5.6 Proving Triangle Congruence by ASA and AAS

Texas Essential Knowledge and Skills
G.5.A
G.6.B

Essential Question

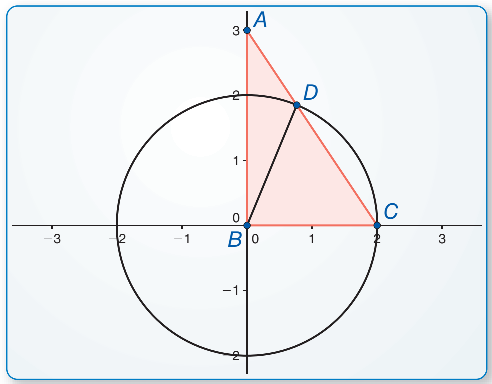
What information is sufficient to determine whether two triangles are congruent?

EXPLORATION 1 Determining Whether SSA Is Sufficient

Work with a partner.

a. Use dynamic geometry software to construct $\triangle ABC$. Construct the triangle so that vertex B is at the origin, $\overline{AB}$ has a length of 3 units, and $\overline{BC}$ has a length of 2 units.

b. Construct a circle with a radius of 2 units centered at the origin. Locate point D where the circle intersects $\overline{AC}$. Draw $\overline{BD}$.



Sample

Points
$A(0, 3)$
$B(0, 0)$
$C(2, 0)$
$D(0.77, 1.85)$
Segments
$AB = 3$
$AC = 3.61$
$BC = 2$
$AD = 1.38$
Angle
$m\angle A = 33.69°$

c. $\triangle ABC$ and $\triangle ABD$ have two congruent sides and a nonincluded congruent angle. Name them.

d. Is $\triangle ABC \cong \triangle ABD$? Explain your reasoning.

e. Is SSA sufficient to determine whether two triangles are congruent? Explain your reasoning.

MAKING MATHEMATICAL ARGUMENTS

To be proficient in math, you need to recognize and use counterexamples.

EXPLORATION 2 Determining Valid Congruence Theorems

Work with a partner. Use dynamic geometry software to determine which of the following are valid triangle congruence theorems. For those that are not valid, write a counterexample. Explain your reasoning.

Possible Congruence Theorem	Valid or not valid?
SSS	
SSA	
SAS	
AAS	
ASA	
AAA	

Communicate Your Answer

3. What information is sufficient to determine whether two triangles are congruent?

4. Is it possible to show that two triangles are congruent using more than one congruence theorem? If so, give an example.

5.6 Lesson

What You Will Learn

- Use the ASA and AAS Congruence Theorems.

Core Vocabulary

Previous
congruent figures
rigid motion

Using the ASA and AAS Congruence Theorems

Theorem

Theorem 5.10 Angle-Side-Angle (ASA) Congruence Theorem

If two angles and the included side of one triangle are congruent to two angles and the included side of a second triangle, then the two triangles are congruent.

If $\angle A \cong \angle D$, $\overline{AC} \cong \overline{DF}$, and $\angle C \cong \angle F$, then $\triangle ABC \cong \triangle DEF$.

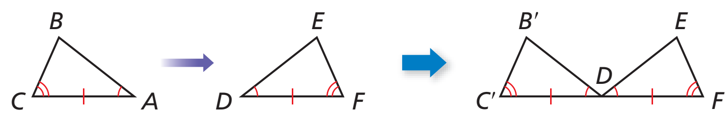
Proof p. 274

PROOF Angle-Side-Angle (ASA) Congruence Theorem

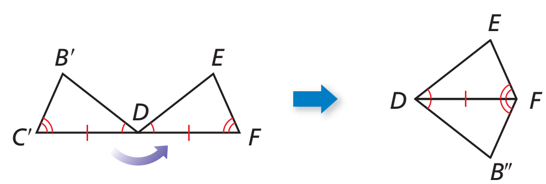
Given $\angle A \cong \angle D$, $\overline{AC} \cong \overline{DF}$, $\angle C \cong \angle F$

