5.4 Equilateral and Isosceles Triangles

Texas Essential Knowledge and Skills
G.5.C
G.6.D

Essential Question

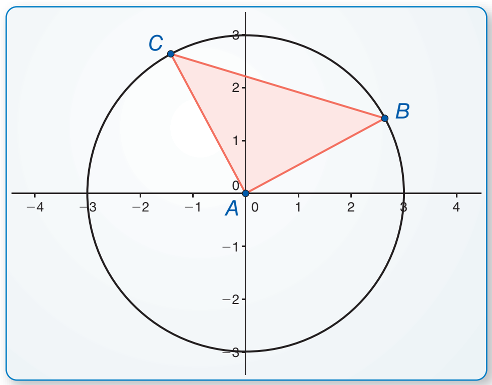
What conjectures can you make about the side lengths and angle measures of an isosceles triangle?

EXPLORATION 1 Writing a Conjecture about Isosceles Triangles

Work with a partner. Use dynamic geometry software.

a. Construct a circle with a radius of 3 units centered at the origin.

b. Construct $\triangle ABC$ so that B and C are on the circle and A is at the origin.



Sample

Points
$A(0, 0)$
$B(2.64, 1.42)$
$C(-1.42, 2.64)$
Segments
$AB = 3$
$AC = 3$
$BC = 4.24$
Angles
$m\angle A = 90°$
$m\angle B = 45°$
$m\angle C = 45°$

MAKING MATHEMATICAL ARGUMENTS

To be proficient in math, you need to make conjectures and build a logical progression of statements to explore the truth of your conjectures.

c. Recall that a triangle is *isosceles* if it has at least two congruent sides. Explain why $\triangle ABC$ is an isosceles triangle.

d. What do you observe about the angles of $\triangle ABC$?

e. Repeat parts (a)–(d) with several other isosceles triangles using circles of different radii. Keep track of your observations by copying and completing the table below. Then write a conjecture about the angle measures of an isosceles triangle.

		A	B	C	AB	AC	BC	$m\angle A$	$m\angle B$	$m\angle C$
Sample	**1.**	(0, 0)	(2.64, 1.42)	(−1.42, 2.64)	3	3	4.24	90°	45°	45°
	2.	(0, 0)								
	3.	(0, 0)								
	4.	(0, 0)								
	5.	(0, 0)								

f. Write the converse of the conjecture you wrote in part (e). Is the converse true?

Communicate Your Answer

2. What conjectures can you make about the side lengths and angle measures of an isosceles triangle?

3. How would you prove your conclusion in Exploration 1(e)? in Exploration 1(f)?

5.4 Lesson

Core Vocabulary

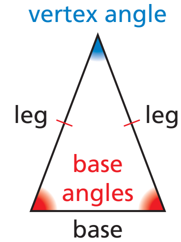
legs, *p. 256*
vertex angle, *p. 256*
base, *p. 256*
base angles, *p. 256*

What You Will Learn

- Use the Base Angles Theorem.
- Use isosceles and equilateral triangles.

Using the Base Angles Theorem

A triangle is isosceles when it has at least two congruent sides. When an isosceles triangle has exactly two congruent sides, these two sides are the **legs**. The angle formed by the legs is the **vertex angle**. The third side is the **base** of the isosceles triangle. The two angles adjacent to the base are called **base angles**.



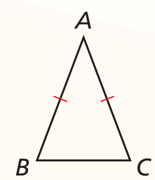
Theorems

Theorem 5.6 Base Angles Theorem

If two sides of a triangle are congruent, then the angles opposite them are congruent.

If $\overline{AB} \cong \overline{AC}$, then $\angle B \cong \angle C$.

Proof p. 256; Ex. 33, p. 272

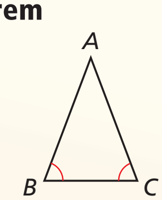


Theorem 5.7 Converse of the Base Angles Theorem

If two angles of a triangle are congruent, then the sides opposite them are congruent.

