11.1 Circumference and Arc Length

Essential Question

How can you find the length of a circular arc?

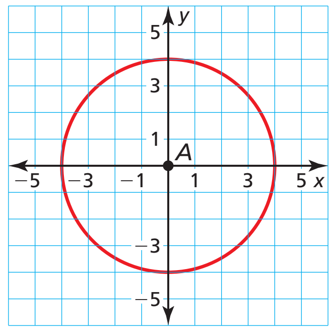
TEXAS ESSENTIAL KNOWLEDGE AND SKILLS

G.12.B
G.12.D

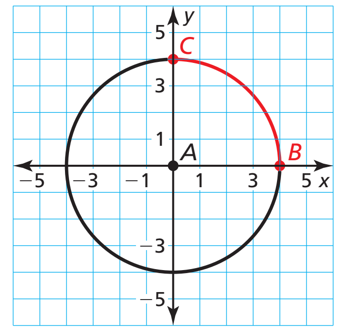
EXPLORATION 1 Finding the Length of a Circular Arc

Work with a partner. Find the length of each red circular arc.

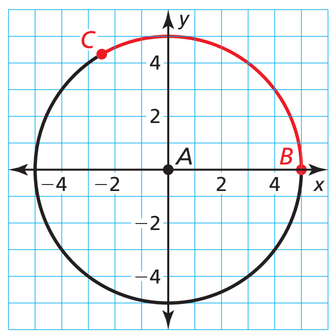
a. entire circle



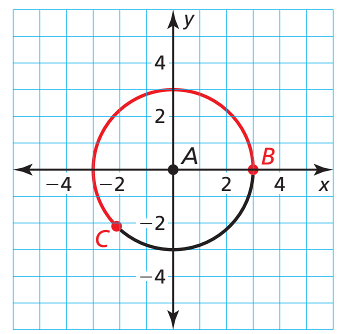
b. one-fourth of a circle



c. one-third of a circle

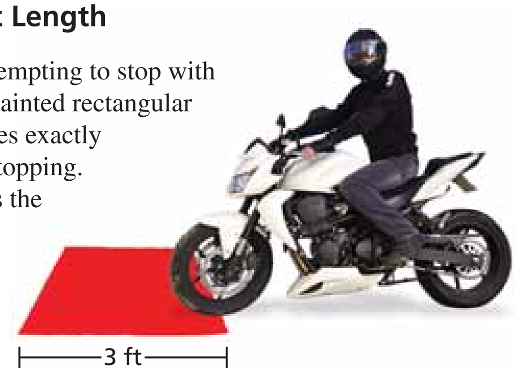


d. five-eighths of a circle



EXPLORATION 2 Using Arc Length

Work with a partner. The rider is attempting to stop with the front tire of the motorcycle in the painted rectangular box for a skills test. The front tire makes exactly one-half additional revolution before stopping. The diameter of the tire is 25 inches. Is the front tire still in contact with the painted box? Explain.



ANALYZING MATHEMATICAL RELATIONSHIPS

To be proficient in math, you need to notice if calculations are repeated and look both for general methods and for shortcuts.

Communicate Your Answer

3. How can you find the length of a circular arc?

4. A motorcycle tire has a diameter of 24 inches. Approximately how many inches does the motorcycle travel when its front tire makes three-fourths of a revolution?

11.1 Lesson

Core Vocabulary

circumference, *p. 598*
arc length, *p. 599*
radian, *p. 601*

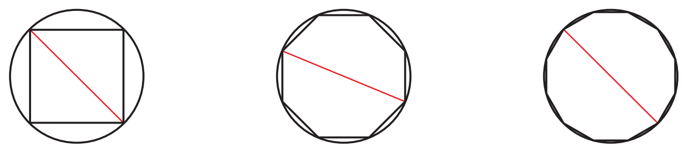
Previous
circle
diameter
radius

What You Will Learn

- Use the formula for circumference.
- Use arc lengths to find measures.
- Solve real-life problems.
- Measure angles in radians.