Prove $\triangle ABC \cong \triangle DEF$

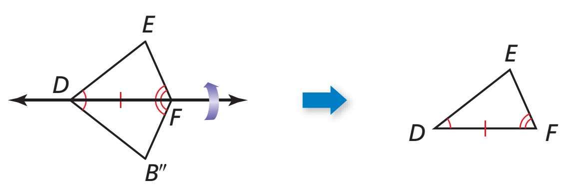
First, translate $\triangle ABC$ so that point A maps to point D, as shown below.



This translation maps $\triangle ABC$ to $\triangle DB'C'$. Next, rotate $\triangle DB'C'$ counterclockwise through $\angle C'DF$ so that the image of $\overrightarrow{DC'}$ coincides with $\overrightarrow{DF}$, as shown below.



Because $\overline{DC'} \cong \overline{DF}$, the rotation maps point C' to point F. So, this rotation maps $\triangle DB'C'$ to $\triangle DB''F$. Now, reflect $\triangle DB''F$ in the line through points D and F, as shown below.



Because points D and F lie on $\overleftrightarrow{DF}$, this reflection maps them onto themselves. Because a reflection preserves angle measure and $\angle B''DF \cong \angle EDF$, the reflection maps $\overrightarrow{DB''}$ to $\overrightarrow{DE}$. Similarly, because $\angle B''FD \cong \angle EFD$, the reflection maps $\overrightarrow{FB''}$ to $\overrightarrow{FE}$. The image of B'' lies on $\overrightarrow{DE}$ and $\overrightarrow{FE}$. Because $\overrightarrow{DE}$ and $\overrightarrow{FE}$ only have point E in common, the image of B'' must be E. So, this reflection maps $\triangle DB''F$ to $\triangle DEF$.

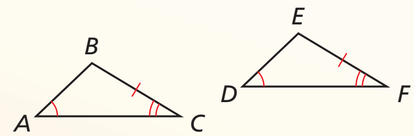
Because you can map $\triangle ABC$ to $\triangle DEF$ using a composition of rigid motions, $\triangle ABC \cong \triangle DEF$.

Theorem

Theorem 5.11 Angle-Angle-Side (AAS) Congruence Theorem

If two angles and a non-included side of one triangle are congruent to two angles and the corresponding non-included side of a second triangle, then the two triangles are congruent.

If $\angle A \cong \angle D$, $\angle C \cong \angle F$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.

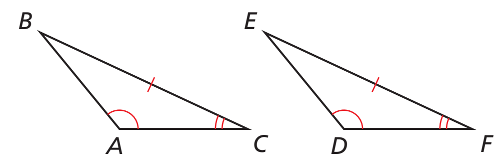


Proof p. 275

PROOF Angle-Angle-Side (AAS) Congruence Theorem

Given $\angle A \cong \angle D$,
$\angle C \cong \angle F$,
$\overline{BC} \cong \overline{EF}$

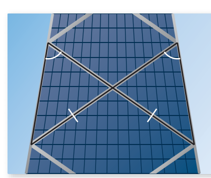
Prove $\triangle ABC \cong \triangle DEF$



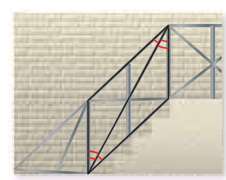
You are given $\angle A \cong \angle D$ and $\angle C \cong \angle F$. By the Third Angles Theorem (Theorem 5.4), $\angle B \cong \angle E$. You are given $\overline{BC} \cong \overline{EF}$. So, two pairs of angles and their included sides are congruent. By the ASA Congruence Theorem, $\triangle ABC \cong \triangle DEF$.

EXAMPLE 1 Identifying Congruent Triangles

Can the triangles be proven congruent with the information given in the diagram? If so, state the theorem you would use.

a. 

b. 

c. 