If $\angle B \cong \angle C$, then $\overline{AB} \cong \overline{AC}$.

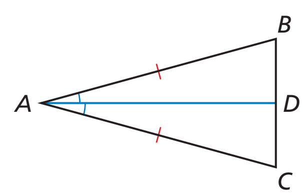
Proof Ex. 27, p. 279



PROOF Base Angles Theorem

Given $\overline{AB} \cong \overline{AC}$

Prove $\angle B \cong \angle C$



Plan for Proof

a. Draw $\overline{AD}$ so that it bisects $\angle CAB$.

b. Use the SAS Congruence Theorem to show that $\triangle ADB \cong \triangle ADC$.

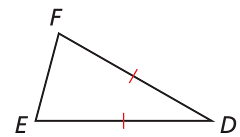
c. Use properties of congruent triangles to show that $\angle B \cong \angle C$.

Plan in Action

	STATEMENTS	REASONS
a.	**1.** Draw $\overline{AD}$, the angle bisector of $\angle CAB$.	**1.** Construction of angle bisector
	2. $\angle CAD \cong \angle BAD$	**2.** Definition of angle bisector
	3. $\overline{AB} \cong \overline{AC}$	**3.** Given
	4. $\overline{DA} \cong \overline{DA}$	**4.** Reflexive Property of Congruence (Thm. 2.1)
b.	**5.** $\triangle ADB \cong \triangle ADC$	**5.** SAS Congruence Theorem (Thm. 5.5)
c.	**6.** $\angle B \cong \angle C$	**6.** Corresponding parts of congruent triangles are congruent.

EXAMPLE 1 Using the Base Angles Theorem

In $\triangle DEF$, $\overline{DE} \cong \overline{DF}$. Name two congruent angles.



SOLUTION

▶ $\overline{DE} \cong \overline{DF}$, so by the Base Angles Theorem, $\angle E \cong \angle F$.

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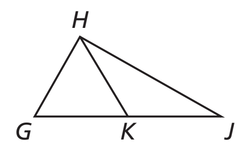
Copy and complete the statement.



1. If $\overline{HG} \cong \overline{HK}$, then $\angle$____ $\cong \angle$____.

2. If $\angle KHJ \cong \angle KJH$, then ____ $\cong$ ____.

Recall that an equilateral triangle has three congruent sides.

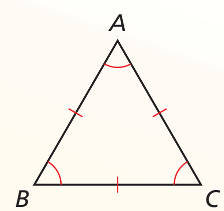
Corollaries

READING
The corollaries state that a triangle is *equilateral* if and only if it is *equiangular*.

Corollary 5.2 Corollary to the Base Angles Theorem

If a triangle is equilateral, then it is equiangular.

Proof Ex. 37, p. 262; Ex. 10, p. 357



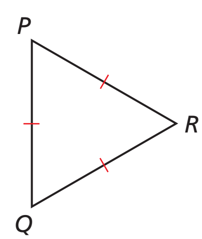
Corollary 5.3 Corollary to the Converse of the Base Angles Theorem

If a triangle is equiangular, then it is equilateral.

Proof Ex. 39, p. 262

EXAMPLE 2 Finding Measures in a Triangle

Find the measures of $\angle P$, $\angle Q$, and $\angle R$.



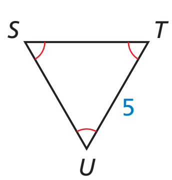
SOLUTION

The diagram shows that $\triangle PQR$ is equilateral. So, by the Corollary to the Base Angles Theorem, $\triangle PQR$ is equiangular. So, $m\angle P = m\angle Q = m\angle R$.

$3(m\angle P) = 180^\circ$ Triangle Sum Theorem (Theorem 5.1)

$m\angle P = 60^\circ$ Divide each side by 3.

▶ The measures of $\angle P$, $\angle Q$, and $\angle R$ are all 60°.



