE = 5$



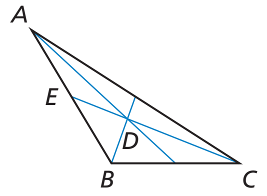
8. $DE = 11$



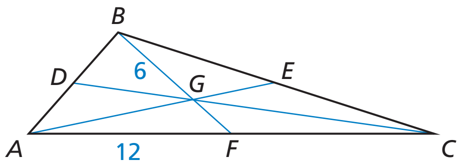
9. $DE = 9$



10. $DE = 15$



In Exercises 11–14, point G is the centroid of $\triangle ABC$. $BG = 6$, $AF = 12$, and $AE = 15$. Find the length of the segment.



11. $\overline{FC}$

12. $\overline{BF}$

13. $\overline{AG}$

14. $\overline{GE}$

In Exercises 15–18, find the coordinates of the centroid of the triangle with the given vertices. *(See Example 2.)*

15. $A(2, 3)$, $B(8, 1)$, $C(5, 7)$

16. $F(1, 5)$, $G(-2, 7)$, $H(-6, 3)$

17. $S(5, 5)$, $T(11, -3)$, $U(-1, 1)$

18. $X(1, 4)$, $Y(7, 2)$, $Z(2, 3)$

In Exercises 19–22, tell whether the orthocenter is *inside*, *on*, or *outside* the triangle. Then find the coordinates of the orthocenter. *(See Example 3.)*

19. $L(0, 5)$, $M(3, 1)$, $N(8, 1)$

20. $X(-3, 2)$, $Y(5, 2)$, $Z(-3, 6)$

21. $A(-4, 0)$, $B(1, 0)$, $C(-1, 3)$

22. $T(-2, 1)$, $U(2, 1)$, $V(0, 4)$

CONSTRUCTION In Exercises 23–26, draw the indicated triangle and find its centroid and orthocenter.

23. isosceles right triangle

24. obtuse scalene triangle

25. right scalene triangle

26. acute isosceles triangle

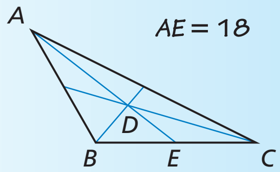
ERROR ANALYSIS In Exercises 27 and 28, describe and correct the error in finding DE. Point D is the centroid of $\triangle ABC$.

27. ✗

$DE = \frac{2}{3}AE$

$DE = \frac{2}{3}(18)$

$DE = 12$

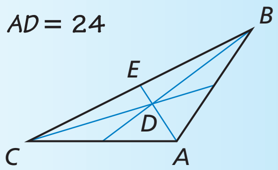


28. ✗

$DE = \frac{2}{3}AD$

$DE = \frac{2}{3}(24)$

$DE = 16$



PROOF In Exercises 29 and 30, write a proof of the statement. *(See Example 4.)*

29. The angle bisector from the vertex angle to the base of an isosceles triangle is also a median.

30. The altitude from the vertex angle to the base of an isosceles triangle is also a perpendicular bisector.

CRITICAL THINKING In Exercises 31–36, complete the statement with *always*, *sometimes*, or *never*. Explain your reasoning.

31. The centroid is __________ on the triangle.

32. The orthocenter is __________ outside the triangle.

33. A median is __________ the same line segment as a perpendicular bisector.

34. An altitude is __________ the same line segment as an angle bisector.

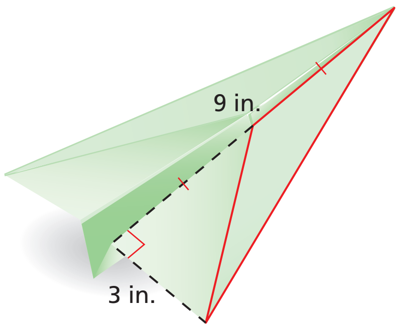
35. The centroid and orthocenter are __________ the same point.

36. The centroid is __________ formed by the intersection of the three medians.

37. **WRITING** Compare an altitude of a triangle with a perpendicular bisector of a triangle.

38. **WRITING** Compare a median, an altitude, and an angle bisector of a triangle.

39. **MODELING WITH MATHEMATICS** Find the area of the triangular part of the paper airplane wing that is outlined in red. Which special segment of the triangle did you use?

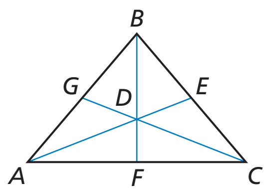


40. **ANALYZING RELATIONSHIPS** Copy and complete the statement for $\triangle DEF$ with centroid K and medians $\overline{DH}$, $\overline{EJ}$, and $\overline{FG}$.

a. $EJ =$ _____ KJ **b.** $DK =$ _____ KH

c. $FG =$ _____ KF **d.** $KG =$ _____ FG

MATHEMATICAL CONNECTIONS In Exercises 41–44, point D is the centroid of $\triangle ABC$. Use the given information to find the value of x.



41. $BD = 4x + 5$ and $BF = 9x$

42. $GD = 2x - 8$ and $GC = 3x + 3$

43. $AD = 5x$ and $DE = 3x - 2$

44. $DF = 4x - 1$ and $BD = 6x + 4$

45. **MATHEMATICAL CONNECTIONS** Graph the lines on the same coordinate plane. Find the centroid of the triangle formed by their intersections.

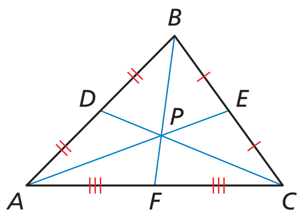
$y_1 = 3x - 4$

$y_2 = \frac{3}{4}x + 5$

$y_3 = -\frac{3}{2}x - 4$

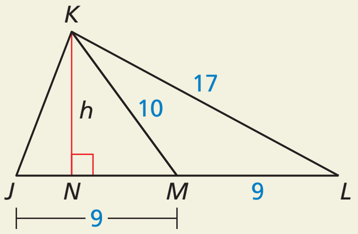
46. **CRITICAL THINKING** In what type(s) of triangles can a vertex be one of the points of concurrency of the triangle? Explain your reasoning.

47. WRITING EQUATIONS Use the numbers and symbols to write three different equations for PE.



PE	AE	AP	$+$	$-$
$=$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$

48. HOW DO YOU SEE IT? Use the figure.



a. What type of segment is $\overline{KM}$? Which point of concurrency lies on $\overline{KM}$?

b. What type of segment is $\overline{KN}$? Which point of concurrency lies on $\overline{KN}$?

c. Compare the areas of $\triangle JKM$ and $\triangle KLM$. Do you think the areas of the triangles formed by the median of any triangle will always compare this way? Explain your reasoning.

49. MAKING AN ARGUMENT Your friend claims that it is possible for the circumcenter, incenter, centroid, and orthocenter to all be the same point. Do you agree? Explain your reasoning.

50. DRAWING CONCLUSIONS The center of gravity of a triangle, the point where a triangle can balance on the tip of a pencil, is one of the four points of concurrency. Draw and cut out a large scalene triangle on a piece of cardboard. Which of the four points of concurrency is the center of gravity? Explain.

51. PROOF Prove that a median of an equilateral triangle is also an angle bisector, perpendicular bisector, and altitude.

52. THOUGHT PROVOKING Construct an acute scalene triangle. Find the orthocenter, centroid, and circumcenter. What can you conclude about the three points of concurrency?

53. CONSTRUCTION Follow the steps to construct a nine-point circle. Why is it called a nine-point circle?

Step 1 Construct a large acute scalene triangle.

Step 2 Find the orthocenter and circumcenter of the triangle.

Step 3 Find the midpoint between the orthocenter and circumcenter.

Step 4 Find the midpoint between each vertex and the orthocenter.

Step 5 Construct a circle. Use the midpoint in Step 3 as the center of the circle, and the distance from the center to the midpoint of a side of the triangle as the radius.

54. PROOF Prove the statements in parts (a)−(c).

Given $\overline{LP}$ and $\overline{MQ}$ are medians of scalene $\triangle LMN$. Point R is on $\overrightarrow{LP}$ such that $\overline{LP} \cong \overline{PR}$. Point S is on $\overrightarrow{MQ}$ such that $\overline{MQ} \cong \overline{QS}$.

Prove **a.** $\overline{NS} \cong \overline{NR}$

b. $\overline{NS}$ and $\overline{NR}$ are both parallel to $\overline{LM}$.

c. R, N, and S are collinear.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Determine whether $\overline{AB}$ is parallel to $\overline{CD}$.

55. $A(5, 6)$, $B(-1, 3)$, $C(-4, 9)$, $D(-16, 3)$

56. $A(-3, 6)$, $B(5, 4)$, $C(-14, -10)$, $D(-2, -7)$

57. $A(6, -3)$, $B(5, 2)$, $C(-4, -4)$, $D(-5, 2)$

58. $A(-5, 6)$, $B(-7, 2)$, $C(7, 1)$, $D(4, -5)$

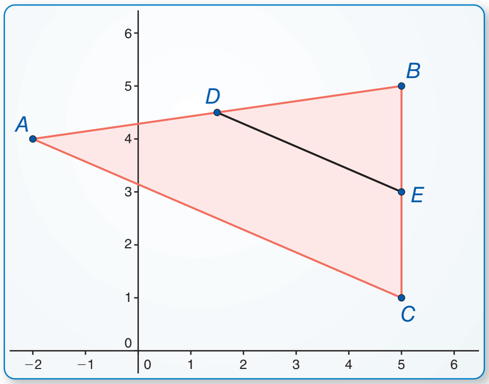
6.4 The Triangle Midsegment Theorem

Essential Question How are the midsegments of a triangle related to the sides of the triangle?

EXPLORATION 1 Midsegments of a Triangle

Work with a partner. Use dynamic geometry software. Draw any $\triangle ABC$.

a. Plot midpoint D of $\overline{AB}$ and midpoint E of $\overline{BC}$. Draw $\overline{DE}$, which is a *midsegment* of $\triangle ABC$.



Sample

Points
$A(-2, 4)$
$B(5, 5)$
$C(5, 1)$
$D(1.5, 4.5)$
$E(5, 3)$

Segments
$BC = 4$
$AC = 7.62$
$AB = 7.07$
$DE = ?$

b. Compare the slope and length of $\overline{DE}$ with the slope and length of $\overline{AC}$.

c. Write a conjecture about the relationships between the midsegments and sides of a triangle. Test your conjecture by drawing the other midsegments of $\triangle ABC$, dragging vertices to change $\triangle ABC$, and noting whether the relationships hold.