COMMON ERROR

You need at least one pair of congruent corresponding sides to prove two triangles are congruent.

SOLUTION

a. The vertical angles are congruent, so two pairs of angles and a pair of non-included sides are congruent. The triangles are congruent by the AAS Congruence Theorem.

b. There is not enough information to prove the triangles are congruent, because no sides are known to be congruent.

c. Two pairs of angles and their included sides are congruent. The triangles are congruent by the ASA Congruence Theorem.

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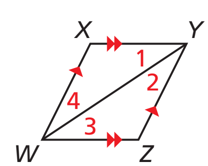
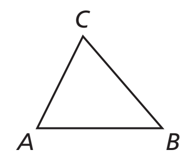
1. Can the triangles be proven congruent with the information given in the diagram? If so, state the theorem you would use.



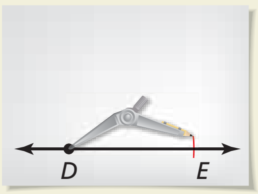
CONSTRUCTION Copying a Triangle Using ASA

Construct a triangle that is congruent to $\triangle ABC$ using the ASA Congruence Theorem. Use a compass and straightedge.



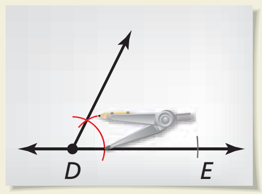
SOLUTION

Step 1



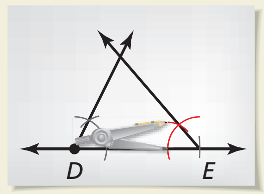
Construct a side Construct $\overline{DE}$ so that it is congruent to $\overline{AB}$.

Step 2



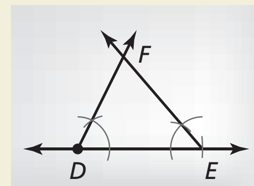
Construct an angle Construct $\angle D$ with vertex D and side $\overrightarrow{DE}$ so that it is congruent to $\angle A$.

Step 3



Construct an angle Construct $\angle E$ with vertex E and side $\overrightarrow{ED}$ so that it is congruent to $\angle B$.

Step 4



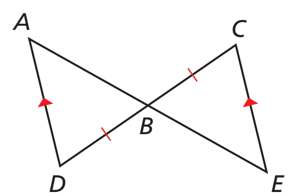
Label a point Label the intersection of the sides of $\angle D$ and $\angle E$ that you constructed in Steps 2 and 3 as F. By the ASA Congruence Theorem, $\triangle ABC \cong \triangle DEF$.

EXAMPLE 2 Using the ASA Congruence Theorem

Write a proof.

Given $\overline{AD} \parallel \overline{EC}$, $\overline{BD} \cong \overline{BC}$

Prove $\triangle ABD \cong \triangle EBC$



SOLUTION

STATEMENTS	REASONS
1. $\overline{AD} \parallel \overline{EC}$	**1.** Given
A **2.** $\angle D \cong \angle C$	**2.** Alternate Interior Angles Theorem (Thm. 3.2)
S **3.** $\overline{BD} \cong \overline{BC}$	**3.** Given
A **4.** $\angle ABD \cong \angle EBC$	**4.** Vertical Angles Congruence Theorem (Thm 2.6)
5. $\triangle ABD \cong \triangle EBC$	**5.** ASA Congruence Theorem

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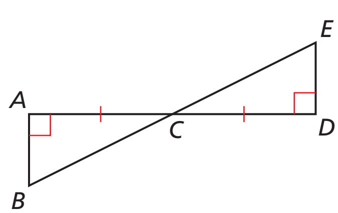
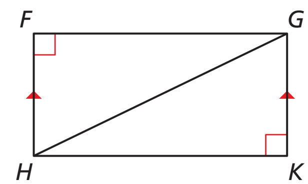
2. In the diagram, $\overline{AB} \perp \overline{AD}$, $\overline{DE} \perp \overline{AD}$, and $\overline{AC} \cong \overline{DC}$. Prove $\triangle ABC \cong \triangle DEC$.