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3. Find the length of $\overline{ST}$ for the triangle at the left.

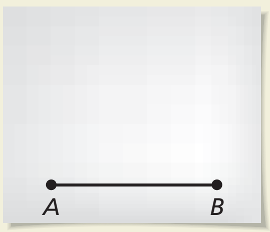
Using Isosceles and Equilateral Triangles

CONSTRUCTION Constructing an Equilateral Triangle

Construct an equilateral triangle that has side lengths congruent to $\overline{AB}$. Use a compass and straightedge.

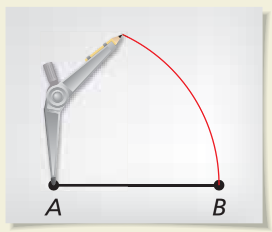
SOLUTION

Step 1



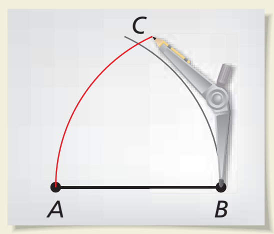
Copy a segment Copy $\overline{AB}$.

Step 2



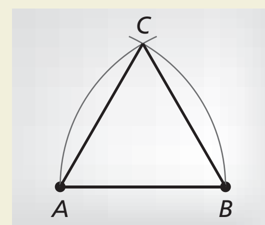
Draw an arc Draw an arc with center A and radius AB.

Step 3



Draw an arc Draw an arc with center B and radius AB. Label the intersection of the arcs from Steps 2 and 3 as C.

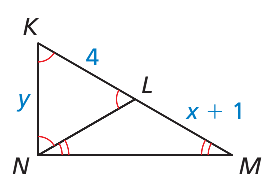
Step 4



Draw a triangle Draw $\triangle ABC$. Because $\overline{AB}$ and $\overline{AC}$ are radii of the same circle, $\overline{AB} \cong \overline{AC}$. Because $\overline{AB}$ and $\overline{BC}$ are radii of the same circle, $\overline{AB} \cong \overline{BC}$. By the Transitive Property of Congruence (Theorem 2.1), $\overline{AC} \cong \overline{BC}$. So, $\triangle ABC$ is equilateral.

EXAMPLE 3 Using Isosceles and Equilateral Triangles

Find the values of x and y in the diagram.



COMMON ERROR

You cannot use N to refer to $\angle LNM$ because three angles have N as their vertex.

SOLUTION

Step 1 Find the value of y. Because $\triangle KLN$ is equiangular, it is also equilateral and $\overline{KN} \cong \overline{KL}$. So, $y = 4$.

Step 2 Find the value of x. Because $\angle LNM \cong \angle LMN$, $\overline{LN} \cong \overline{LM}$, and $\triangle LMN$ is isosceles. You also know that $LN = 4$ because $\triangle KLN$ is equilateral.

$LN = LM$	Definition of congruent segments
$4 = x + 1$	Substitute 4 for LN and $x + 1$ for LM.
$3 = x$	Subtract 1 from each side.

EXAMPLE 4 Solving a Multi-Step Problem

In the lifeguard tower, $\overline{PS} \cong \overline{QR}$ and $\angle QPS \cong \angle PQR$.

a. Explain how to prove that $\triangle QPS \cong \triangle PQR$.

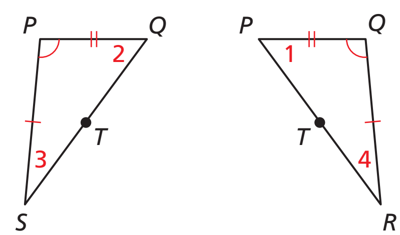
b. Explain why $\triangle PQT$ is isosceles.

COMMON ERROR

When you redraw the triangles so that they do not overlap, be careful to copy all given information and labels correctly.