CONSTRUCTING VIABLE ARGUMENTS

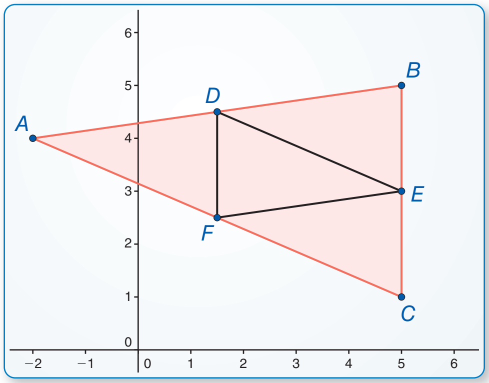
To be proficient in math, you need to make conjectures and build a logical progression of statements to explore the truth of your conjectures.

EXPLORATION 2 Midsegments of a Triangle

Work with a partner. Use dynamic geometry software. Draw any $\triangle ABC$.

a. Draw all three midsegments of $\triangle ABC$.

b. Use the drawing to write a conjecture about the triangle formed by the midsegments of the original triangle.



Sample

Points
$A(-2, 4)$
$B(5, 5)$
$C(5, 1)$
$D(1.5, 4.5)$
$E(5, 3)$

Segments
$BC = 4$
$AC = 7.62$
$AB = 7.07$
$DE = ?$
$DF = ?$
$EF = ?$

Communicate Your Answer

3. How are the midsegments of a triangle related to the sides of the triangle?

4. In $\triangle RST$, $\overline{UV}$ is the midsegment connecting the midpoints of $\overline{RS}$ and $\overline{ST}$. Given $UV = 12$, find RT.

6.4 Lesson

Core Vocabulary

midsegment of a triangle, *p. 298*

Previous
midpoint
parallel
slope
coordinate proof

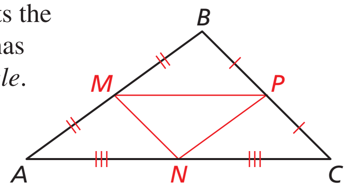
What You Will Learn

- Use midsegments of triangles in the coordinate plane.
- Use the Triangle Midsegment Theorem to find distances.

Using the Midsegment of a Triangle

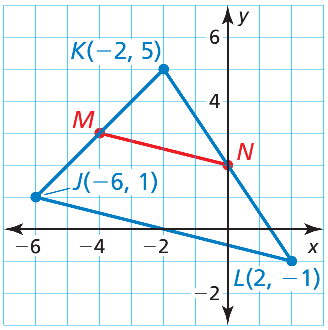
A **midsegment of a triangle** is a segment that connects the midpoints of two sides of the triangle. Every triangle has three midsegments, which form the *midsegment triangle*.

The midsegments of $\triangle ABC$ at the right are $\overline{MP}$, $\overline{MN}$, and $\overline{NP}$. The *midsegment triangle* is $\triangle MNP$.



EXAMPLE 1 **Using Midsegments in the Coordinate Plane**

In $\triangle JKL$, show that midsegment $\overline{MN}$ is parallel to $\overline{JL}$ and that $MN = \frac{1}{2}JL$.



READING

In the figure for Example 1, midsegment $\overline{MN}$ can be called "the midsegment opposite $\overline{JL}$."

SOLUTION

Step 1 Find the coordinates of M and N by finding the midpoints of $\overline{JK}$ and $\overline{KL}$.

$$M\left(\frac{-6 + (-2)}{2}, \frac{1 + 5}{2}\right) = M\left(\frac{-8}{2}, \frac{6}{2}\right) = M(-4, 3)$$

$$N\left(\frac{-2 + 2}{2}, \frac{5 + (-1)}{2}\right) = N\left(\frac{0}{2}, \frac{4}{2}\right) = N(0, 2)$$

Step 2 Find and compare the slopes of $\overline{MN}$ and $\overline{JL}$.

$$\text{slope of } \overline{MN} = \frac{2 - 3}{0 - (-4)} = -\frac{1}{4} \qquad \text{slope of } \overline{JL} = \frac{-1 - 1}{2 - (-6)} = -\frac{2}{8} = -\frac{1}{4}$$

- Because the slopes are the same, $\overline{MN}$ is parallel to $\overline{JL}$.

Step 3 Find and compare the lengths of $\overline{MN}$ and $\overline{JL}$.

$$MN = \sqrt{[0 - (-4)]^2 + (2 - 3)^2} = \sqrt{16 + 1} = \sqrt{17}$$

$$JL = \sqrt{[2 - (-6)]^2 + (-1 - 1)^2} = \sqrt{64 + 4} = \sqrt{68} = 2\sqrt{17}$$

- Because $\sqrt{17} = \frac{1}{2}\left(2\sqrt{17}\right)$, $MN = \frac{1}{2}JL$.

Monitoring Progress

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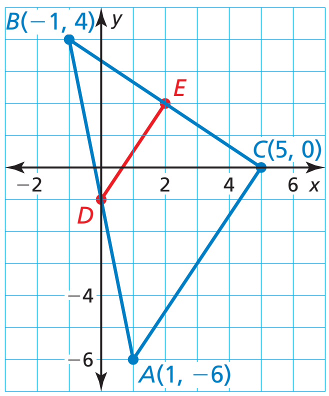
Use the graph of $\triangle ABC$.



1. In $\triangle ABC$, show that midsegment $\overline{DE}$ is parallel to $\overline{AC}$ and that $DE = \frac{1}{2}AC$.
2. Find the coordinates of the endpoints of midsegment $\overline{EF}$, which is opposite $\overline{AB}$. Show that $\overline{EF}$ is parallel to $\overline{AB}$ and that $EF = \frac{1}{2}AB$.

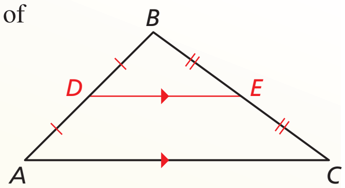
Using the Triangle Midsegment Theorem

Theorem

Theorem 6.8 Triangle Midsegment Theorem

The segment connecting the midpoints of two sides of a triangle is parallel to the third side and is half as long as that side.

$\overline{DE}$ is a midsegment of $\triangle ABC$, $\overline{DE} \parallel \overline{AC}$, and $DE = \frac{1}{2}AC$.



Proof Example 2, p. 299; Monitoring Progress Question 3, p. 299; Ex. 22, p. 302

STUDY TIP

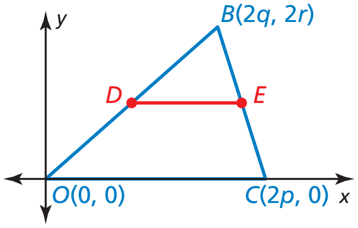
When assigning coordinates, try to choose coordinates that make some of the computations easier. In Example 2, you can avoid fractions by using $2p$, $2q$, and $2r$.

EXAMPLE 2 Proving the Triangle Midsegment Theorem

Write a coordinate proof of the Triangle Midsegment Theorem for one midsegment.

Given $\overline{DE}$ is a midsegment of $\triangle OBC$.

Prove $\overline{DE} \parallel \overline{OC}$ and $DE = \frac{1}{2}OC$



SOLUTION

Step 1 Place $\triangle OBC$ in a coordinate plane and assign coordinates. Because you are finding midpoints, use $2p$, $2q$, and $2r$. Then find the coordinates of D and E.

$$D\left(\frac{2q+0}{2}, \frac{2r+0}{2}\right) = D(q, r) \qquad E\left(\frac{2q+2p}{2}, \frac{2r+0}{2}\right) = E(q+p, r)$$

Step 2 Prove $\overline{DE} \parallel \overline{OC}$. The y-coordinates of D and E are the same, so $\overline{DE}$ has a slope of 0. $\overline{OC}$ is on the x-axis, so its slope is 0.

▶ Because their slopes are the same, $\overline{DE} \parallel \overline{OC}$.

Step 3 Prove $DE = \frac{1}{2}OC$. Use the Ruler Postulate (Post. 1.1) to find DE and OC.

$$DE = |(q+p) - q| = p \qquad OC = |2p - 0| = 2p$$

▶ Because $p = \frac{1}{2}(2p)$, $DE = \frac{1}{2}OC$.

Monitoring Progress

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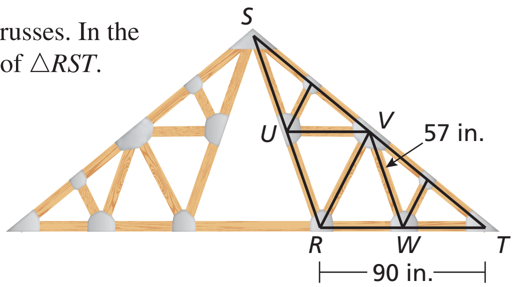
3. In Example 2, find the coordinates of F, the midpoint of $\overline{OC}$. Show that $\overline{FE} \parallel \overline{OB}$ and $FE = \frac{1}{2}OB$.

STEM VIDEO

EXAMPLE 3 Using the Triangle Midsegment Theorem

Triangles are used for strength in roof trusses. In the diagram, $\overline{UV}$ and $\overline{VW}$ are midsegments of $\triangle RST$. Find UV and RS.



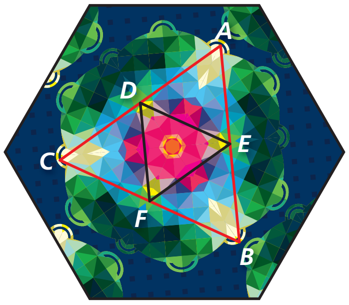
SOLUTION

$UV = \frac{1}{2} \cdot RT = \frac{1}{2}(90 \text{ in.}) = 45 \text{ in.}$

$RS = 2 \cdot VW = 2(57 \text{ in.}) = 114 \text{ in.}$

EXAMPLE 4 Using the Triangle Midsegment Theorem

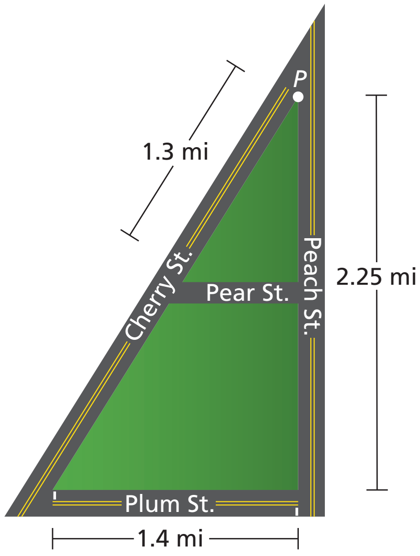
In the kaleidoscope image, $\overline{AE} \cong \overline{BE}$ and $\overline{AD} \cong \overline{CD}$. Show that $\overline{CB} \parallel \overline{DE}$.



SOLUTION

Because $\overline{AE} \cong \overline{BE}$ and $\overline{AD} \cong \overline{CD}$, E is the midpoint of $\overline{AB}$ and D is the midpoint of $\overline{AC}$ by definition. Then $\overline{DE}$ is a midsegment of $\triangle ABC$ by definition and $\overline{CB} \parallel \overline{DE}$ by the Triangle Midsegment Theorem.