Using the Formula for Circumference

The **circumference** of a circle is the distance around the circle. Consider a regular polygon inscribed in a circle. As the number of sides increases, the polygon approximates the circle and the ratio of the perimeter of the polygon to the diameter of the circle approaches $\pi \approx 3.14159. . ..$

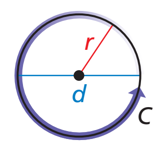


For all circles, the ratio of the circumference C to the diameter d is the same. This ratio is $\frac{C}{d} = \pi$. Solving for C yields the formula for the circumference of a circle, $C = \pi d$. Because $d = 2r$, you can also write the formula as $C = \pi(2r) = 2\pi r$.

Core Concept

Circumference of a Circle

The circumference C of a circle is $C = \pi d$ or $C = 2\pi r$, where d is the diameter of the circle and r is the radius of the circle.



$C = \pi d = 2\pi r$

USING PRECISE MATHEMATICAL LANGUAGE

You have sometimes used 3.14 to approximate the value of π. Throughout this book, you should use the π key on a calculator, then round to the hundredths place unless instructed otherwise.

EXAMPLE 1 Using the Formula for Circumference

Find each indicated measure.

a. circumference of a circle with a radius of 9 centimeters

b. radius of a circle with a circumference of 26 meters

SOLUTION

a. $C = 2\pi r$

$= 2 \cdot \pi \cdot 9$

$= 18\pi$

≈ 56.55

- The circumference is about 56.55 centimeters.

b. $C = 2\pi r$

$26 = 2\pi r$

$\frac{26}{2\pi} = r$

$4.14 \approx r$

- The radius is about 4.14 meters.

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1. Find the circumference of a circle with a diameter of 5 inches.

2. Find the diameter of a circle with a circumference of 17 feet.

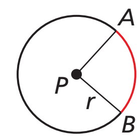
Using Arc Lengths to Find Measures

An **arc length** is a portion of the circumference of a circle. You can use the measure of the arc (in degrees) to find its length (in linear units).

Core Concept

Arc Length

In a circle, the ratio of the length of a given arc to the circumference is equal to the ratio of the measure of the arc to 360°.



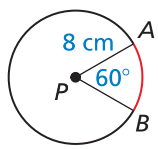
$$\frac{\text{Arc length of } \widehat{AB}}{2\pi r} = \frac{m\widehat{AB}}{360^\circ}, \text{ or}$$

$$\text{Arc length of } \widehat{AB} = \frac{m\widehat{AB}}{360^\circ} \cdot 2\pi r$$

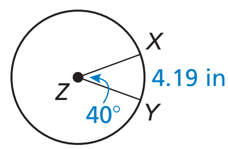
EXAMPLE 2 Using Arc Lengths to Find Measures

Find each indicated measure.

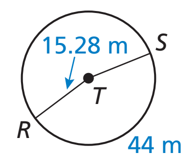
a. arc length of $\widehat{AB}$



b. circumference of $\odot Z$



c. $m\widehat{RS}$



SOLUTION

a. $\text{Arc length of } \widehat{AB} = \frac{60^\circ}{360^\circ} \cdot 2\pi(8)$

$\approx 8.38 \text{ cm}$

b. $\frac{\text{Arc length of } \widehat{XY}}{C} = \frac{m\widehat{XY}}{360^\circ}$

$\frac{4.19}{C} = \frac{40^\circ}{360^\circ}$

$\frac{4.19}{C} = \frac{1}{9}$

$37.71 \text{ in.} = C$

c. $\frac{\text{Arc length of } \widehat{RS}}{2\pi r} = \frac{m\widehat{RS}}{360^\circ}$

$\frac{44}{2\pi(15.28)} = \frac{m\widehat{RS}}{360^\circ}$

$360^\circ \cdot \frac{44}{2\pi(15.28)} = m\widehat{RS}$

$165^\circ \approx m\widehat{RS}$

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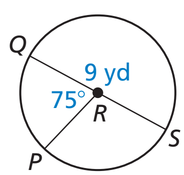
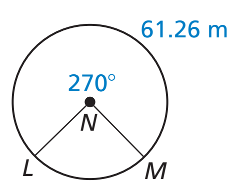
Find the indicated measure.