SOLUTION

a. Draw and label $\triangle QPS$ and $\triangle PQR$ so that they do not overlap. You can see that $\overline{PQ} \cong \overline{QP}$, $\overline{PS} \cong \overline{QR}$, and $\angle QPS \cong \angle PQR$. So, by the SAS Congruence Theorem (Theorem 5.5), $\triangle QPS \cong \triangle PQR$.



b. From part (a), you know that $\angle 1 \cong \angle 2$ because corresponding parts of congruent triangles are congruent. By the Converse of the Base Angles Theorem, $\overline{PT} \cong \overline{QT}$, and $\triangle PQT$ is isosceles.

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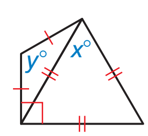
4. Find the values of x and y in the diagram.



5. In Example 4, show that $\triangle PTS \cong \triangle QTR$.

5.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

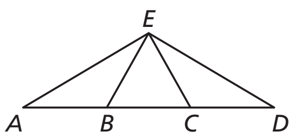
Vocabulary and Core Concept Check

1. **VOCABULARY** Describe how to identify the *vertex angle* of an isosceles triangle.

2. **WRITING** What is the relationship between the base angles of an isosceles triangle? Explain.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, copy and complete the statement. State which theorem you used. *(See Example 1.)*



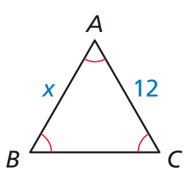
3. If $\overline{AE} \cong \overline{DE}$, then $\angle$___ $\cong \angle$___.

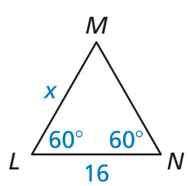
4. If $\overline{AB} \cong \overline{EB}$, then $\angle$___ $\cong \angle$___.

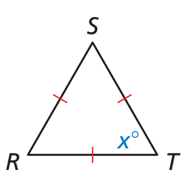
5. If $\angle D \cong \angle CED$, then ___ $\cong$ ___.

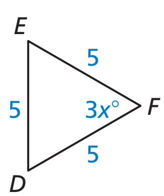
6. If $\angle EBC \cong \angle ECB$, then ___ $\cong$ ___.

In Exercises 7–10, find the value of *x*. *(See Example 2.)*

7. 

8. 

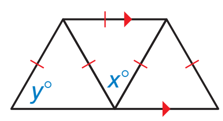
9. 

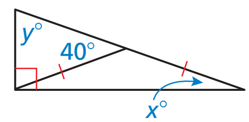
10. 

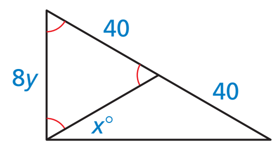
11. **MODELING WITH MATHEMATICS** The dimensions of a sports pennant are given in the diagram. Find the values of *x* and *y*.

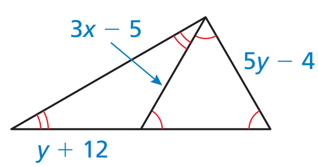
12. **MODELING WITH MATHEMATICS** A logo in an advertisement is an equilateral triangle with a side length of 7 centimeters. Sketch the logo and give the measure of each side.

In Exercises 13–16, find the values of *x* and *y*. *(See Example 3.)*

13. 

14. 

15. 

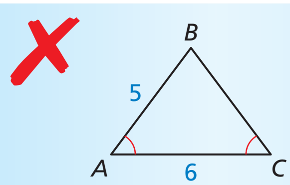
16. 

CONSTRUCTION In Exercises 17 and 18, construct an equilateral triangle whose sides are the given length.

17. 3 inches

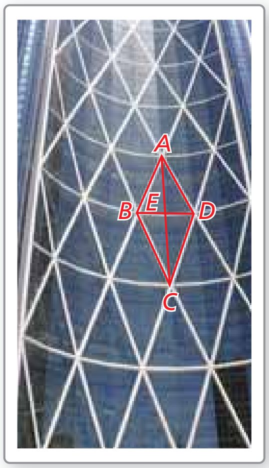
18. 1.25 inches

19. **ERROR ANALYSIS** Describe and correct the error in finding the length of $\overline{BC}$.