EXAMPLE 5 Modeling with Mathematics



Pear Street intersects Cherry Street and Peach Street at their midpoints. Your home is at point P. You leave your home and jog down Cherry Street to Plum Street, over Plum Street to Peach Street, up Peach Street to Pear Street, over Pear Street to Cherry Street, and then back home up Cherry Street. About how many miles do you jog?

SOLUTION

1. **Understand the Problem** You know the distances from your home to Plum Street along Peach Street, from Peach Street to Cherry Street along Plum Street, and from Pear Street to your home along Cherry Street. You need to find the other distances on your route, then find the total number of miles you jog.
2. **Make a Plan** By definition, you know that Pear Street is a midsegment of the triangle formed by the other three streets. Use the Triangle Midsegment Theorem to find the length of Pear Street and the definition of midsegment to find the length of Cherry Street. Then add the distances along your route.
3. **Solve the Problem**

   length of Pear Street $= \frac{1}{2} \cdot$ (length of Plum St.) $= \frac{1}{2}(1.4 \text{ mi}) = 0.7$ mi

   length of Cherry Street $= 2 \cdot$ (length from P to Pear St.) $= 2(1.3 \text{ mi}) = 2.6$ mi

   distance along your route: $2.6 + 1.4 + \frac{1}{2}(2.25) + 0.7 + 1.3 = 7.125$

   ▶ So, you jog about 7 miles.
4. **Look Back** Use compatible numbers to check that your answer is reasonable.

   total distance:

   $$2.6 + 1.4 + \frac{1}{2}(2.25) + 0.7 + 1.3 \approx 2.5 + 1.5 + 1 + 0.5 + 1.5 = 7 \checkmark$$

Monitoring Progress

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4. Copy the diagram in Example 3. Draw and name the third midsegment. Then find the length of $\overline{VS}$ when the length of the third midsegment is 81 inches.
5. In Example 4, if F is the midpoint of $\overline{CB}$, what do you know about $\overline{DF}$?
6. **WHAT IF?** In Example 5, you jog down Peach Street to Plum Street, over Plum Street to Cherry Street, up Cherry Street to Pear Street, over Pear Street to Peach Street, and then back home up Peach Street. Do you jog more miles in Example 5? Explain.

Dynamic Solutions available at *BigIdeasMath.com*

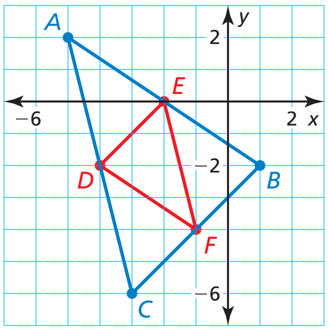
Vocabulary and Core Concept Check

1. VOCABULARY The ___________ of a triangle is a segment that connects the midpoints of two sides of the triangle.

2. COMPLETE THE SENTENCE If $\overline{DE}$ is the midsegment opposite $\overline{AC}$ in $\triangle ABC$, then $\overline{DE} \parallel \overline{AC}$ and $DE =$ ___AC by the Triangle Midsegment Theorem (Theorem 6.8).

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, use the graph of $\triangle ABC$ with midsegments $\overline{DE}$, $\overline{EF}$, and $\overline{DF}$. *(See Example 1.)*



3. Find the coordinates of points D, E, and F.

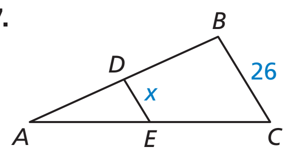
4. Show that $\overline{DE}$ is parallel to $\overline{CB}$ and that $DE = \frac{1}{2}CB$.

5. Show that $\overline{EF}$ is parallel to $\overline{AC}$ and that $EF = \frac{1}{2}AC$.

6. Show that $\overline{DF}$ is parallel to $\overline{AB}$ and that $DF = \frac{1}{2}AB$.

In Exercises 7–10, $\overline{DE}$ is a midsegment of $\triangle ABC$. Find the value of x. *(See Example 3.)*

7.



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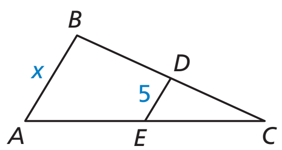

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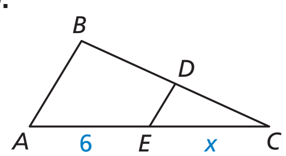

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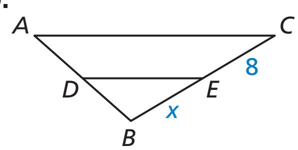
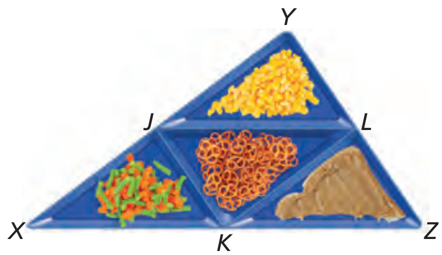


In Exercises 11–16, $\overline{XJ} \cong \overline{JY}$, $\overline{YL} \cong \overline{LZ}$, and $\overline{XK} \cong \overline{KZ}$. Copy and complete the statement. *(See Example 4.)*



11. $\overline{JK} \parallel$ ___

12. $\overline{JL} \parallel$ ___

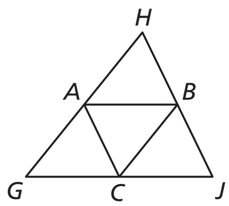
13. $\overline{XY} \parallel$ ___

14. $\overline{JY} \cong$ ___ $\cong$ ___

15. $\overline{JL} \cong$ ___ $\cong$ ___

16. $\overline{JK} \cong$ ___ $\cong$ ___

MATHEMATICAL CONNECTIONS **In Exercises 17–19, use $\triangle GHJ$, where A, B, and C are midpoints of the sides.**

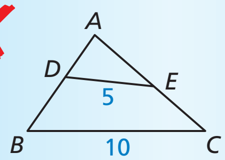


17. When $AB = 3x + 8$ and $GJ = 2x + 24$, what is AB?

18. When $AC = 3y - 5$ and $HJ = 4y + 2$, what is HB?

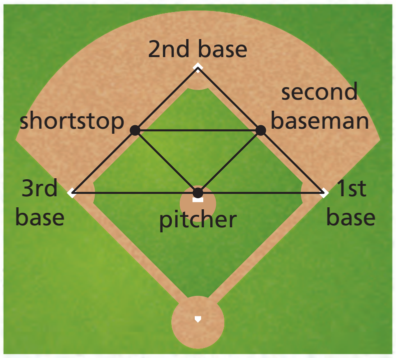
19. When $GH = 7z - 1$ and $CB = 4z - 3$, what is GA?

20. ERROR ANALYSIS Describe and correct the error.



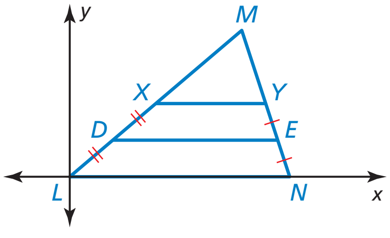
$DE = \frac{1}{2}BC$, so by the Triangle Midsegment Theorem (Thm. 6.8), $\overline{AD} \cong \overline{DB}$ and $\overline{AE} \cong \overline{EC}$.

21. MODELING WITH MATHEMATICS The distance between consecutive bases on a baseball field is 90 feet. A second baseman stands halfway between first base and second base, a shortstop stands halfway between second base and third base, and a pitcher stands halfway between first base and third base. Find the distance between the shortstop and the pitcher. *(See Example 5.)*



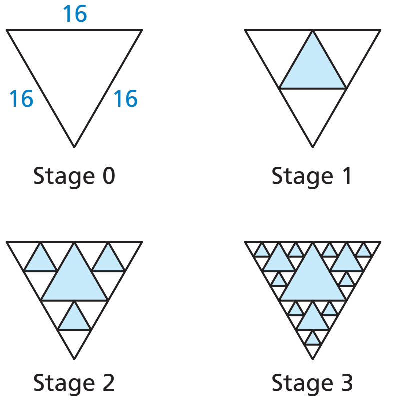
22. PROVING A THEOREM Use the figure from Example 2 to prove the Triangle Midsegment Theorem (Theorem 6.8) for midsegment $\overline{DF}$, where F is the midpoint of $\overline{OC}$. *(See Example 2.)*

23. CRITICAL THINKING $\overline{XY}$ is a midsegment of $\triangle LMN$. Suppose $\overline{DE}$ is called a "quarter segment" of $\triangle LMN$. What do you think an "eighth segment" would be? Make conjectures about the properties of a quarter segment and an eighth segment. Use variable coordinates to verify your conjectures.



24. THOUGHT PROVOKING Find a real-life object that uses midsegments as part of its structure. Print a photograph of the object and identify the midsegments of one of the triangles in the structure.

25. ABSTRACT REASONING To create the design shown, shade the triangle formed by the three midsegments of the triangle. Then repeat the process for each unshaded triangle.

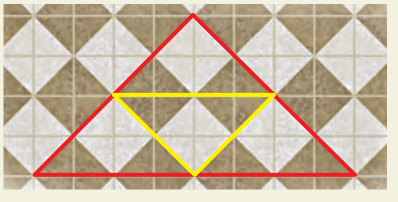


a. What is the perimeter of the shaded triangle in Stage 1?

b. What is the total perimeter of all the shaded triangles in Stage 2?

c. What is the total perimeter of all the shaded triangles in Stage 3?

26. HOW DO YOU SEE IT? Explain how you know that the yellow triangle is the midsegment triangle of the red triangle in the pattern of floor tiles shown.



27. ATTENDING TO PRECISION The points $P(2, 1)$, $Q(4, 5)$, and $R(7, 4)$ are the midpoints of the sides of a triangle. Graph the three midsegments. Then show how to use your graph and the properties of midsegments to draw the original triangle. Give the coordinates of each vertex.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find a counterexample to show that the conjecture is false.

28. The difference of two numbers is always less than the greater number.

29. An isosceles triangle is always equilateral.

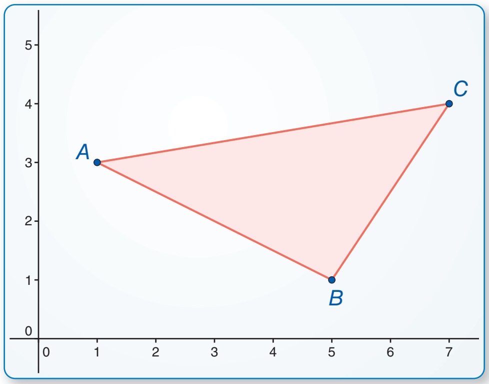
6.5 Indirect Proof and Inequalities in One Triangle

Essential Question How are the sides related to the angles of a triangle? How are any two sides of a triangle related to the third side?

