10.1 Simplifying Radical Expressions

Essential Question How can you multiply square roots?

EXPLORATION 1 Operations with Square Roots

Work with a partner. For each operation with square roots, compare the results obtained using the two indicated orders of operations. What can you conclude?

a. Square Roots and Addition

Is $\sqrt{36} + \sqrt{64}$ equal to $\sqrt{36 + 64}$?

Is $\sqrt{25} + \sqrt{144}$ equal to $\sqrt{25 + 144}$?

In general, is $\sqrt{a} + \sqrt{b}$ equal to $\sqrt{a + b}$? Explain your reasoning.

b. Square Roots and Multiplication

Is $\sqrt{4} \cdot \sqrt{9}$ equal to $\sqrt{4 \cdot 9}$?

Is $\sqrt{25} \cdot \sqrt{100}$ equal to $\sqrt{25 \cdot 100}$?

In general, is $\sqrt{a} \cdot \sqrt{b}$ equal to $\sqrt{a \cdot b}$? Explain your reasoning.

c. Square Roots and Subtraction

Is $\sqrt{64} - \sqrt{36}$ equal to $\sqrt{64 - 36}$?

Is $\sqrt{169} - \sqrt{144}$ equal to $\sqrt{169 - 144}$?

In general, is $\sqrt{a} - \sqrt{b}$ equal to $\sqrt{a - b}$? Explain your reasoning.

REASONING ABSTRACTLY

To be proficient in math, you need to recognize and use counterexamples.

EXPLORATION 2 Writing Counterexamples

Work with a partner. A **counterexample** is an example that proves that a general statement is *not* true. For each general statement in Exploration 1 that is not true, write a counterexample different from the examples given.

Communicate Your Answer

3. How can you multiply square roots?

4. Give an example of multiplying square roots that is different from the examples in Exploration 1.

5. Write an algebraic rule for the product of square roots.

10.1 Lesson

What You Will Learn

- Use properties of radicals to simplify expressions.
- Perform operations with radicals.

Core Vocabulary

counterexample, *p. 379*
radical expression, *p. 380*
simplest form of a radical, *p. 380*
like radicals, *p. 382*

Previous
radicand
perfect cube

Using Properties of Radicals

A **radical expression** is an expression that contains a radical. An expression involving a radical with index n is in **simplest form** when these three conditions are met.

- No radicands have perfect nth powers as factors other than 1.
- No radicands contain fractions.
- No radicals appear in the denominator of a fraction.

You can use the property below to simplify radical expressions involving square roots.

Core Concept

Product Property of Square Roots

Words The square root of a product equals the product of the square roots of the factors.

Numbers $\sqrt{9 \cdot 5} = \sqrt{9} \cdot \sqrt{5} = 3\sqrt{5}$

Algebra $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, where $a, b \geq 0$

STUDY TIP

There can be more than one way to factor a radicand. An efficient method is to find the greatest perfect square factor.

EXAMPLE 1 Using the Product Property of Square Roots

a. $\sqrt{108} = \sqrt{36 \cdot 3}$ Factor using the greatest perfect square factor.

$= \sqrt{36} \cdot \sqrt{3}$ Product Property of Square Roots

$= 6\sqrt{3}$ Simplify.

b. $\sqrt{98} = \sqrt{49 \cdot 2}$ Factor using the greatest perfect square factor.

$= \sqrt{49} \cdot \sqrt{2}$ Product Property of Square Roots

$= 7\sqrt{2}$ Simplify.

c. $\sqrt{9x^3} = \sqrt{9 \cdot x^2 \cdot x}$ Factor using the greatest perfect square factor.

$= \sqrt{9} \cdot \sqrt{x^2} \cdot \sqrt{x}$ Product Property of Square Roots

$= 3x\sqrt{x}$ Simplify.

d. $-\sqrt{500t^6} = -\sqrt{100 \cdot 5 \cdot t^6}$ Factor using the greatest perfect square factor.

$= -\sqrt{100} \cdot \sqrt{t^6} \cdot \sqrt{5}$ Product Property of Square Roots

$= -10t^3\sqrt{5}$ Simplify.