EXAMPLE 3 Using the AAS Congruence Theorem

Write a proof.



Given $\overline{HF} \parallel \overline{GK}$, $\angle F$ and $\angle K$ are right angles.

Prove $\triangle HFG \cong \triangle GKH$

SOLUTION

STATEMENTS	REASONS
1. $\overline{HF} \parallel \overline{GK}$	**1.** Given
A **2.** $\angle GHF \cong \angle HGK$	**2.** Alternate Interior Angles Theorem (Theorem 3.2)
3. $\angle F$ and $\angle K$ are right angles.	**3.** Given
A **4.** $\angle F \cong \angle K$	**4.** Right Angles Congruence Theorem (Theorem 2.3)
S **5.** $\overline{HG} \cong \overline{GH}$	**5.** Reflexive Property of Congruence (Theorem 2.1)
6. $\triangle HFG \cong \angle GKH$	**6.** AAS Congruence Theorem

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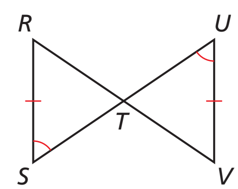
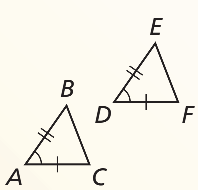
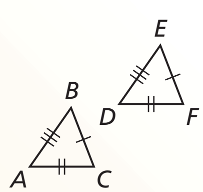
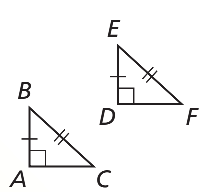
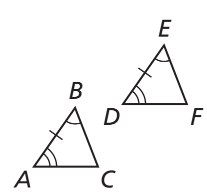
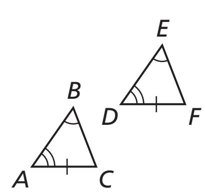
3. In the diagram, $\angle S \cong \angle U$ and $\overline{RS} \cong \overline{VU}$. Prove $\triangle RST \cong \triangle VUT$.



Concept Summary

Triangle Congruence Theorems

You have learned five methods for proving that triangles are congruent.

SAS	SSS	HL (right △s only)	ASA	AAS
				
Two sides and the included angle are congruent.	All three sides are congruent.	The hypotenuse and one of the legs are congruent.	Two angles and the included side are congruent.	Two angles and a non-included side are congruent.

In the Exercises, you will prove three additional theorems about the congruence of right triangles: **H**ypotenuse-**A**ngle, **L**eg-**L**eg, and **A**ngle-**L**eg.

5.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

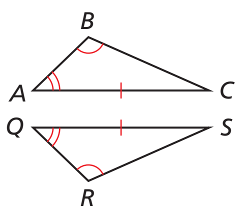
1. **WRITING** How are the AAS Congruence Theorem (Theorem 5.11) and the ASA Congruence Theorem (Theorem 5.10) similar? How are they different?

2. **WRITING** You know that a pair of triangles has two pairs of congruent corresponding angles. What other information do you need to show that the triangles are congruent?

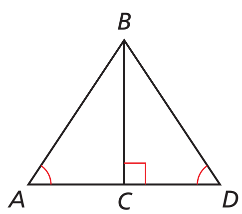
Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, decide whether enough information is given to prove that the triangles are congruent. If so, state the theorem you would use. *(See Example 1.)*