EXPLORATION 1 Comparing Angle Measures and Side Lengths

Work with a partner. Use dynamic geometry software. Draw any scalene $\triangle ABC$.

a. Find the side lengths and angle measures of the triangle.



Sample

Points
$A(1, 3)$
$B(5, 1)$
$C(7, 4)$
Segments
$BC = ?$
$AC = ?$
$AB = ?$

Angles
$m\angle A = ?$
$m\angle B = ?$
$m\angle C = ?$

b. Order the side lengths. Order the angle measures. What do you observe?

c. Drag the vertices of $\triangle ABC$ to form new triangles. Record the side lengths and angle measures in a table. Write a conjecture about your findings.

ATTENDING TO PRECISION

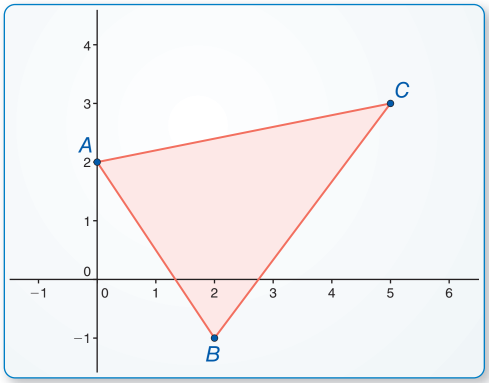
To be proficient in math, you need to express numerical answers with a degree of precision appropriate for the content.

EXPLORATION 2 A Relationship of the Side Lengths of a Triangle

Work with a partner. Use dynamic geometry software. Draw any $\triangle ABC$.

a. Find the side lengths of the triangle.

b. Compare each side length with the sum of the other two side lengths.



Sample

Points
$A(0, 2)$
$B(2, -1)$
$C(5, 3)$
Segments
$BC = ?$
$AC = ?$
$AB = ?$

c. Drag the vertices of $\triangle ABC$ to form new triangles and repeat parts (a) and (b). Organize your results in a table. Write a conjecture about your findings.

Communicate Your Answer

3. How are the sides related to the angles of a triangle? How are any two sides of a triangle related to the third side?

4. Is it possible for a triangle to have side lengths of 3, 4, and 10? Explain.

6.5 Lesson

What You Will Learn

- Write indirect proofs.
- List sides and angles of a triangle in order by size.
- Use the Triangle Inequality Theorem to find possible side lengths of triangles.

Core Vocabulary

indirect proof, *p. 304*

Previous
proof
inequality

Writing an Indirect Proof

Suppose a student looks around the cafeteria, concludes that hamburgers are not being served, and explains as follows.

At first, I assumed that we are having hamburgers because today is Tuesday, and Tuesday is usually hamburger day.

There is always ketchup on the table when we have hamburgers, so I looked for the ketchup, but I didn't see any.

So, my assumption that we are having hamburgers must be false.

The student uses *indirect* reasoning. In an **indirect proof**, you start by making the temporary assumption that the desired conclusion is false. By then showing that this assumption leads to a logical impossibility, you prove the original statement true *by contradiction*.

Core Concept

How to Write an Indirect Proof (Proof by Contradiction)

Step 1 Identify the statement you want to prove. Assume temporarily that this statement is false by assuming that its opposite is true.

Step 2 Reason logically until you reach a contradiction.

Step 3 Point out that the desired conclusion must be true because the contradiction proves the temporary assumption false.

EXAMPLE 1 Writing an Indirect Proof

Write an indirect proof that in a given triangle, there can be at most one right angle.

Given $\triangle ABC$

Prove $\triangle ABC$ can have at most one right angle.

SOLUTION

Step 1 Assume temporarily that $\triangle ABC$ has two right angles. Then assume $\angle A$ and $\angle B$ are right angles.

Step 2 By the definition of right angle, $m\angle A = m\angle B = 90°$. By the Triangle Sum Theorem (Theorem 5.1), $m\angle A + m\angle B + m\angle C = 180°$. Using the Substitution Property of Equality, $90° + 90° + m\angle C = 180°$. So, $m\angle C = 0°$ by the Subtraction Property of Equality. A triangle cannot have an angle measure of 0°. So, this contradicts the given information.

Step 3 So, the assumption that $\triangle ABC$ has two right angles must be false, which proves that $\triangle ABC$ can have at most one right angle.

READING

You have reached a *contradiction* when you have two statements that cannot both be true at the same time.

Monitoring Progress

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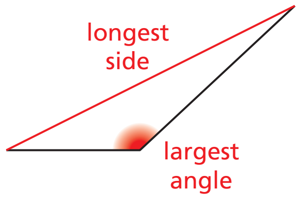
1. Write an indirect proof that a scalene triangle cannot have two congruent angles.

Relating Sides and Angles of a Triangle

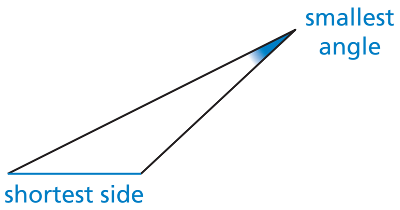
EXAMPLE 2 Relating Side Length and Angle Measure

Draw an obtuse scalene triangle. Find the largest angle and longest side and mark them in red. Find the smallest angle and shortest side and mark them in blue. What do you notice?

SOLUTION



The longest side and largest angle are opposite each other.



The shortest side and smallest angle are opposite each other.

COMMON ERROR

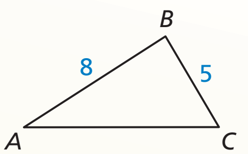
Be careful not to confuse the symbol ∠ meaning *angle* with the symbol < meaning *is less than*. Notice that the bottom edge of the angle symbol is horizontal.

The relationships in Example 2 are true for all triangles, as stated in the two theorems below. These relationships can help you decide whether a particular arrangement of side lengths and angle measures in a triangle may be possible.

Theorems

Theorem 6.9 Triangle Longer Side Theorem

If one side of a triangle is longer than another side, then the angle opposite the longer side is larger than the angle opposite the shorter side.

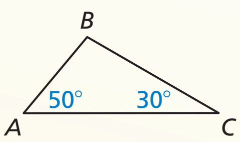


Proof Ex. 43, p. 310

$AB > BC$, so $m\angle C > m\angle A$.

Theorem 6.10 Triangle Larger Angle Theorem

If one angle of a triangle is larger than another angle, then the side opposite the larger angle is longer than the side opposite the smaller angle.



Proof p. 305

$m\angle A > m\angle C$, so $BC > AB$.

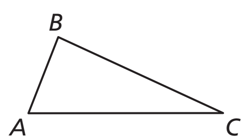
COMMON ERROR

Be sure to consider all cases when assuming the opposite is true.

PROOF Triangle Larger Angle Theorem

Given $m\angle A > m\angle C$

Prove $BC > AB$



Indirect Proof

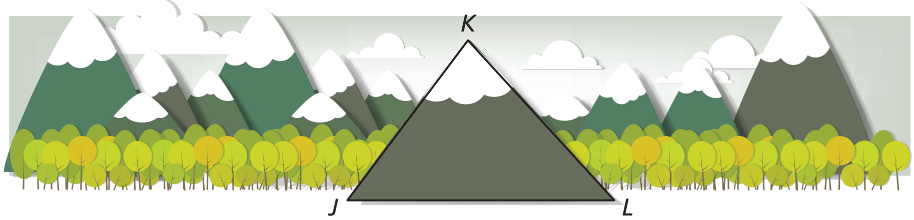
Step 1 Assume temporarily that $BC \ngtr AB$. Then it follows that either $BC < AB$ or $BC = AB$.

Step 2 If $BC < AB$, then $m\angle A < m\angle C$ by the Triangle Longer Side Theorem.
If $BC = AB$, then $m\angle A = m\angle C$ by the Base Angles Theorem (Thm. 5.6).

Step 3 Both conclusions contradict the given statement that $m\angle A > m\angle C$. So, the temporary assumption that $BC \ngtr AB$ cannot be true. This proves that $BC > AB$.

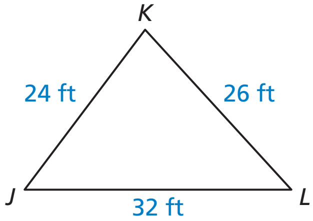
EXAMPLE 3 Ordering Angle Measures of a Triangle

You are constructing a stage prop that shows a large triangular mountain. The bottom edge of the mountain is about 32 feet long, the left slope is about 24 feet long, and the right slope is about 26 feet long. List the angles of $\triangle JKL$ in order from smallest to largest.



SOLUTION

Draw the triangle that represents the mountain. Label the side lengths.



The sides from shortest to longest are $\overline{JK}$, $\overline{KL}$, and $\overline{JL}$. The angles opposite these sides are $\angle L$, $\angle J$, and $\angle K$, respectively.

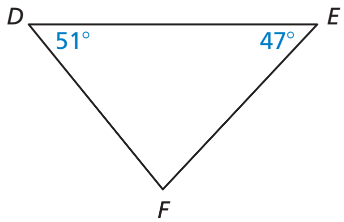
▶ So, by the Triangle Longer Side Theorem, the angles from smallest to largest are $\angle L$, $\angle J$, and $\angle K$.

EXAMPLE 4 Ordering Side Lengths of a Triangle

List the sides of $\triangle DEF$ in order from shortest to longest.

SOLUTION

First, find $m\angle F$ using the Triangle Sum Theorem (Theorem 5.1).



$$m\angle D + m\angle E + m\angle F = 180^\circ$$

$$51^\circ + 47^\circ + m\angle F = 180^\circ$$

$$m\angle F = 82^\circ$$

The angles from smallest to largest are $\angle E$, $\angle D$, and $\angle F$. The sides opposite these angles are $\overline{DF}$, $\overline{EF}$, and $\overline{DE}$, respectively.

▶ So, by the Triangle Larger Angle Theorem, the sides from shortest to longest are $\overline{DF}$, $\overline{EF}$, and $\overline{DE}$.

Monitoring Progress

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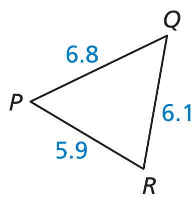
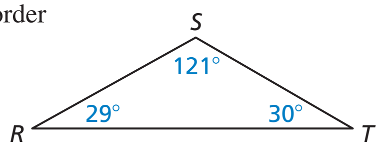
2. List the angles of $\triangle PQR$ in order from smallest to largest.