Because $\angle A \cong \angle C$,
$\overline{AC} \cong \overline{BC}$.
So, $BC = 6$.

20. **PROBLEM SOLVING** The diagram represents part of the exterior of the Bow Tower in Calgary, Alberta, Canada. In the diagram, $\triangle ABD$ and $\triangle CBD$ are congruent equilateral triangles. *(See Example 4.)*



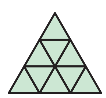
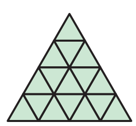
a. Explain why $\triangle ABC$ is isosceles.

b. Explain why $\angle BAE \cong \angle BCE$.

c. Show that $\triangle ABE$ and $\triangle CBE$ are congruent.

d. Find the measure of $\angle BAE$.

21. **FINDING A PATTERN** In the pattern shown, each small triangle is an equilateral triangle with an area of 1 square unit.

Triangle	Area
	1 square unit
	
	
	

a. Explain how you know that any triangle made out of equilateral triangles is equilateral.

b. Find the areas of the first four triangles in the pattern.

c. Describe any patterns in the areas. Predict the area of the seventh triangle in the pattern. Explain your reasoning.

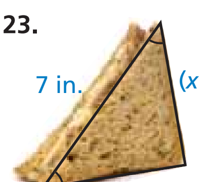
22. **REASONING** The base of isosceles $\triangle XYZ$ is $\overline{YZ}$. What can you prove? Select all that apply.

Ⓐ $\overline{XY} \cong \overline{XZ}$ Ⓑ $\angle X \cong \angle Y$

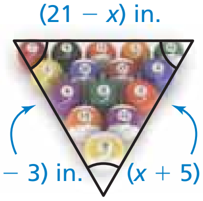
Ⓒ $\angle Y \cong \angle Z$ Ⓓ $\overline{YZ} \cong \overline{ZX}$

In Exercises 23 and 24, find the perimeter of the triangle.

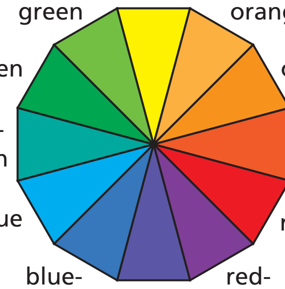
23.



24.



MODELING WITH MATHEMATICS **In Exercises 25–28, use the diagram based on the color wheel. The 12 triangles in the diagram are isosceles triangles with congruent vertex angles.**



25. Complementary colors lie directly opposite each other on the color wheel. Explain how you know that the yellow triangle is congruent to the purple triangle.

26. The measure of the vertex angle of the yellow triangle is 30°. Find the measures of the base angles.

27. Trace the color wheel. Then form a triangle whose vertices are the midpoints of the bases of the red, yellow, and blue triangles. (These colors are the *primary colors*.) What type of triangle is this?

28. Other triangles can be formed on the color wheel that are congruent to the triangle in Exercise 27. The colors on the vertices of these triangles are called *triads*. What are the possible triads?

29. **CRITICAL THINKING** Are isosceles triangles always acute triangles? Explain your reasoning.

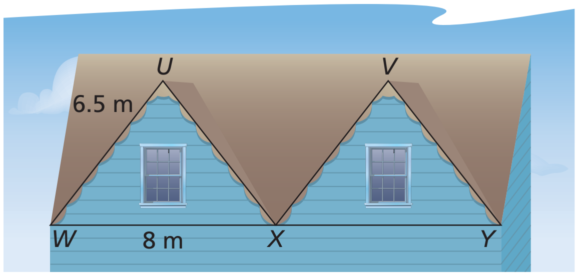
30. **CRITICAL THINKING** Is it possible for an equilateral triangle to have an angle measure other than 60°? Explain your reasoning.