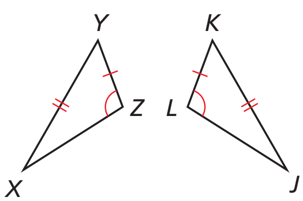
3. $\triangle ABC$, $\triangle QRS$



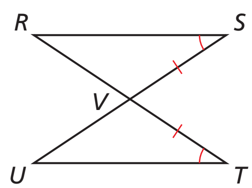
4. $\triangle ABC$, $\triangle DBC$



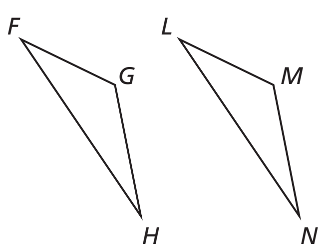
5. $\triangle XYZ$, $\triangle JKL$



6. $\triangle RSV$, $\triangle UTV$



In Exercises 7 and 8, state the third congruence statement that is needed to prove that $\triangle FGH \cong \triangle LMN$ using the given theorem.



7. **Given** $\overline{GH} \cong \overline{MN}$, $\angle G \cong \angle M$, ___ $\cong$ ____

   Use the AAS Congruence Theorem (Thm. 5.11).

8. **Given** $\overline{FG} \cong \overline{LM}$, $\angle G \cong \angle M$, ___ $\cong$ ____

   Use the ASA Congruence Theorem (Thm. 5.10).

In Exercises 9–12, decide whether you can use the given information to prove that $\triangle ABC \cong \triangle DEF$. Explain your reasoning.

9. $\angle A \cong \angle D$, $\angle C \cong \angle F$, $\overline{AC} \cong \overline{DF}$

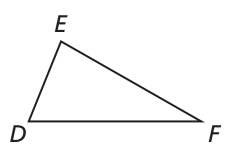
10. $\angle C \cong \angle F$, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$

11. $\angle B \cong \angle E$, $\angle C \cong \angle F$, $\overline{AC} \cong \overline{DE}$

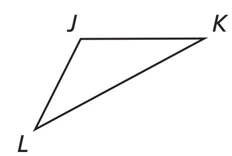
12. $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\overline{BC} \cong \overline{EF}$

CONSTRUCTION **In Exercises 13 and 14, construct a triangle that is congruent to the given triangle using the ASA Congruence Theorem (Theorem 5.10). Use a compass and straightedge.**

13.

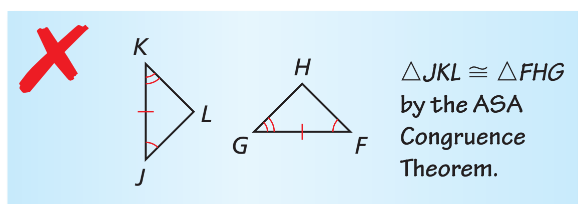


14.



ERROR ANALYSIS **In Exercises 15 and 16, describe and correct the error.**

15.



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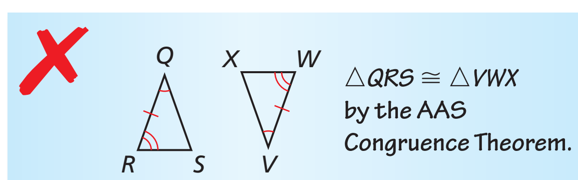
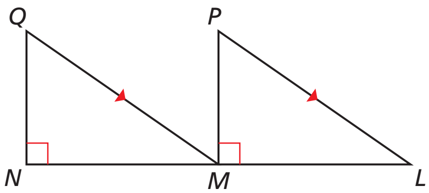


PROOF **In Exercises 17 and 18, prove that the triangles are congruent using the ASA Congruence Theorem (Theorem 5.10).** *(See Example 2.)*

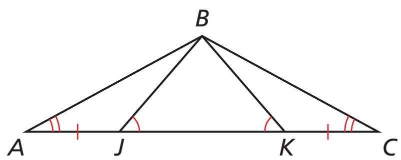
17. **Given** M is the midpoint of $\overline{NL}$.
$\overline{NL} \perp \overline{NQ}$, $\overline{NL} \perp \overline{MP}$, $\overline{QM} \parallel \overline{PL}$