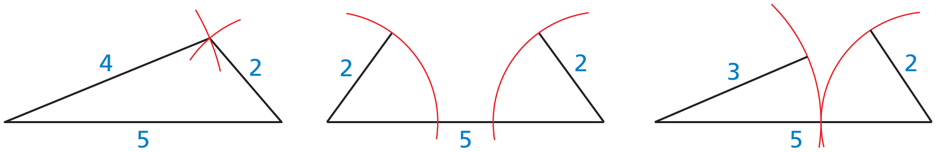


3. List the sides of $\triangle RST$ in order from shortest to longest.



Using the Triangle Inequality Theorem

Not every group of three segments can be used to form a triangle. The lengths of the segments must fit a certain relationship. For example, three attempted triangle constructions using segments with given lengths are shown below. Only the first group of segments forms a triangle.

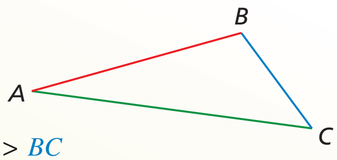


When you start with the longest side and attach the other two sides at its endpoints, you can see that the other two sides are not long enough to form a triangle in the second and third figures. This leads to the *Triangle Inequality Theorem*.

Theorem

Theorem 6.11 Triangle Inequality Theorem

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.



$AB + BC > AC$ $\quad$ $AC + BC > AB$ $\quad$ $AB + AC > BC$

Proof Ex. 47, p. 310

EXAMPLE 5 Finding Possible Side Lengths

A triangle has one side of length 14 and another side of length 9. Describe the possible lengths of the third side. Is it possible for the third side to have a length of 24? Explain.

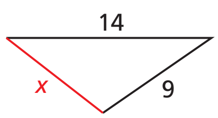
SOLUTION

Let x represent the length of the third side. Draw diagrams to help visualize the small and large values of x. Then use the Triangle Inequality Theorem to write and solve inequalities.

READING

You can combine the two inequalities, $x > 5$ and $x < 23$, to write the compound inequality $5 < x < 23$. This can be read as *x is between 5 and 23.*

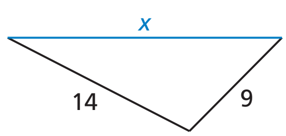
Small values of *x*



$x + 9 > 14$

$x > 5$

Large values of *x*



$9 + 14 > x$

$23 > x$, or $x < 23$

▶ The length of the third side must be greater than 5 and less than 23. Since $9 + 14 < 24$, it is not possible for the triangle to have a third side of length 24.

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4. A triangle has one side of length 12 inches and another side of length 20 inches. Describe the possible lengths of the third side.

Decide whether it is possible to construct a triangle with the given side lengths. Explain your reasoning.

5. 4 ft, 9 ft, 10 ft $\quad$ **6.** 8 m, 9 m, 18 m $\quad$ **7.** 5 cm, 7 cm, 12 cm

6.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** Why is an indirect proof also called a *proof by contradiction*?

2. **WRITING** How can you tell which side of a triangle is the longest from the angle measures of the triangle? How can you tell which side is the shortest?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, write the first step in an indirect proof of the statement. *(See Example 1.)*

3. If $WV + VU \neq 12$ inches and $VU = 5$ inches, then $WV \neq 7$ inches.

4. If x and y are odd integers, then xy is odd.

5. In $\triangle ABC$, if $m\angle A = 100°$, then $\angle B$ is not a right angle.

6. In $\triangle JKL$, if M is the midpoint of $\overline{KL}$, then $\overline{JM}$ is a median.

In Exercises 7 and 8, determine which two statements contradict each other. Explain your reasoning.

7. Ⓐ $\triangle LMN$ is a right triangle.
   Ⓑ $\angle L \cong \angle N$
   Ⓒ $\triangle LMN$ is equilateral.

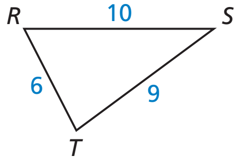
8. Ⓐ Both $\angle X$ and $\angle Y$ have measures greater than 20°.
   Ⓑ Both $\angle X$ and $\angle Y$ have measures less than 30°.
   Ⓒ $m\angle X + m\angle Y = 62°$

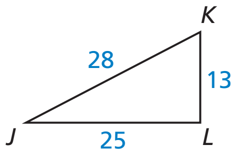
In Exercises 9 and 10, use a ruler and protractor to draw the given type of triangle. Mark the largest angle and longest side in red and the smallest angle and shortest side in blue. What do you notice? *(See Example 2.)*

9. acute scalene

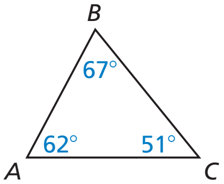
10. right scalene

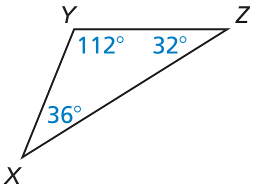
In Exercises 11 and 12, list the angles of the given triangle from smallest to largest. *(See Example 3.)*

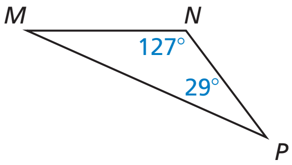
11.


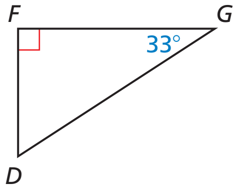
12.


In Exercises 13–16, list the sides of the given triangle from shortest to longest. *(See Example 4.)*

13.


14.


15.


16.


In Exercises 17–20, describe the possible lengths of the third side of the triangle given the lengths of the other two sides. *(See Example 5.)*

17. 5 inches, 12 inches

18. 12 feet, 18 feet

19. 2 feet, 40 inches

20. 25 meters, 25 meters

In Exercises 21–24, is it possible to construct a triangle with the given side lengths? If not, explain why not.

21. 6, 7, 11

22. 3, 6, 9

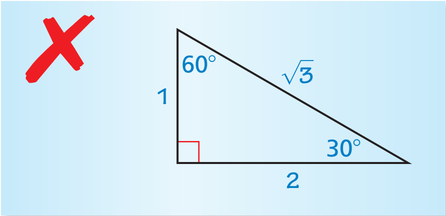
23. 28, 17, 46

24. 35, 120, 125

25. **ERROR ANALYSIS** Describe and correct the error in writing the first step of an indirect proof.

> ✗ Show that $\angle A$ is obtuse.
>
> **Step 1** Assume temporarily that $\angle A$ is acute.

26. ERROR ANALYSIS Describe and correct the error in labeling the side lengths 1, 2, and $\sqrt{3}$ on the triangle.



27. REASONING You are a lawyer representing a client who has been accused of a crime. The crime took place in Los Angeles, California. Security footage shows your client in New York at the time of the crime. Explain how to use indirect reasoning to prove your client is innocent.

28. REASONING Your class has fewer than 30 students. The teacher divides your class into two groups. The first group has 15 students. Use indirect reasoning to show that the second group must have fewer than 15 students.

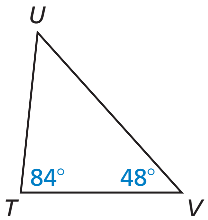
29. PROBLEM SOLVING Which statement about $\triangle TUV$ is false?

Ⓐ $UV > TU$

Ⓑ $UV + TV > TU$

Ⓒ $UV < TV$

Ⓓ $\triangle TUV$ is isosceles.



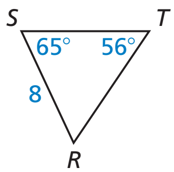
30. PROBLEM SOLVING In $\triangle RST$, which is a possible side length for ST? Select all that apply.

Ⓐ 7

Ⓑ 8

Ⓒ 9

Ⓓ 10



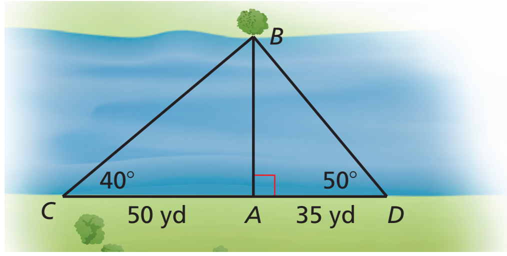
31. PROOF Write an indirect proof that an odd number is not divisible by 4.

32. PROOF Write an indirect proof of the statement "In $\triangle QRS$, if $m\angle Q + m\angle R = 90°$, then $m\angle S = 90°$."

33. WRITING Explain why the hypotenuse of a right triangle must always be longer than either leg.

34. CRITICAL THINKING Is it possible to decide if three side lengths form a triangle without checking all three inequalities shown in the Triangle Inequality Theorem (Theorem 6.11)? Explain your reasoning.

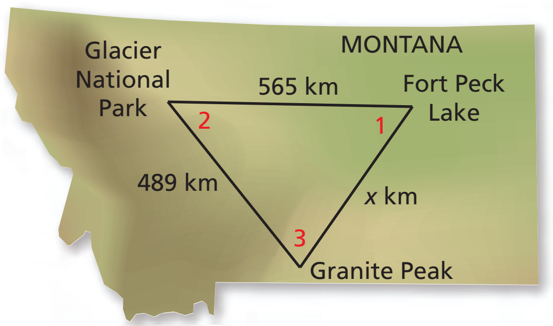
35. MODELING WITH MATHEMATICS You can estimate the width of the river from point A to the tree at point B by measuring the angle to the tree at several locations along the riverbank. The diagram shows the results for locations C and D.



a. Using $\triangle BCA$ and $\triangle BDA$, determine the possible widths of the river. Explain your reasoning.

b. What could you do if you wanted a closer estimate?

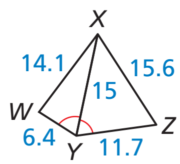
36. MODELING WITH MATHEMATICS You travel from Fort Peck Lake to Glacier National Park and from Glacier National Park to Granite Peak.



a. Write two inequalities to represent the possible distances from Granite Peak back to Fort Peck Lake.

b. How is your answer to part (a) affected if you know that $m\angle 2 < m\angle 1$ and $m\angle 2 < m\angle 3$?

37. REASONING In the figure, $\overline{XY}$ bisects $\angle WYZ$. List all six angles of $\triangle XYZ$ and $\triangle WXY$ in order from smallest to largest. Explain your reasoning.

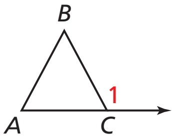


38. MATHEMATICAL CONNECTIONS In $\triangle DEF$, $m\angle D = (x + 25)°$, $m\angle E = (2x - 4)°$, and $m\angle F = 63°$. List the side lengths and angle measures of the triangle in order from least to greatest.

39. ANALYZING RELATIONSHIPS Another triangle inequality relationship is given by the Exterior Angle Inequality Theorem. It states:

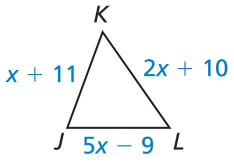
The measure of an exterior angle of a triangle is greater than the measure of either of the nonadjacent interior angles.

Explain how you know that $m\angle 1 > m\angle A$ and $m\angle 1 > m\angle B$ in $\triangle ABC$ with exterior angle $\angle 1$.