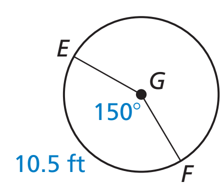
3. arc length of $\widehat{PQ}$



4. circumference of $\odot N$



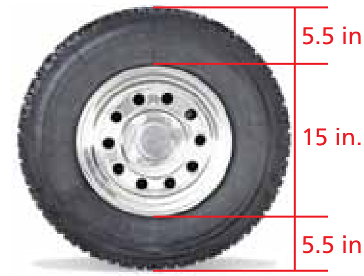
5. radius of $\odot G$



Solving Real-Life Problems

EXAMPLE 3 Using Circumference to Find Distance Traveled

The dimensions of a car tire are shown. To the nearest foot, how far does the tire travel when it makes 15 revolutions?



SOLUTION

Step 1 Find the diameter of the tire.

$$d = 15 + 2(5.5) = 26 \text{ in.}$$

Step 2 Find the circumference of the tire.

$$C = \pi d = \pi \cdot 26 = 26\pi \text{ in.}$$

Step 3 Find the distance the tire travels in 15 revolutions. In one revolution, the tire travels a distance equal to its circumference. In 15 revolutions, the tire travels a distance equal to 15 times its circumference.

Distance traveled = Number of revolutions • Circumference

$$= 15 \cdot 26\pi \approx 1225.2 \text{ in.}$$

Step 4 Use unit analysis. Change 1225.2 inches to feet.

$$1225.2 \cancel{\text{in.}} \cdot \frac{1 \text{ ft}}{12 \cancel{\text{in.}}} = 102.1 \text{ ft}$$

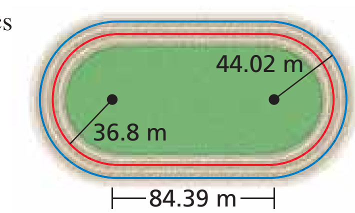
▶ The tire travels approximately 102 feet.

COMMON ERROR

Always pay attention to units. In Example 3, you need to convert units to get a correct answer.

EXAMPLE 4 Using Arc Length to Find Distances

The curves at the ends of the track shown are 180° arcs of circles. The radius of the arc for a runner on the red path shown is 36.8 meters. About how far does this runner travel to go once around the track? Round to the nearest tenth of a meter.



SOLUTION

The path of the runner on the red path is made of two straight sections and two semicircles. To find the total distance, find the sum of the lengths of each part.

Distance = 2 • Length of each straight section + 2 • Length of each semicircle

$$= 2(84.39) + 2\left(\tfrac{1}{2} \cdot 2\pi \cdot 36.8\right)$$

$$\approx 400.0$$

▶ The runner on the red path travels about 400.0 meters.

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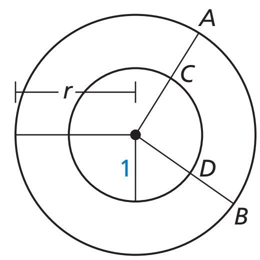
6. A car tire has a diameter of 28 inches. How many revolutions does the tire make while traveling 500 feet?

7. In Example 4, the radius of the arc for a runner on the blue path is 44.02 meters, as shown in the diagram. About how far does this runner travel to go once around the track? Round to the nearest tenth of a meter.

Measuring Angles in Radians

Recall that in a circle, the ratio of the length of a given arc to the circumference is equal to the ratio of the measure of the arc to 360°. To see why, consider the diagram.



A circle of radius 1 has circumference 2π, so the arc length of $\overparen{CD}$ is $\frac{m\overparen{CD}}{360^\circ} \cdot 2\pi$.

Recall that all circles are similar and corresponding lengths of similar figures are proportional. Because $m\overparen{AB} = m\overparen{CD}$, $\overparen{AB}$ and $\overparen{CD}$ are corresponding arcs. So, you can write the following proportion.

$$\frac{\text{Arc length of } \overparen{AB}}{\text{Arc length of } \overparen{CD}} = \frac{r}{1}$$

$$\text{Arc length of } \overparen{AB} = r \cdot \text{Arc length of } \overparen{CD}$$

$$\text{Arc length of } \overparen{AB} = r \cdot \frac{m\overparen{CD}}{360^\circ} \cdot 2\pi$$

This form of the equation shows that the arc length associated with a central angle is *proportional to the radius* of the circle. The constant of proportionality, $\frac{m\overparen{CD}}{360^\circ} \cdot 2\pi$, is defined to be the **radian** measure of the central angle associated with the arc.