Prove $\triangle NQM \cong \triangle MPL$



18. **Given** $\overline{AJ} \cong \overline{KC}$, $\angle BJK \cong \angle BKJ$, $\angle A \cong \angle C$

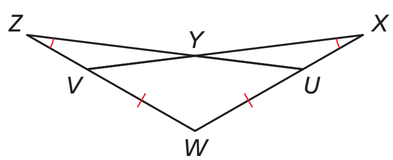
Prove $\triangle ABK \cong \triangle CBJ$



PROOF **In Exercises 19 and 20, prove that the triangles are congruent using the AAS Congruence Theorem (Theorem 5.11).** *(See Example 3.)*

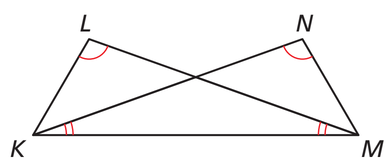
19. **Given** $\overline{VW} \cong \overline{UW}$, $\angle X \cong \angle Z$

Prove $\triangle XWV \cong \triangle ZWU$



20. **Given** $\angle NKM \cong \angle LMK$, $\angle L \cong \angle N$

Prove $\triangle NMK \cong \triangle LKM$



PROOF **In Exercises 21–23, write a paragraph proof for the theorem about right triangles.**

21. **Hypotenuse-Angle (HA) Congruence Theorem** If an angle and the hypotenuse of a right triangle are congruent to an angle and the hypotenuse of a second right triangle, then the triangles are congruent.

22. **Leg-Leg (LL) Congruence Theorem** If the legs of a right triangle are congruent to the legs of a second right triangle, then the triangles are congruent.

23. **Angle-Leg (AL) Congruence Theorem** If an angle and a leg of a right triangle are congruent to an angle and a leg of a second right triangle, then the triangles are congruent.

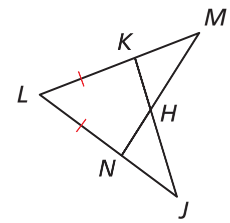
24. **REASONING** What additional information do you need to prove $\triangle JKL \cong \triangle MNL$ by the ASA Congruence Theorem (Theorem 5.10)?

Ⓐ $\overline{KM} \cong \overline{KJ}$

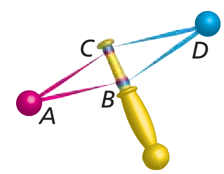
Ⓑ $\overline{KH} \cong \overline{NH}$

Ⓒ $\angle M \cong \angle J$

Ⓓ $\angle LKJ \cong \angle LNM$



25. **MATHEMATICAL CONNECTIONS** This toy contains $\triangle ABC$ and $\triangle DBC$. Can you conclude that $\triangle ABC \cong \triangle DBC$ from the given angle measures? Explain.



$m\angle ABC = (8x - 32)°$

$m\angle DBC = (4y - 24)°$

$m\angle BCA = (5x + 10)°$

$m\angle BCD = (3y + 2)°$

$m\angle CAB = (2x - 8)°$

$m\angle CDB = (y - 6)°$

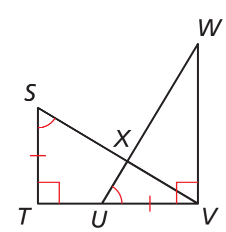
26. **REASONING** Which of the following congruence statements are true? Select all that apply.

Ⓐ $\overline{TU} \cong \overline{UV}$

Ⓑ $\triangle STV \cong \triangle XVW$

Ⓒ $\triangle TVS \cong \triangle VWU$

Ⓓ $\triangle VST \cong \triangle VUW$



27. **PROVING A THEOREM** Prove the Converse of the Base Angles Theorem (Theorem 5.7). (*Hint:* Draw an auxiliary line inside the triangle.)