STUDY TIP

In this course, whenever a variable appears in the radicand, assume that it has only *nonnegative* values.

Monitoring Progress

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Simplify the expression.

1. $\sqrt{24}$ **2.** $-\sqrt{80}$ **3.** $\sqrt{49x^3}$ **4.** $\sqrt{75n^5}$

You can extend the Product Property of Square Roots to other radicals, such as cube roots. When using this *property of cube roots*, the radicands may contain negative numbers.

STUDY TIP

To write a cube root in simplest form, find factors of the radicand that are perfect cubes.

EXAMPLE 2 Using the Product Property of Cube Roots

a. $\sqrt[3]{-128} = \sqrt[3]{-64 \cdot 2}$ Factor using the greatest perfect cube factor.

$= \sqrt[3]{-64} \cdot \sqrt[3]{2}$ Product Property of Cube Roots

$= -4\sqrt[3]{2}$ Simplify.

b. $\sqrt[3]{81} = \sqrt[3]{27 \cdot 3}$ Factor using the greatest perfect cube factor.

$= \sqrt[3]{27} \cdot \sqrt[3]{3}$ Product Property of Cube Roots

$= 3\sqrt[3]{3}$ Simplify.

c. $\sqrt[3]{72} = \sqrt[3]{8 \cdot 9}$ Factor using the greatest perfect cube factor.

$= \sqrt[3]{8} \cdot \sqrt[3]{9}$ Product Property of Cube Roots

$= 2\sqrt[3]{9}$ Simplify.

d. $\sqrt[3]{125x^7} = \sqrt[3]{125 \cdot x^6 \cdot x}$ Factor using the greatest perfect cube factors.

$= \sqrt[3]{125} \cdot \sqrt[3]{x^6} \cdot \sqrt[3]{x}$ Product Property of Cube Roots

$= 5x^2\sqrt[3]{x}$ Simplify.

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Simplify the expression.

5. $\sqrt[3]{54}$

6. $\sqrt[3]{486}$

7. $\sqrt[3]{-256}$

8. $\sqrt[3]{432}$

9. $\sqrt[3]{-162}$

10. $\sqrt[3]{16x^4}$

Performing Operations with Radicals

Radicals with the same index and radicand are called **like radicals**. You can add and subtract like radicals the same way you combine like terms by using the Distributive Property.

STUDY TIP

Do not assume that radicals with different radicands cannot be added or subtracted. Always check to see whether you can simplify the radicals. In some cases, the radicals will become like radicals.

EXAMPLE 3 Adding and Subtracting Radicals

a. $5\sqrt{7} + \sqrt{11} - 8\sqrt{7} = 5\sqrt{7} - 8\sqrt{7} + \sqrt{11}$ Commutative Property of Addition

$= (5 - 8)\sqrt{7} + \sqrt{11}$ Distributive Property

$= -3\sqrt{7} + \sqrt{11}$ Subtract.

b. $10\sqrt{5} + \sqrt{20} = 10\sqrt{5} + \sqrt{4 \cdot 5}$ Factor using the greatest perfect square factor.

$= 10\sqrt{5} + \sqrt{4} \cdot \sqrt{5}$ Product Property of Square Roots

$= 10\sqrt{5} + 2\sqrt{5}$ Simplify.

$= (10 + 2)\sqrt{5}$ Distributive Property

$= 12\sqrt{5}$ Add.

c. $6\sqrt[3]{2} + 2\sqrt[3]{2} = (6 + 2)\sqrt[3]{2}$ Distributive Property

$= 8\sqrt[3]{2}$ Add.

You can multiply like or unlike radicals using the Product Properties, as long as they have the same index.

EXAMPLE 4 Multiplying Radicals

a. $\sqrt{50} \cdot \sqrt{24} = \sqrt{1200}$ Product Property of Square Roots

$= \sqrt{400 \cdot 3}$ Factor using the greatest perfect square factor.