31. **MATHEMATICAL CONNECTIONS** The lengths of the sides of a triangle are $3t$, $5t - 12$, and $t + 20$. Find the values of t that make the triangle isosceles. Explain your reasoning.

32. **MATHEMATICAL CONNECTIONS** The measure of an exterior angle of an isosceles triangle is $x°$. Write expressions representing the possible angle measures of the triangle in terms of x.

33. **WRITING** Explain why the measure of the vertex angle of an isosceles triangle must be an even number of degrees when the measures of all the angles of the triangle are whole numbers.

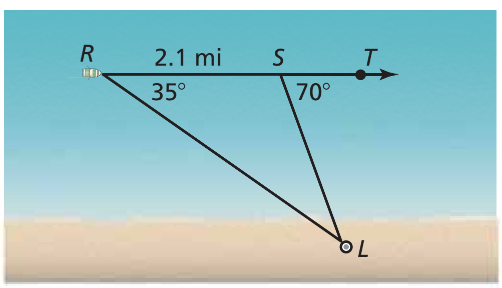
34. **PROBLEM SOLVING** The triangular faces of the peaks on a roof are congruent isosceles triangles with vertex angles U and V.



a. Name two angles congruent to $\angle WUX$. Explain your reasoning.

b. Find the distance between points U and V.

35. **PROBLEM SOLVING** A boat is traveling parallel to the shore along $\overrightarrow{RT}$. When the boat is at point R, the captain measures the angle to the lighthouse as 35°. After the boat has traveled 2.1 miles, the captain measures the angle to the lighthouse to be 70°.



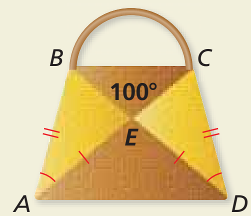
a. Find SL. Explain your reasoning.

b. Explain how to find the distance between the boat and the shoreline.

36. **THOUGHT PROVOKING** The postulates and theorems in this book represent Euclidean geometry. In spherical geometry, all points are points on the surface of a sphere. A line is a circle on the sphere whose diameter is equal to the diameter of the sphere. In spherical geometry, do all equiangular triangles have the same angle measures? Justify your answer.

37. **PROVING A COROLLARY** Prove that the Corollary to the Base Angles Theorem (Corollary 5.2) follows from the Base Angles Theorem (Theorem 5.6).

38. **HOW DO YOU SEE IT?** You are designing fabric purses to sell at the school fair.



a. Explain why $\triangle ABE \cong \triangle DCE$.

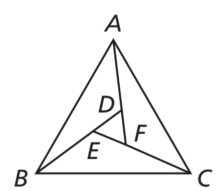
b. Name the isosceles triangles in the purse.

c. Name three angles that are congruent to $\angle EAD$.

39. **PROVING A COROLLARY** Prove that the Corollary to the Converse of the Base Angles Theorem (Corollary 5.3) follows from the Converse of the Base Angles Theorem (Theorem 5.7).

40. **MAKING AN ARGUMENT** The coordinates of two points are $T(0, 6)$ and $U(6, 0)$. Your friend claims that points T, U, and V will always be the vertices of an isosceles triangle when V is any point on the line $y = x$. Is your friend correct? Explain your reasoning.

41. **PROOF** Use the diagram to prove that $\triangle DEF$ is equilateral.



Given $\triangle ABC$ is equilateral.
$\angle CAD \cong \angle ABE \cong \angle BCF$

Prove $\triangle DEF$ is equilateral.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Use the given property to complete the statement. *(Section 2.5)*

42. Reflexive Property of Congruence (Theorem 2.1): ____ $\cong \overline{SE}$

43. Symmetric Property of Congruence (Theorem 2.1): If ____ $\cong$ ____, then $\overline{RS} \cong \overline{JK}$.

44. Transitive Property of Congruence (Theorem 2.1): If $\overline{EF} \cong \overline{PQ}$, and $\overline{PQ} \cong \overline{UV}$, then ____ $\cong$ ____.