In a circle of radius 1, the radian measure of a given central angle can be thought of as the length of the arc associated with the angle. The radian measure of a complete circle (360°) is exactly 2π radians, because the circumference of a circle of radius 1 is exactly 2π. You can use this fact to convert from degree measure to radian measure and vice versa.

Core Concept

Converting between Degrees and Radians

Degrees to radians

Multiply degree measure by

$$\frac{2\pi \text{ radians}}{360^\circ}, \text{ or } \frac{\pi \text{ radians}}{180^\circ}.$$

Radians to degrees

Multiply radian measure by

$$\frac{360^\circ}{2\pi \text{ radians}}, \text{ or } \frac{180^\circ}{\pi \text{ radians}}.$$

EXAMPLE 5 Converting between Degree and Radian Measure

a. Convert 45° to radians.

b. Convert $\frac{3\pi}{2}$ radians to degrees.

SOLUTION

a. $45^\circ \cdot \frac{\pi \text{ radians}}{180^\circ} = \frac{\pi}{4}$ radian

▶ So, $45^\circ = \frac{\pi}{4}$ radian.

b. $\frac{3\pi}{2}$ radians $\cdot \frac{180^\circ}{\pi \text{ radians}} = 270^\circ$

▶ So, $\frac{3\pi}{2}$ radians $= 270^\circ$.

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8. Convert 15° to radians.

9. Convert $\frac{4\pi}{3}$ radians to degrees.

11.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Describe the difference between an arc measure and an arc length.

2. **WHICH ONE DOESN'T BELONG?** Which phrase does *not* belong with the other three? Explain your reasoning.

the distance around a circle	π times twice the radius	π times the diameter

the distance from the center to any point on the circle

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, find the indicated measure. *(See Examples 1 and 2.)*

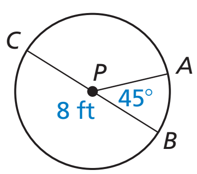
3. circumference of a circle with a radius of 6 inches

4. diameter of a circle with a circumference of 63 feet

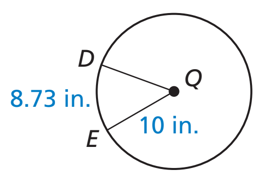
5. radius of a circle with a circumference of 28π

6. exact circumference of a circle with a diameter of 5 inches

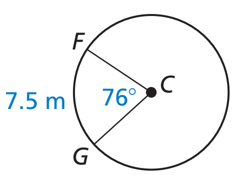
7. arc length of $\overset{\frown}{AB}$



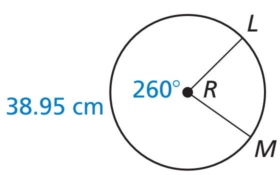
8. $m\overset{\frown}{DE}$



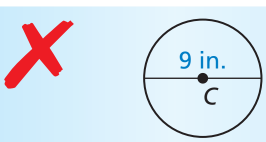
9. circumference of $\odot C$



10. radius of $\odot R$

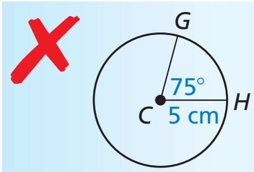


11. **ERROR ANALYSIS** Describe and correct the error in finding the circumference of $\odot C$.



$C = 2\pi r$
$= 2\pi(9)$
$= 18\pi$ in.

12. **ERROR ANALYSIS** Describe and correct the error in finding the length of $\overset{\frown}{GH}$.



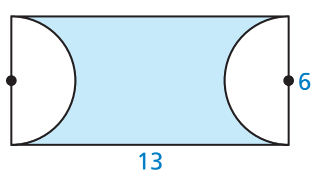
Arc length of $\overset{\frown}{GH}$
$= m\overset{\frown}{GH} \cdot 2\pi r$
$= 75 \cdot 2\pi(5)$
$= 750\pi$ cm

13. **PROBLEM SOLVING** A measuring wheel is used to calculate the length of a path. The diameter of the wheel is 8 inches. The wheel makes 87 complete revolutions along the length of the path. To the nearest foot, how long is the path? *(See Example 3.)*

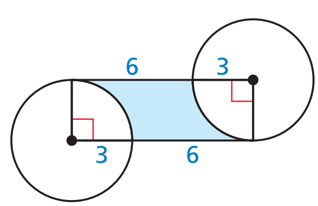
14. **PROBLEM SOLVING** The radius of the front wheel of your bicycle is 32.5 centimeters. You ride 40 meters. How many complete revolutions does the front wheel make?