28. **MAKING AN ARGUMENT** Your friend claims to be able to rewrite any proof that uses the AAS Congruence Theorem (Thm. 5.11) as a proof that uses the ASA Congruence Theorem (Thm. 5.10). Is this possible? Explain your reasoning.

29. **MODELING WITH MATHEMATICS** When a light ray from an object meets a mirror, it is reflected back to your eye. For example, in the diagram, a light ray from point C is reflected at point D and travels back to point A. The *law of reflection* states that the angle of incidence, $\angle CDB$, is congruent to the angle of reflection, $\angle ADB$.

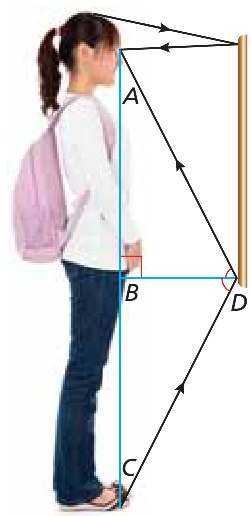
a. Prove that $\triangle ABD$ is congruent to $\triangle CBD$.

Given $\angle CDB \cong \angle ADB$, $\overline{DB} \perp \overline{AC}$

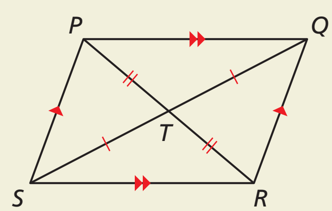
Prove $\triangle ABD \cong \triangle CBD$

b. Verify that $\triangle ACD$ is isosceles.

c. Does moving away from the mirror have any effect on the amount of his or her reflection a person sees? Explain.



30. **HOW DO YOU SEE IT?** Name as many pairs of congruent triangles as you can from the diagram. Explain how you know that each pair of triangles is congruent.



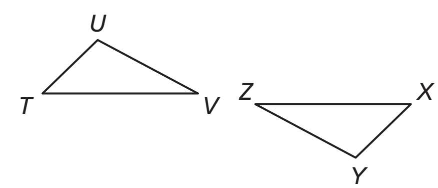
31. **CONSTRUCTION** Construct a triangle. Show that there is no AAA congruence rule by constructing a second triangle that has the same angle measures but is not congruent.

32. **THOUGHT PROVOKING** Graph theory is a branch of mathematics that studies vertices and the way they are connected. In graph theory, two polygons are *isomorphic* if there is a one-to-one mapping from one polygon's vertices to the other polygon's vertices that preserves adjacent vertices. In graph theory, are any two triangles isomorphic? Explain your reasoning.

33. **MATHEMATICAL CONNECTIONS** Six statements are given about $\triangle TUV$ and $\triangle XYZ$.

$\overline{TU} \cong \overline{XY}$ $\quad$ $\overline{UV} \cong \overline{YZ}$ $\quad$ $\overline{TV} \cong \overline{XZ}$

$\angle T \cong \angle X$ $\quad$ $\angle U \cong \angle Y$ $\quad$ $\angle V \cong \angle Z$



a. List all combinations of three given statements that would provide enough information to prove that $\triangle TUV$ is congruent to $\triangle XYZ$.

b. You choose three statements at random. What is the probability that the statements you choose provide enough information to prove that the triangles are congruent?

Maintaining Mathematical Proficiency

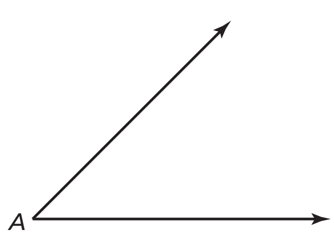
Reviewing what you learned in previous grades and lessons

Find the coordinates of the midpoint of the line segment with the given endpoints. *(Section 1.3)*

34. $C(1, 0)$ and $D(5, 4)$ **35.** $J(-2, 3)$ and $K(4, -1)$ **36.** $R(-5, -7)$ and $S(2, -4)$

Use a compass and straightedge to copy the angle. *(Section 1.5)*

37.



38.