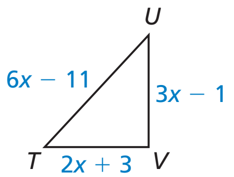


MATHEMATICAL CONNECTIONS **In Exercises 40 and 41, describe the possible values of *x*.**

40.

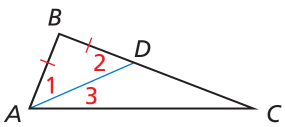


41.



42. HOW DO YOU SEE IT? Your house is on the corner of Hill Street and Eighth Street. The library is on the corner of View Street and Seventh Street. What is the shortest route to get from your house to the library? Explain your reasoning.

43. PROVING A THEOREM Use the diagram to prove the Triangle Longer Side Theorem (Theorem 6.9).



Given $BC > AB$, $BD = BA$

Prove $m\angle BAC > m\angle C$

44. USING STRUCTURE The length of the base of an isosceles triangle is ℓ. Describe the possible lengths for each leg. Explain your reasoning.

45. MAKING AN ARGUMENT Your classmate claims to have drawn a triangle with one side length of 13 inches and a perimeter of 2 feet. Is this possible? Explain your reasoning.

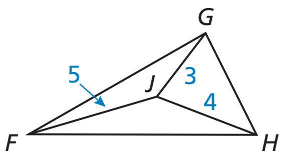
46. THOUGHT PROVOKING Cut two pieces of string that are each 24 centimeters long. Construct an isosceles triangle out of one string and a scalene triangle out of the other. Measure and record the side lengths. Then classify each triangle by its angles.

47. PROVING A THEOREM Prove the Triangle Inequality Theorem (Theorem 6.11).

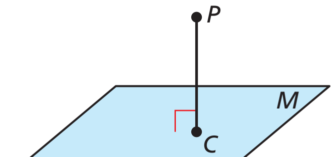
Given $\triangle ABC$

Prove $AB + BC > AC$, $AC + BC > AB$, and $AB + AC > BC$

48. ATTENDING TO PRECISION The perimeter of $\triangle HGF$ must be between what two integers? Explain your reasoning.



49. PROOF Write an indirect proof that a perpendicular segment is the shortest segment from a point to a plane.



Given $\overline{PC} \perp$ plane M

Prove $\overline{PC}$ is the shortest segment from P to plane M.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

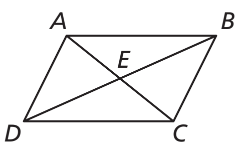
Name the included angle between the pair of sides given.

50. $\overline{AE}$ and $\overline{BE}$

51. $\overline{AC}$ and $\overline{DC}$

52. $\overline{AD}$ and $\overline{DC}$

53. $\overline{CE}$ and $\overline{BE}$



6.6 Inequalities in Two Triangles

Essential Question If two sides of one triangle are congruent to two sides of another triangle, what can you say about the third sides of the triangles?

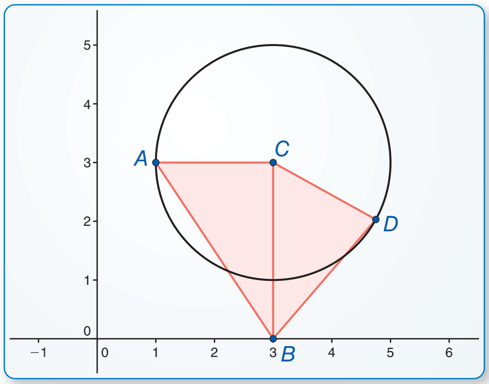
EXPLORATION 1 **Comparing Measures in Triangles**

Work with a partner. Use dynamic geometry software.

a. Draw $\triangle ABC$, as shown below.

b. Draw the circle with center $C(3, 3)$ through the point $A(1, 3)$.

c. Draw $\triangle DBC$ so that D is a point on the circle.



Sample
Points
$A(1, 3)$
$B(3, 0)$
$C(3, 3)$
$D(4.75, 2.03)$
Segments
$BC = 3$
$AC = 2$
$DC = 2$
$AB = 3.61$
$DB = 2.68$

d. Which two sides of $\triangle ABC$ are congruent to two sides of $\triangle DBC$? Justify your answer.

e. Compare the lengths of $\overline{AB}$ and $\overline{DB}$. Then compare the measures of $\angle ACB$ and $\angle DCB$. Are the results what you expected? Explain.

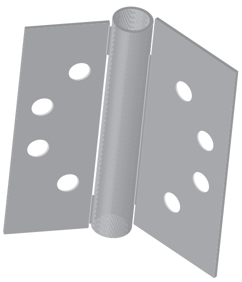
f. Drag point D to several locations on the circle. At each location, repeat part (e). Copy and record your results in the table below.

	D	AC	BC	AB	BD	$m\angle ACB$	$m\angle BCD$
1.	(4.75, 2.03)	2	3				
2.		2	3				
3.		2	3				
4.		2	3				
5.		2	3				

g. Look for a pattern of the measures in your table. Then write a conjecture that summarizes your observations.

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to make conjectures and build a logical progression of statements to explore the truth of your conjectures.



Communicate Your Answer

2. If two sides of one triangle are congruent to two sides of another triangle, what can you say about the third sides of the triangles?

3. Explain how you can use the hinge shown at the left to model the concept described in Question 2.

6.6 Lesson

Core Vocabulary

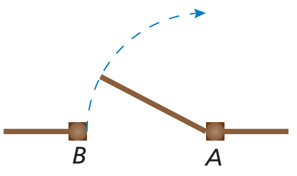
Previous
indirect proof
inequality

What You Will Learn

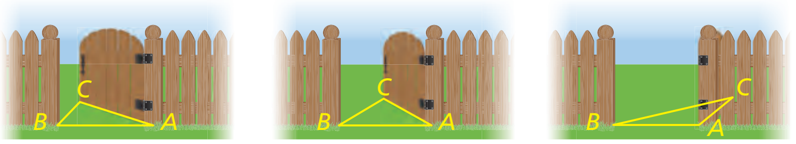
- Compare measures in triangles.
- Solve real-life problems using the Hinge Theorem.

Comparing Measures in Triangles

Imagine a gate between fence posts A and B that has hinges at A and swings open at B.



As the gate swings open, you can think of $\triangle ABC$, with side $\overline{AC}$ formed by the gate itself, side $\overline{AB}$ representing the distance between the fence posts, and side $\overline{BC}$ representing the opening between post B and the outer edge of the gate.



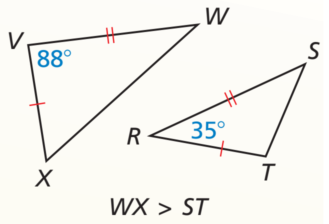
Notice that as the gate opens wider, both the measure of $\angle A$ and the distance BC increase. This suggests the *Hinge Theorem*.

Theorems

Theorem 6.12 Hinge Theorem

If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first is larger than the included angle of the second, then the third side of the first is longer than the third side of the second.

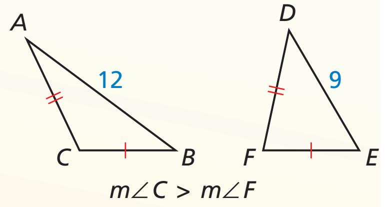
Proof BigIdeasMath.com



Theorem 6.13 Converse of the Hinge Theorem

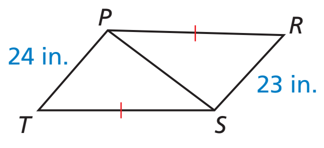
If two sides of one triangle are congruent to two sides of another triangle, and the third side of the first is longer than the third side of the second, then the included angle of the first is larger than the included angle of the second.

Proof Example 3, p. 313



EXAMPLE 1 Using the Converse of the Hinge Theorem

Given that $\overline{ST} \cong \overline{PR}$, how does $m\angle PST$ compare to $m\angle SPR$?



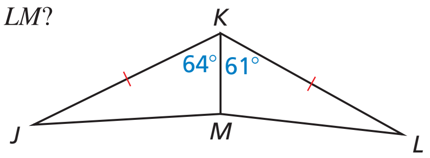
SOLUTION

You are given that $\overline{ST} \cong \overline{PR}$, and you know that $\overline{PS} \cong \overline{PS}$ by the Reflexive Property of Congruence (Theorem 2.1). Because 24 inches > 23 inches, $PT > SR$. So, two sides of $\triangle STP$ are congruent to two sides of $\triangle PRS$ and the third side of $\triangle STP$ is longer.

- By the Converse of the Hinge Theorem, $m\angle PST > m\angle SPR$.

EXAMPLE 2 Using the Hinge Theorem

Given that $\overline{JK} \cong \overline{LK}$, how does JM compare to LM?



SOLUTION

You are given that $\overline{JK} \cong \overline{LK}$, and you know that $\overline{KM} \cong \overline{KM}$ by the Reflexive Property of Congruence (Theorem 2.1). Because $64° > 61°$, $m\angle JKM > m\angle LKM$. So, two sides of $\triangle JKM$ are congruent to two sides of $\triangle LKM$, and the included angle in $\triangle JKM$ is larger.

▶ By the Hinge Theorem, $JM > LM$.

Monitoring Progress

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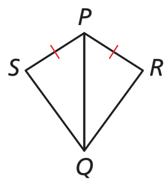
Use the diagram.



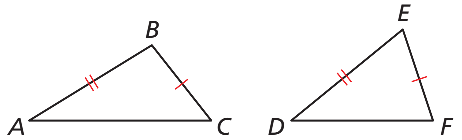
1. If $PR = PS$ and $m\angle QPR > m\angle QPS$, which is longer, $\overline{SQ}$ or $\overline{RQ}$?
2. If $PR = PS$ and $RQ < SQ$, which is larger, $\angle RPQ$ or $\angle SPQ$?

EXAMPLE 3 Proving the Converse of the Hinge Theorem

Write an indirect proof of the Converse of the Hinge Theorem.

Given $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $AC > DF$

Prove $m\angle B > m\angle E$



Indirect Proof

Step 1 Assume temporarily that $m\angle B \not> m\angle E$. Then it follows that either $m\angle B < m\angle E$ or $m\angle B = m\angle E$.

Step 2 If $m\angle B < m\angle E$, then $AC < DF$ by the Hinge Theorem.

If $m\angle B = m\angle E$, then $\angle B \cong \angle E$. So, $\triangle ABC \cong \triangle DEF$ by the SAS Congruence Theorem (Theorem 5.5) and $AC = DF$.

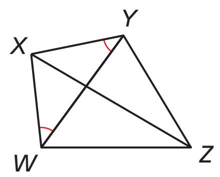
Step 3 Both conclusions contradict the given statement that $AC > DF$. So, the temporary assumption that $m\angle B \not> m\angle E$ cannot be true. This proves that $m\angle B > m\angle E$.

EXAMPLE 4 Proving Triangle Relationships

Write a paragraph proof.