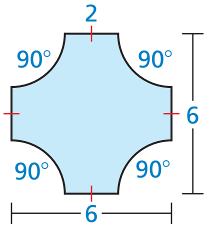
In Exercises 15–18, find the perimeter of the shaded region. *(See Example 4.)*

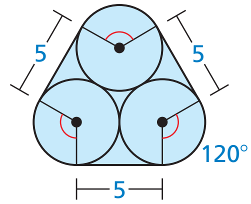
15.



16.



17. 

18. 

In Exercises 19–22, convert the angle measure. *(See Example 5.)*

19. Convert 70° to radians.

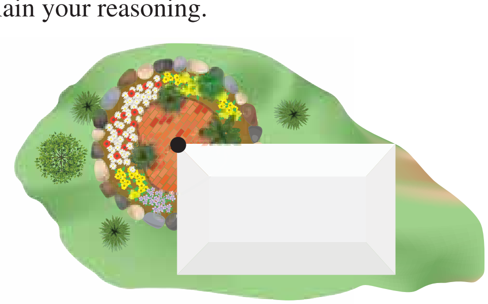
20. Convert 300° to radians.

21. Convert $\frac{11\pi}{12}$ radians to degrees.

22. Convert $\frac{\pi}{8}$ radian to degrees.

23. **PROBLEM SOLVING** The London Eye is a Ferris wheel in London, England, that travels at a speed of 0.26 meter per second. How many minutes does it take the London Eye to complete one full revolution?

24. **PROBLEM SOLVING** You are planning to plant a circular garden adjacent to one of the corners of a building, as shown. You can use up to 38 feet of fence to make a border around the garden. What radius (in feet) can the garden have? Choose all that apply. Explain your reasoning.



(A) 7 (B) 8 (C) 9 (D) 10

In Exercises 25 and 26, find the circumference of the circle with the given equation. Write the circumference in terms of π.

25. $x^2 + y^2 = 16$

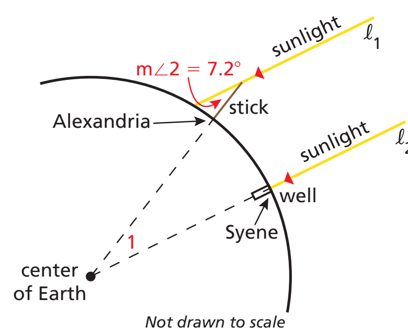
26. $(x + 2)^2 + (y - 3)^2 = 9$

27. **USING STRUCTURE** A semicircle has endpoints $(-2, 5)$ and $(2, 8)$. Find the arc length of the semicircle.

28. **REASONING** $\widehat{EF}$ is an arc on a circle with radius r. Let $x°$ be the measure of $\widehat{EF}$. Describe the effect on the length of $\widehat{EF}$ if you (a) double the radius of the circle, and (b) double the measure of $\widehat{EF}$.

29. **MAKING AN ARGUMENT** Your friend claims that it is possible for two arcs with the same measure to have different arc lengths. Is your friend correct? Explain your reasoning.

30. **PROBLEM SOLVING** Over 2000 years ago, the Greek scholar Eratosthenes estimated Earth's circumference by assuming that the Sun's rays were parallel. He chose a day when the Sun shone straight down into a well in the city of Syene. At noon, he measured the angle the Sun's rays made with a vertical stick in the city of Alexandria. Eratosthenes assumed that the distance from Syene to Alexandria was equal to about 575 miles. Explain how Eratosthenes was able to use this information to estimate Earth's circumference. Then estimate Earth's circumference.