$= \sqrt{400} \cdot \sqrt{3}$ Product Property of Square Roots

$= 20\sqrt{3}$ Simplify.

b. $\sqrt[3]{6} \cdot \sqrt[3]{18} = \sqrt[3]{108}$ Product Property of Cube Roots

$= \sqrt[3]{27 \cdot 4}$ Factor using the greatest perfect cube factor.

$= \sqrt[3]{27} \cdot \sqrt[3]{4}$ Product Property of Cube Roots

$= 3\sqrt[3]{4}$ Simplify.

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Simplify the expression.

11. $3\sqrt{2} - \sqrt{6} + 10\sqrt{2}$

12. $4\sqrt{7} - 6\sqrt{63}$

13. $4\sqrt[3]{5} - 11\sqrt[3]{5}$

14. $\sqrt{42} \cdot \sqrt{21}$

15. $\sqrt{15} \cdot \sqrt{45}$

16. $\sqrt[3]{44} \cdot \sqrt[3]{6}$

EXAMPLE 5 Multiplying Radicals

Simplify $\sqrt{5}\left(\sqrt{3}-\sqrt{75}\right)$.

SOLUTION

$$
\begin{aligned}
\sqrt{5}\left(\sqrt{3}-\sqrt{75}\right) &= \sqrt{5}\cdot\sqrt{3}-\sqrt{5}\cdot\sqrt{75} && \text{Distributive Property}\\
&= \sqrt{15}-\sqrt{375} && \text{Product Property of Square Roots}\\
&= \sqrt{15}-5\sqrt{15} && \text{Simplify.}\\
&= (1-5)\sqrt{15} && \text{Distributive Property}\\
&= -4\sqrt{15} && \text{Subtract.}
\end{aligned}
$$

EXAMPLE 6 Modeling with Mathematics

The ratio of the length to the width of a *golden rectangle* is $\left(1+\sqrt{5}\right):2$. The dimensions of the face of the Parthenon in Greece form a golden rectangle. What is the perimeter of the this golden rectangle?

SOLUTION

1. **Understand the Problem** Think of the length and height of the Parthenon as the length and width of a golden rectangle. The height of the rectangular face is 19 meters. You know the ratio of the length to the height. Find the length ℓ and then find the perimeter using the formula $P = 2\ell + 2w$.

2. **Make a Plan** Use the ratio $\left(1+\sqrt{5}\right):2$ to write a proportion and solve for ℓ.

3. **Solve the Problem**

$$
\begin{aligned}
\frac{1+\sqrt{5}}{2} &= \frac{\ell}{19} && \text{Write a proportion.}\\
2\ell &= 19\left(1+\sqrt{5}\right) && \text{Cross Products Property}\\
\ell &= \frac{19\left(1+\sqrt{5}\right)}{2} && \text{Divide each side by 2.}\\
\ell &= \frac{19}{2}+\frac{19}{2}\sqrt{5} && \text{Simplify.}
\end{aligned}
$$

The perimeter is $P = 2\left(\frac{19}{2}+\frac{19}{2}\sqrt{5}\right)+2(19)$

$$
\begin{aligned}
&= 19 + 19\sqrt{5} + 38\\
&= 57 + 19\sqrt{5}\\
&\approx 99.49.
\end{aligned}
$$

▶ The perimeter is about 99.5 meters.

4. **Look Back** The golden ratio is slightly more than 1.5 and the height is slightly less than 20, so the perimeter should be around $2(20) + 2(1.5)(20) = 100$. So, your answer is reasonable.

Monitoring Progress

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Simplify the expression.

17. $\sqrt{3}\left(8\sqrt{2}+7\sqrt{32}\right)$

18. $\sqrt[3]{-4}\left(\sqrt[3]{2}-\sqrt[3]{16}\right)$

19. The dimensions of a dance floor form a golden rectangle. The shorter side of the dance floor is 50 feet. What is the length of the longer side of the dance floor?

10.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** An example that proves a general statement is *not* true is called a ________________.

2. **VOCABULARY** List the conditions that must be met for a radical expression to be in simplest form.