Given $\angle XWY \cong \angle XYW$, $WZ > YZ$

Prove $m\angle WXZ > m\angle YXZ$



Paragraph Proof Because $\angle XWY \cong \angle XYW$, $\overline{XY} \cong \overline{XW}$ by the Converse of the Base Angles Theorem (Theorem 5.7). By the Reflexive Property of Congruence (Theorem 2.1), $\overline{XZ} \cong \overline{XZ}$. Because $WZ > YZ$, $m\angle WXZ > m\angle YXZ$ by the Converse of the Hinge Theorem.

Monitoring Progress

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3. Write a temporary assumption you can make to prove the Hinge Theorem indirectly. What two cases does that assumption lead to?

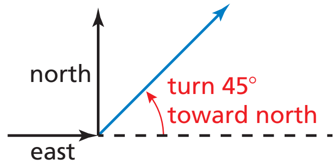
Solving Real-Life Problems

EXAMPLE 5 **Solving a Real-Life Problem**

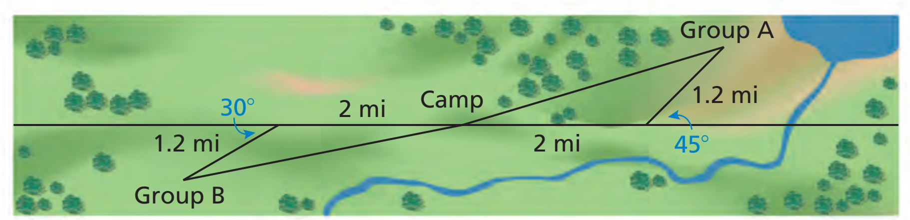
Two groups of bikers leave the same camp heading in opposite directions. Each group travels 2 miles, then changes direction and travels 1.2 miles. Group A starts due east and then turns 45° toward north. Group B starts due west and then turns 30° toward south. Which group is farther from camp? Explain your reasoning.

SOLUTION

1. **Understand the Problem** You know the distances and directions that the groups of bikers travel. You need to determine which group is farther from camp. You can interpret a turn of 45° toward north, as shown.



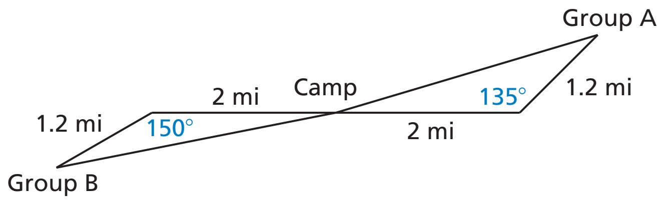
2. **Make a Plan** Draw a diagram that represents the situation and mark the given measures. The distances that the groups bike and the distances back to camp form two triangles. The triangles have two congruent side lengths of 2 miles and 1.2 miles. Include the third side of each triangle in the diagram.



3. **Solve the Problem** Use linear pairs to find the included angles for the paths that the groups take.

Group A: $180° - 45° = 135°$ **Group B:** $180° - 30° = 150°$

The included angles are 135° and 150°.



Because $150° > 135°$, the distance Group B is from camp is greater than the distance Group A is from camp by the Hinge Theorem.

▶ So, Group B is farther from camp.

4. **Look Back** Because the included angle for Group A is 15° less than the included angle for Group B, you can reason that Group A would be closer to camp than Group B. So, Group B is farther from camp.

Monitoring Progress

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4. **WHAT IF?** In Example 5, Group C leaves camp and travels 2 miles due north, then turns 40° toward east and travels 1.2 miles. Compare the distances from camp for all three groups.

6.6 Exercises

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Vocabulary and Core Concept Check

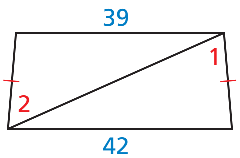
1. **WRITING** Explain why Theorem 6.12 is named the "Hinge Theorem."

2. **COMPLETE THE SENTENCE** In $\triangle ABC$ and $\triangle DEF$, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $AC < DF$. So $m\angle$_____ $>$ $m\angle$_____ by the Converse of the Hinge Theorem (Theorem 6.13).

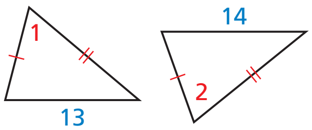
Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, copy and complete the statement with <, >, or =. Explain your reasoning. *(See Example 1.)*

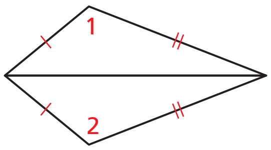
3. $m\angle 1$ _____ $m\angle 2$



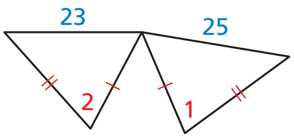
4. $m\angle 1$ _____ $m\angle 2$



5. $m\angle 1$ _____ $m\angle 2$

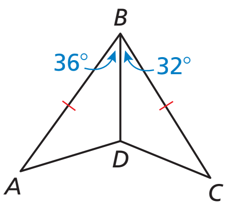


6. $m\angle 1$ _____ $m\angle 2$

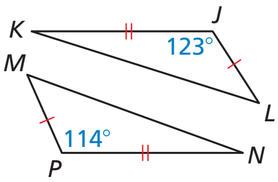


In Exercises 7–10, copy and complete the statement with <, >, or =. Explain your reasoning. *(See Example 2.)*

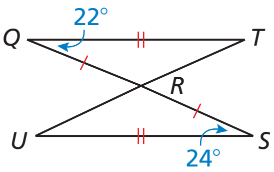
7. AD _____ CD



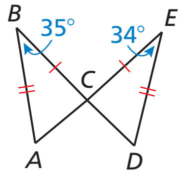
8. MN _____ LK



9. TR _____ UR



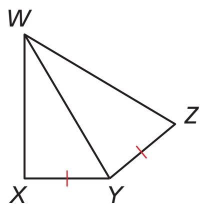
10. AC _____ DC



PROOF In Exercises 11 and 12, write a proof. *(See Example 4.)*

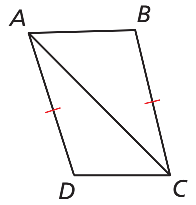
11. **Given** $\overline{XY} \cong \overline{YZ}$, $m\angle WYZ > m\angle WYX$

Prove $WZ > WX$



12. **Given** $\overline{BC} \cong \overline{DA}$, $DC < AB$

Prove $m\angle BCA > m\angle DAC$



In Exercises 13 and 14, you and your friend leave on different flights from the same airport. Determine which flight is farther from the airport. Explain your reasoning. *(See Example 5.)*

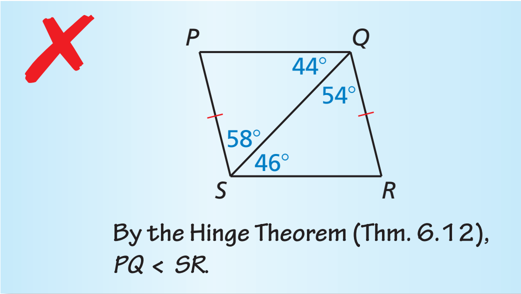
13. **Your flight:** Flies 100 miles due west, then turns 20° toward north and flies 50 miles.

Friend's flight: Flies 100 miles due north, then turns 30° toward east and flies 50 miles.

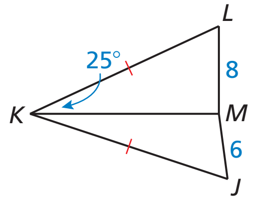
14. **Your flight:** Flies 210 miles due south, then turns 70° toward west and flies 80 miles.

Friend's flight: Flies 80 miles due north, then turns 50° toward east and flies 210 miles.

15. **ERROR ANALYSIS** Describe and correct the error in using the Hinge Theorem (Theorem 6.12).

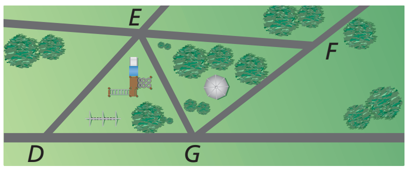


16. **REPEATED REASONING** Which is a possible measure for $\angle JKM$? Select all that apply.



Ⓐ 15° Ⓑ 22° Ⓒ 25° Ⓓ 35°

17. **DRAWING CONCLUSIONS** The path from E to F is longer than the path from E to D. The path from G to D is the same length as the path from G to F. What can you conclude about the angles of the paths? Explain your reasoning.

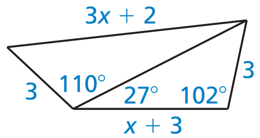


18. **ABSTRACT REASONING** In $\triangle EFG$, the bisector of $\angle F$ intersects the bisector of $\angle G$ at point H. Explain why $\overline{FG}$ must be longer than $\overline{FH}$ or $\overline{HG}$.

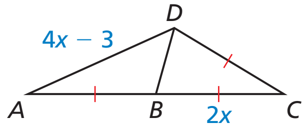
19. **ABSTRACT REASONING** $\overline{NR}$ is a median of $\triangle NPQ$, and $NQ > NP$. Explain why $\angle NRQ$ is obtuse.

MATHEMATICAL CONNECTIONS **In Exercises 20 and 21, write and solve an inequality for the possible values of x.**

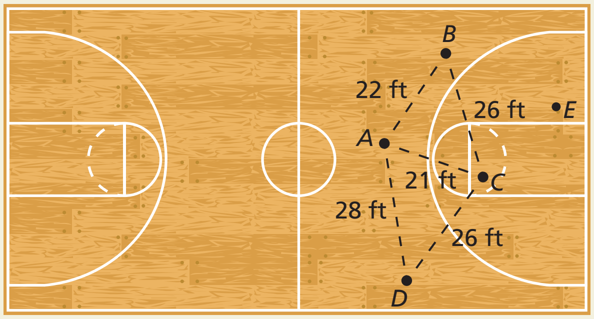
20.



21. **Given** B is the midpoint of $\overline{AC}$.



22. **HOW DO YOU SEE IT?** In the diagram, triangles are formed by the locations of the players on the basketball court. The dashed lines represent the possible paths of the basketball as the players pass. How does $m\angle ACB$ compare with $m\angle ACD$?



23. **CRITICAL THINKING** In $\triangle ABC$, the altitudes from B and C meet at point D, and $m\angle BAC > m\angle BDC$. What is true about $\triangle ABC$? Justify your answer.

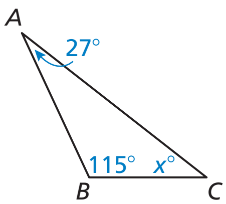
24. **THOUGHT PROVOKING** The postulates and theorems in this book represent Euclidean geometry. In spherical geometry, all points are on the surface of a sphere. A line is a circle on the sphere whose diameter is equal to the diameter of the sphere. In spherical geometry, state an inequality involving the sum of the angles of a triangle. Find a formula for the area of a triangle in spherical geometry.

