6.4 Exponential Growth and Decay

Essential Question What are some of the characteristics of exponential growth and exponential decay functions?

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
A.9.B
A.9.C
A.9.D

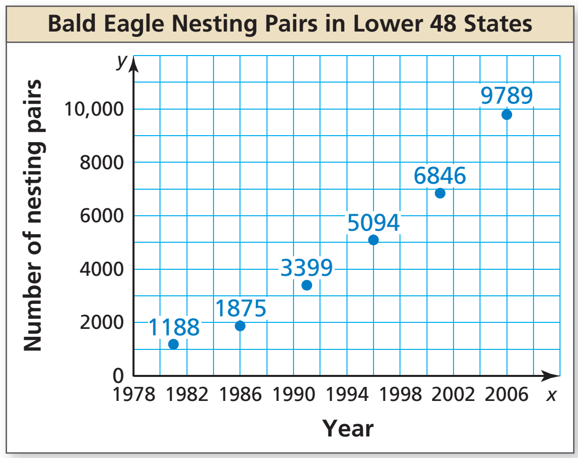
EXPLORATION 1 Predicting a Future Event

Work with a partner. It is estimated, that in 1782, there were about 100,000 nesting pairs of bald eagles in the United States. By the 1960s, this number had dropped to about 500 nesting pairs. In 1967, the bald eagle was declared an endangered species in the United States. With protection, the nesting pair population began to increase. Finally, in 2007, the bald eagle was removed from the list of endangered and threatened species.

Describe the pattern shown in the graph. Is it exponential growth? Assume the pattern continues. When will the population return to that of the late 1700s? Explain your reasoning.

APPLYING MATHEMATICS

To be proficient in math, you need to apply the mathematics you know to solve problems arising in everyday life.



EXPLORATION 2 Describing a Decay Pattern

Work with a partner. A forensic pathologist was called to estimate the time of death of a person. At midnight, the body temperature was 80.5°F and the room temperature was a constant 60°F. One hour later, the body temperature was 78.5°F.

a. By what percent did the difference between the body temperature and the room temperature drop during the hour?

b. Assume that the original body temperature was 98.6°F. Use the percent decrease found in part (a) to make a table showing the decreases in body temperature. Use the table to estimate the time of death.

Communicate Your Answer

3. What are some of the characteristics of exponential growth and exponential decay functions?

4. Use the Internet or some other reference to find an example of each type of function. Your examples should be different than those given in Explorations 1 and 2.

a. exponential growth **b.** exponential decay

6.4 Lesson

Core Vocabulary

exponential growth, *p. 300*
exponential growth function, *p. 300*
exponential decay, *p. 301*
exponential decay function, *p. 301*
compound interest, *p. 303*

What You Will Learn

- Use and identify exponential growth and decay functions.
- Interpret and rewrite exponential growth and decay functions.
- Solve real-life problems involving exponential growth and decay.

Exponential Growth and Decay Functions

Exponential growth occurs when a quantity increases by the same factor over equal intervals of time.

STUDY TIP

Notice that an exponential growth function is of the form $y = ab^x$, where b is replaced by $1 + r$ and x is replaced by t.

Core Concept

Exponential Growth Functions

A function of the form $y = a(1 + r)^t$, where $a > 0$ and $r > 0$, is an **exponential growth function**.

initial amount
rate of growth (in decimal form)

final amount $\rightarrow y = a\underbrace{(1 + r)}_{\text{growth factor}}{}^t \leftarrow$ time

EXAMPLE 1 Using an Exponential Growth Function

The inaugural attendance of an annual music festival is 150,000. The attendance y increases by 8% each year.

a. Write an exponential growth function that represents the attendance after t years.

b. How many people will attend the festival in the fifth year? Round your answer to the nearest thousand.

SOLUTION

a. The initial amount is 150,000, and the rate of growth is 8%, or 0.08.

$y = a(1 + r)^t$	Write the exponential growth function.
$= 150{,}000(1 + 0.08)^t$	Substitute 150,000 for a and 0.08 for r.
$= 150{,}000(1.08)^t$	Add.

▶ The festival attendance can be represented by $y = 150{,}000(1.08)^t$.

b. The value $t = 4$ represents the fifth year because $t = 0$ represents the first year.

$y = 150{,}000(1.08)^t$	Write the exponential growth function.
$= 150{,}000(1.08)^4$	Substitute 4 for t.
$\approx 204{,}073$	Use a calculator.

▶ About 204,000 people will attend the festival in the fifth year.

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1. A website has 500,000 members in 2010. The number y of members increases by 15% each year. (a) Write an exponential growth function that represents the website membership t years after 2010. (b) How many members will there be in 2016? Round your answer to the nearest ten thousand.

Exponential decay occurs when a quantity decreases by the same factor over equal intervals of time.