3. **WRITING** Are the expressions $3\sqrt{2x}$ and $\sqrt{18x}$ equivalent? Explain your reasoning.

4. **WHICH ONE DOESN'T BELONG?** Which expression does *not* belong with the other three? Explain your reasoning.

$-\frac{1}{3}\sqrt{6}$	$6\sqrt{3}$	$\frac{1}{6}\sqrt{3}$	$-3\sqrt{3}$

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, determine whether the expression is in simplest form. If the expression is not in simplest form, explain why.

5. $\sqrt{19}$
6. $\sqrt{14}$
7. $\sqrt{48}$
8. $\sqrt{34}$
9. $\sqrt[3]{72}$
10. $\sqrt{475}$
11. $\sqrt{30} + 9$
12. $6 - \sqrt[3]{54}$

In Exercises 13–20, simplify the expression. *(See Example 1.)*

13. $\sqrt{20}$
14. $\sqrt{32}$
15. $\sqrt{128}$
16. $-\sqrt{72}$
17. $\sqrt{125b}$
18. $\sqrt{4x^2}$
19. $-\sqrt{81m^3}$
20. $\sqrt{48n^5}$

In Exercises 21–28, simplify the expression. *(See Example 2.)*

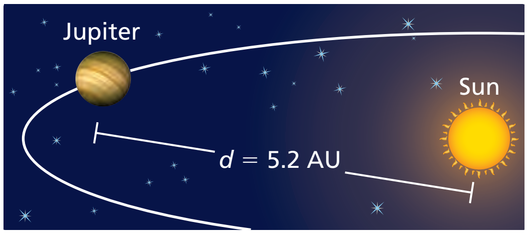
21. $\sqrt[3]{16}$
22. $\sqrt[3]{-108}$
23. $-\sqrt[3]{54}$
24. $\sqrt[3]{4000}$
25. $\sqrt[3]{-250}$
26. $\sqrt[3]{88}$
27. $\sqrt[3]{-64x^5}$
28. $-\sqrt[3]{343n^2}$

ERROR ANALYSIS **In Exercises 29 and 30, describe and correct the error in simplifying the expression.**

29.
$$\sqrt{72} = \sqrt{4 \cdot 18} = \sqrt{4} \cdot \sqrt{18} = 2\sqrt{18}$$

30.
$$\sqrt[3]{-144} = \sqrt[3]{-8 \cdot 9 \cdot 2} = \sqrt[3]{-8} \cdot \sqrt[3]{9} \cdot \sqrt[3]{2} = -6\sqrt[3]{2}$$

31. **MODELING WITH MATHEMATICS** The orbital period of a planet is the time it takes the planet to travel around the Sun. You can find the orbital period P (in Earth years) using the formula $P = \sqrt{d^3}$, where d is the average distance (in astronomical units, abbreviated AU) of the planet from the Sun.



a. Simplify the formula.

b. What is Jupiter's orbital period?

32. **MODELING WITH MATHEMATICS** Research the distance between another planet and the Sun. Repeat part (b) of Exercise 31.

In Exercises 33–36, evaluate the function for the given value of x. Write your answer in simplest form and in decimal form rounded to the nearest hundredth.

33. $h(x) = \sqrt{5x};\ x = 10$ **34.** $g(x) = \sqrt{3x};\ x = 60$

35. $r(x) = \sqrt{2}\left(1 + \sqrt{x}\right);\ x = 6$

36. $p(x) = \sqrt{7x} + \sqrt{5};\ x = 35$

In Exercises 37–40, evaluate the expression when $a = -2$, $b = 8$, and $c = \frac{1}{2}$. Write your answer in simplest form and in decimal form rounded to the nearest hundredth.