Maintaining Mathematical Proficiency

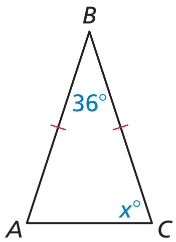
Reviewing what you learned in previous grades and lessons

Find the value of x.

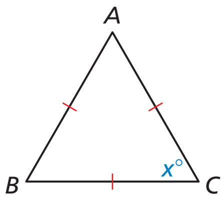
25.



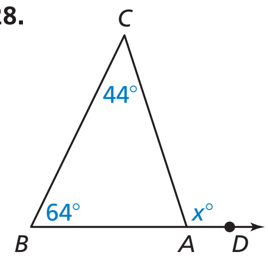
26.



27.



28.



6 Chapter Review

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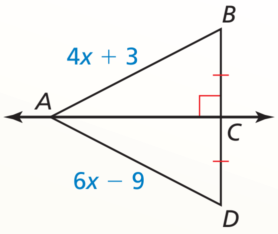
6.1 Perpendicular and Angle Bisectors (pp. 271–278)

Find *AD*.

From the figure, $\overleftrightarrow{AC}$ is the perpendicular bisector of $\overline{BD}$.

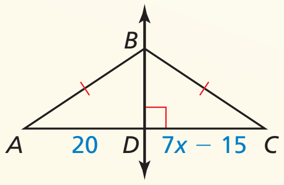


$AB = AD$	Perpendicular Bisector Theorem (Theorem 6.1)
$4x + 3 = 6x - 9$	Substitute.
$x = 6$	Solve for x.

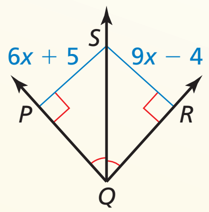
▶ So, $AD = 6(6) - 9 = 27$.

Find the indicated measure. Explain your reasoning.

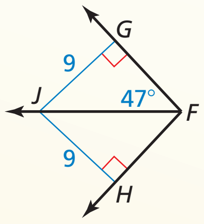
1. *DC*



2. *RS*



3. $m\angle JFH$



6.2 Bisectors of Triangles (pp. 279–288)

Find the coordinates of the circumcenter of △*QRS* with vertices *Q*(3, 3), *R*(5, 7), and *S*(9, 3).

Step 1 Graph △*QRS*.

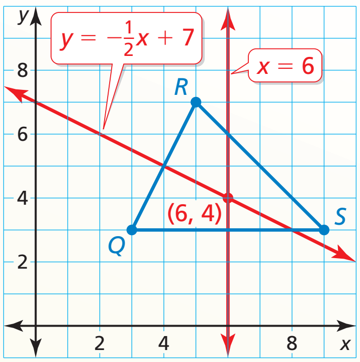
Step 2 Find equations for two perpendicular bisectors.

The midpoint of $\overline{QS}$ is (6, 3). The line through (6, 3) that is perpendicular to $\overline{QS}$ is $x = 6$.

The midpoint of $\overline{QR}$ is (4, 5). The line through (4, 5) that is perpendicular to $\overline{QR}$ is $y = -\frac{1}{2}x + 7$.

Step 3 Find the point where $x = 6$ and $y = -\frac{1}{2}x + 7$ intersect. They intersect at (6, 4).

▶ So, the coordinates of the circumcenter are (6, 4).

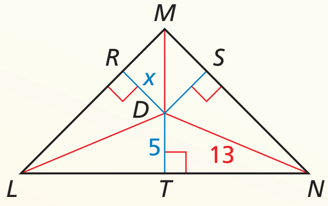


Find the coordinates of the circumcenter of the triangle with the given vertices.

4. *T*(−6, −5), *U*(0, −1), *V*(0, −5)

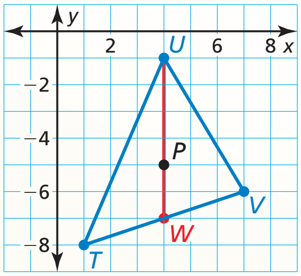
5. *X*(−2, 1), *Y*(2, −3), *Z*(6, −3)

6. Point *D* is the incenter of △*LMN*. Find the value of *x*.



6.3 Medians and Altitudes of Triangles (pp. 289–296)

Find the coordinates of the centroid of $\triangle TUV$ with vertices $T(1, -8)$, $U(4, -1)$, and $V(7, -6)$.



Step 1 Graph $\triangle TUV$.

Step 2 Use the Midpoint Formula to find the midpoint W of $\overline{TV}$. Sketch median $\overline{UW}$.

$$W\left(\frac{1+7}{2}, \frac{-8+(-6)}{2}\right) = (4, -7)$$

Step 3 Find the centroid. It is two-thirds of the distance from each vertex to the midpoint of the opposite side.

The distance from vertex $U(4, -1)$ to $W(4, -7)$ is $-1 - (-7) = 6$ units.

So, the centroid is $\frac{2}{3}(6) = 4$ units down from vertex U on $\overline{UW}$.

▶ So, the coordinates of the centroid P are $(4, -1 - 4)$, or $(4, -5)$.

Find the coordinates of the centroid of the triangle with the given vertices.

7. $A(-10, 3)$, $B(-4, 5)$, $C(-4, 1)$

8. $D(2, -8)$, $E(2, -2)$, $F(8, -2)$

Tell whether the orthocenter of the triangle with the given vertices is *inside*, *on*, or *outside* the triangle. Then find the coordinates of the orthocenter.

9. $G(1, 6)$, $H(5, 6)$, $J(3, 1)$

10. $K(-8, 5)$, $L(-6, 3)$, $M(0, 5)$

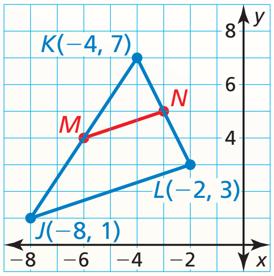
6.4 The Triangle Midsegment Theorem (pp. 297–302)

In $\triangle JKL$, show that midsegment $\overline{MN}$ is parallel to $\overline{JL}$ and that $MN = \frac{1}{2}JL$.

Step 1 Find the coordinates of M and N by finding the midpoints of $\overline{JK}$ and $\overline{KL}$.

$$M\left(\frac{-8+(-4)}{2}, \frac{1+7}{2}\right) = M\left(\frac{-12}{2}, \frac{8}{2}\right) = M(-6, 4)$$

$$N\left(\frac{-4+(-2)}{2}, \frac{7+3}{2}\right) = N\left(\frac{-6}{2}, \frac{10}{2}\right) = N(-3, 5)$$



Step 2 Find and compare the slopes of $\overline{MN}$ and $\overline{JL}$.

$$\text{slope of } \overline{MN} = \frac{5-4}{-3-(-6)} = \frac{1}{3}$$

$$\text{slope of } \overline{JL} = \frac{3-1}{-2-(-8)} = \frac{2}{6} = \frac{1}{3}$$

▶ Because the slopes are the same, $\overline{MN}$ is parallel to $\overline{JL}$.

Step 3 Find and compare the lengths of $\overline{MN}$ and $\overline{JL}$.

$$MN = \sqrt{[-3-(-6)]^2 + (5-4)^2} = \sqrt{9+1} = \sqrt{10}$$

$$JL = \sqrt{[-2-(-8)]^2 + (3-1)^2} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$$

▶ Because $\sqrt{10} = \frac{1}{2}(2\sqrt{10})$, $MN = \frac{1}{2}JL$.

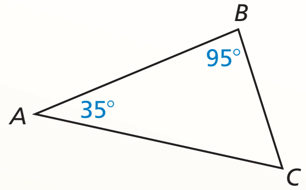
Find the coordinates of the vertices of the midsegment triangle for the triangle with the given vertices.

11. $A(-6, 8)$, $B(-6, 4)$, $C(0, 4)$

12. $D(-3, 1)$, $E(3, 5)$, $F(1, -5)$

6.5 Indirect Proof and Inequalities in One Triangle (pp. 303–310)

a. List the sides of $\triangle ABC$ in order from shortest to longest.



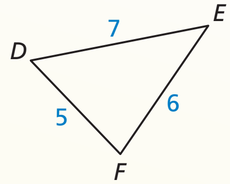
First, find $m\angle C$ using the Triangle Sum Theorem (Thm. 5.1).

$$m\angle A + m\angle B + m\angle C = 180^\circ$$
$$35^\circ + 95^\circ + m\angle C = 180^\circ$$
$$m\angle C = 50^\circ$$

The angles from smallest to largest are $\angle A$, $\angle C$, and $\angle B$. The sides opposite these angles are $\overline{BC}$, $\overline{AB}$, and $\overline{AC}$, respectively.

▶ So, by the Triangle Larger Angle Theorem (Theorem 6.10), the sides from shortest to longest are $\overline{BC}$, $\overline{AB}$, and $\overline{AC}$.

b. List the angles of $\triangle DEF$ in order from smallest to largest.



The sides from shortest to longest are $\overline{DF}$, $\overline{EF}$, and $\overline{DE}$. The angles opposite these sides are $\angle E$, $\angle D$, and $\angle F$, respectively.

▶ So, by the Triangle Longer Side Theorem (Theorem 6.9), the angles from smallest to largest are $\angle E$, $\angle D$, and $\angle F$.

Describe the possible lengths of the third side of the triangle given the lengths of the other two sides.

13. 4 inches, 8 inches

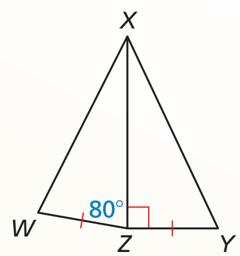
14. 6 meters, 9 meters

15. 11 feet, 18 feet

16. Write an indirect proof of the statement "In $\triangle XYZ$, if $XY = 4$ and $XZ = 8$, then $YZ > 4$."

6.6 Inequalities in Two Triangles (pp. 311–316)

Given that $\overline{WZ} \cong \overline{YZ}$, how does XY compare to XW?

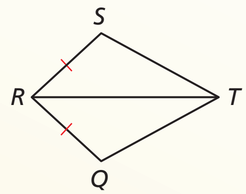


You are given that $\overline{WZ} \cong \overline{YZ}$, and you know that $\overline{XZ} \cong \overline{XZ}$ by the Reflexive Property of Congruence (Theorem 2.1).

Because 90° > 80°, $m\angle XZY > m\angle XZW$. So, two sides of $\triangle XZY$ are congruent to two sides of $\triangle XZW$ and the included angle in $\triangle XZY$ is larger.

▶ By the Hinge Theorem (Theorem 6.12), $XY > XW$.

Use the diagram.



17. If $RQ = RS$ and $m\angle QRT > m\angle SRT$, then how does $\overline{QT}$ compare to $\overline{ST}$?

18. If $RQ = RS$ and $QT > ST$, then how does $\angle QRT$ compare to $\angle SRT$?