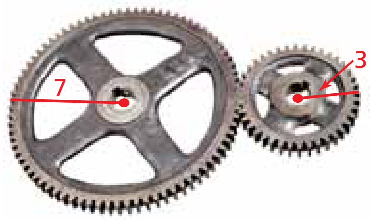
31. **ANALYZING RELATIONSHIPS** In $\odot C$, the ratio of the length of $\widehat{PQ}$ to the length of $\widehat{RS}$ is 2 to 1. What is the ratio of $m\angle PCQ$ to $m\angle RCS$?

(A) 4 to 1 (B) 2 to 1

(C) 1 to 4 (D) 1 to 2

32. **ANALYZING RELATIONSHIPS** A 45° arc in $\odot C$ and a 30° arc in $\odot P$ have the same length. What is the ratio of the radius r_1 of $\odot C$ to the radius r_2 of $\odot P$? Explain your reasoning.

33. **PROBLEM SOLVING** How many revolutions does the smaller gear complete during a single revolution of the larger gear?



34. **USING STRUCTURE** Find the circumference of each circle.

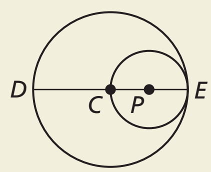
 a. a circle circumscribed about a right triangle whose legs are 12 inches and 16 inches long

 b. a circle circumscribed about a square with a side length of 6 centimeters

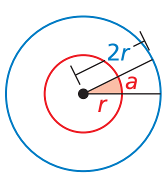
 c. a circle inscribed in an equilateral triangle with a side length of 9 inches

35. **REWRITING A FORMULA** Write a formula in terms of the measure θ (theta) of the central angle (in radians) that can be used to find the length of an arc of a circle. Then use this formula to find the length of an arc of a circle with a radius of 4 inches and a central angle of $\frac{3\pi}{4}$ radians.

36. **HOW DO YOU SEE IT?** Compare the circumference of $\odot P$ to the length of $\widehat{DE}$. Explain your reasoning.



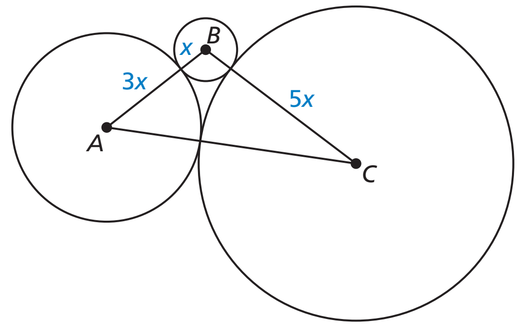
37. **MAKING AN ARGUMENT** In the diagram, the measure of the red shaded angle is 30°. The arc length a is 2. Your classmate claims that it is possible to find the circumference of the blue circle without finding the radius of either circle. Is your classmate correct? Explain your reasoning.



38. **MODELING WITH MATHEMATICS** What is the measure (in radians) of the angle formed by the hands of a clock at each time? Explain your reasoning.

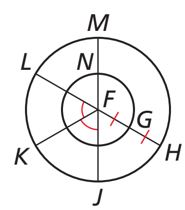
 a. 1:30 P.M. b. 3:15 P.M.

39. **MATHEMATICAL CONNECTIONS** The sum of the circumferences of circles A, B, and C is 63π. Find AC.

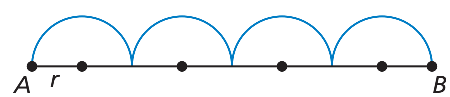


40. **THOUGHT PROVOKING** Is π a rational number? Compare the rational number $\frac{355}{113}$ to π. Find a different rational number that is even closer to π.

41. **PROOF** The circles in the diagram are concentric and $\overline{FG} \cong \overline{GH}$. Prove that $\widehat{JK}$ and $\widehat{NG}$ have the same length.



42. **REPEATED REASONING** $\overline{AB}$ is divided into four congruent segments, and semicircles with radius r are drawn.



 a. What is the sum of the four arc lengths?

 b. What would the sum of the arc lengths be if $\overline{AB}$ was divided into 8 congruent segments? 16 congruent segments? n congruent segments? Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the area of the polygon with the given vertices. *(Section 1.4)*

43. $X(2, 4)$, $Y(8, -1)$, $Z(2, -1)$

44. $L(-3, 1)$, $M(4, 1)$, $N(4, -5)$, $P(-3, -5)$