37. $\sqrt{a^2 + bc}$ **38.** $-\sqrt{4c - 6ab}$

39. $-\sqrt{2a^2 + b^2}$ **40.** $\sqrt{b^2 - 4ac}$

In Exercises 41–48, simplify the expression. *(See Example 3.)*

41. $\sqrt{3} - 2\sqrt{2} + 6\sqrt{2}$ **42.** $\sqrt{5} - 5\sqrt{13} - 8\sqrt{5}$

43. $2\sqrt{6} - 5\sqrt{54}$ **44.** $9\sqrt{32} + \sqrt{2}$

45. $\sqrt{12} + 6\sqrt{3} + 2\sqrt{6}$ **46.** $3\sqrt{7} - 5\sqrt{14} + 2\sqrt{28}$

47. $\sqrt[3]{-81} + 4\sqrt[3]{3}$ **48.** $6\sqrt[3]{128t} - 2\sqrt[3]{2t}$

In Exercises 49–56, simplify the expression. *(See Example 4.)*

49. $\sqrt{5} \cdot \sqrt{245}$ **50.** $\sqrt{8} \cdot \sqrt{32}$

51. $\sqrt{12} \cdot \sqrt{24}$ **52.** $\sqrt{39} \cdot \sqrt{3}$

53. $\sqrt[3]{4} \cdot \sqrt[3]{18}$ **54.** $\sqrt[3]{75} \cdot \sqrt[3]{20}$

55. $\sqrt[3]{81} \cdot \sqrt[3]{3} \cdot \sqrt[3]{8}$ **56.** $\sqrt[3]{49} \cdot \sqrt[3]{14} \cdot \sqrt[3]{2}$

In Exercises 57–64, simplify the expression. *(See Example 5.)*

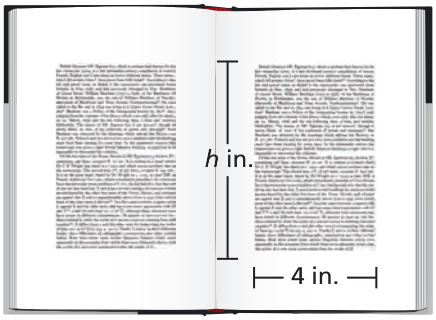
57. $\sqrt{2}\left(\sqrt{45} + \sqrt{5}\right)$ **58.** $\sqrt{3}\left(\sqrt{72} - 3\sqrt{2}\right)$

59. $\sqrt{5}\left(2\sqrt{6x} - \sqrt{96x}\right)$ **60.** $\sqrt{7y}\left(\sqrt{27y} + 5\sqrt{12y}\right)$

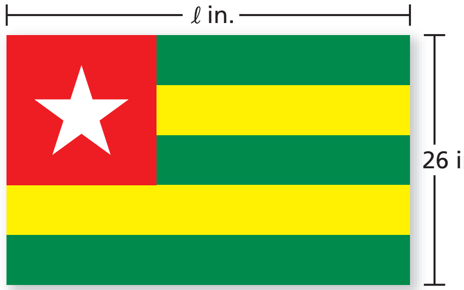
61. $\left(4\sqrt{2} - \sqrt{98}\right)^2$ **62.** $\left(\sqrt{3} + \sqrt{48}\right)\left(\sqrt{20} - \sqrt{5}\right)$

63. $\sqrt[3]{3}\left(\sqrt[3]{4} + \sqrt[3]{32}\right)$ **64.** $\sqrt[3]{2}\left(\sqrt[3]{135} - 4\sqrt[3]{5}\right)$

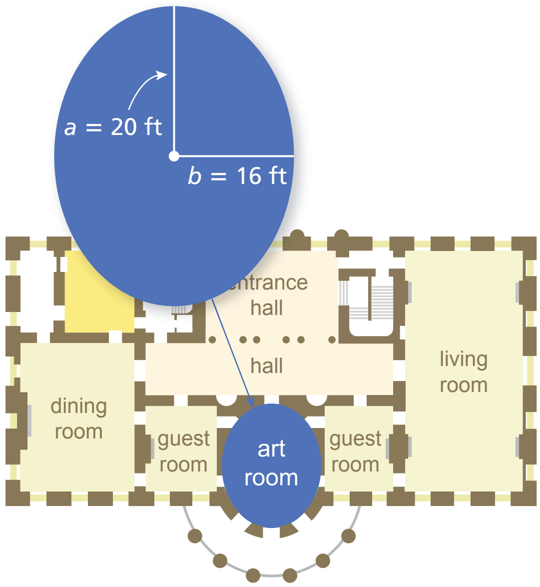
65. **MODELING WITH MATHEMATICS** The text in the book shown forms a golden rectangle. What is the height h of the text? *(See Example 6.)*



66. **MODELING WITH MATHEMATICS** The flag of Togo is approximately the shape of a golden rectangle. What is the length ℓ of the flag?



67. **MODELING WITH MATHEMATICS** The circumference C of the art room in a mansion is approximated by the formula $C \approx 2\pi\sqrt{\frac{a^2 + b^2}{2}}$. Approximate the circumference of the room.



68. **REASONING** Let m be a positive integer. For what values of m will the simplified form of the expression $\sqrt{2^m}$ contain a radical? For what values will it *not* contain a radical? Explain.

REASONING In Exercises 69 and 70, use the table shown.

	2	$\frac{1}{4}$	0	$\sqrt{3}$	$-\sqrt{3}$	π
2						
$\frac{1}{4}$						
0						
$\sqrt{3}$						
$-\sqrt{3}$						
π						

69. Copy and complete the table by (a) finding each sum $\left(2 + 2, 2 + \frac{1}{4}, \text{etc.}\right)$ and (b) finding each product $\left(2 \cdot 2, 2 \cdot \frac{1}{4}, \text{etc.}\right)$.

70. Use your answers in Exercise 69 to determine whether each statement is *always*, *sometimes*, or *never* true. Justify your answer.

a. The sum of a rational number and a rational number is rational.

b. The sum of a rational number and an irrational number is irrational.

c. The sum of an irrational number and an irrational number is irrational.

d. The product of a rational number and a rational number is rational.

e. The product of a nonzero rational number and an irrational number is irrational.

f. The product of an irrational number and an irrational number is irrational.

In Exercises 71–74, simplify the expression.

71. $\sqrt[4]{256y}$

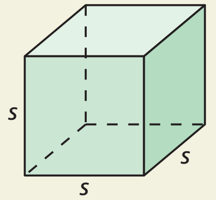
72. $\sqrt[5]{160x^6}$

73. $6\sqrt[4]{9} - \sqrt[5]{9} + 3\sqrt[4]{9}$

74. $\sqrt[5]{2}\left(\sqrt[4]{7} + \sqrt[5]{16}\right)$

75. **REASONING** Let a and b be positive numbers. Explain why $\sqrt{ab}$ lies between a and b on a number line. (*Hint:* Let $a < b$ and multiply each side of $a < b$ by a. Then let $a < b$ and multiply each side by b.)

76. **HOW DO YOU SEE IT?** The edge length s of a cube is an irrational number, the surface area is an irrational number, and the volume is a rational number. Give a possible value of s.



77. **THOUGHT PROVOKING** The ratio of consecutive terms in the Fibonacci sequence gets closer and closer to the golden ratio $\frac{1 + \sqrt{5}}{2}$ as the terms increase. Find the term that follows 610 in the sequence.

78. **CRITICAL THINKING** Determine whether each expression represents a *rational* or an *irrational* number. Justify your answer.

a. $4 + \sqrt{6}$

b. $\frac{\sqrt{48}}{\sqrt{3}}$

c. $\frac{8}{\sqrt{12}}$

d. $\sqrt{3} + \sqrt{7}$

e. $\frac{a}{\sqrt{10} - \sqrt{2}}$, where a is a positive integer

f. $\frac{2 + \sqrt{5}}{2b + \sqrt{5b^2}}$, where b is a positive integer

79. **CRITICAL THINKING** Use the special product pattern $a^2 - b^2 = (a + b)(a - b)$ to write the expression $x - 9$ as a product of two factors. Verify your answer using multiplication.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Graph the linear equation. Identify the x-intercept.

80. $y = x - 4$

81. $y = -2x + 6$

82. $y = -\frac{1}{3}x - 1$

83. $y = \frac{3}{2}x + 6$