Core Concept

Exponential Decay Functions

A function of the form $y = a(1 - r)^t$, where $a > 0$ and $0 < r < 1$, is an **exponential decay function**.

$$\underset{\text{final amount}}{y} = \overset{\text{initial amount}}{a}\underbrace{(1 - \overset{\text{rate of decay (in decimal form)}}{r})}_{\text{decay factor}}{}^{\overset{\text{time}}{t}}$$

STUDY TIP

Notice that an exponential decay function is of the form $y = ab^x$, where b is replaced by $1 - r$ and x is replaced by t.

For exponential growth, the value inside the parentheses is greater than 1 because r is added to 1. For exponential decay, the value inside the parentheses is less than 1 because r is subtracted from 1.

EXAMPLE 2 Identifying Exponential Growth and Decay

Determine whether each table represents an *exponential growth function*, an *exponential decay function*, or *neither*.

a.

x	y
0	270
1	90
2	30
3	10

b.

x	0	1	2	3
y	5	10	20	40

SOLUTION

a.

	x	y	
	0	270	
$+1$	1	90	$\times\frac{1}{3}$
$+1$	2	30	$\times\frac{1}{3}$
$+1$	3	10	$\times\frac{1}{3}$

As x increases by 1, y is multiplied by $\frac{1}{3}$. So, the table represents an exponential decay function.

b.

		$+1$	$+1$	$+1$
x	0	1	2	3
y	5	10	20	40
		$\times 2$	$\times 2$	$\times 2$

As x increases by 1, y is multiplied by 2. So, the table represents an exponential growth function.

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Determine whether the table represents an *exponential growth function*, an *exponential decay function*, or *neither*. Explain.

2.

x	0	1	2	3
y	64	16	4	1

3.

x	1	3	5	7
y	4	11	18	25

Interpreting and Rewriting Exponential Functions

EXAMPLE 3 Interpreting Exponential Functions

Determine whether each function represents *exponential growth* or *exponential decay*. Identify the initial amount and interpret the growth factor or decay factor.

a. The function $y = 0.5(1.07)^t$ represents the value y (in dollars) of a baseball card t years after it is issued.

b. The function $y = 128(0.5)^x$ represents the number y of players left in a video game tournament after x rounds.

SOLUTION

a. The function is of the form $y = a(1 + r)^t$, where $1 + r > 1$, so it represents exponential growth. The initial amount is \$0.50, and the growth factor of 1.07 means that the value of the baseball card increases by 7% each year.

b. The function is of the form $y = a(1 - r)^t$, where $1 - r < 1$, so it represents exponential decay. The initial amount is 128 players, and the decay factor of 0.5 means that 50% of the players are left after each round.

STUDY TIP

You can rewrite exponential expressions and functions using the properties of exponents. Changing the form of an exponential function can reveal important attributes of the function.

EXAMPLE 4 Rewriting Exponential Functions

Rewrite each function in the form $f(x) = ab^x$ to determine whether it represents *exponential growth* or *exponential decay*.

a. $f(x) = 100(0.96)^{x/4}$ **b.** $f(x) = (1.1)^{x-3}$

SOLUTION

a. $f(x) = 100(0.96)^{x/4}$ Write the function.

$= 100(0.96^{1/4})^x$ Power of a Power Property

$\approx 100(0.99)^x$ Evaluate the power.

▶ So, the function represents exponential decay.

b. $f(x) = (1.1)^{x-3}$ Write the function.

$= \dfrac{(1.1)^x}{(1.1)^3}$ Quotient of Powers Property

$\approx 0.75(1.1)^x$ Evaluate the power and simplify.

▶ So, the function represents exponential growth.

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4. The function $y = 10(1.12)^n$ represents the average game attendance y (in thousands of people) of a professional baseball team in its nth season. Determine whether the function represents *exponential growth* or *exponential decay*. Identify the initial amount and interpret the growth factor or decay factor.

Rewrite the function in the form $f(x) = ab^x$ to determine whether it represents *exponential growth* or *exponential decay*.

5. $f(x) = 3(1.02)^{10x}$ **6.** $f(x) = (0.95)^{x+2}$

Solving Real-Life Problems

Exponential growth functions are used in real-life situations involving *compound interest*. Although interest earned is expressed as an *annual* rate, the interest is usually compounded more frequently than once per year. So, the formula $y = a(1 + r)^t$ must be modified for compound interest problems.

STUDY TIP

For interest compounded yearly, you can substitute 1 for n in the formula to get $y = P(1 + r)^t$.

Core Concept

Compound Interest

Compound interest is the interest earned on the principal *and* on previously earned interest. The balance y of an account earning compound interest is

$$y = P\left(1 + \frac{r}{n}\right)^{nt}.$$

P = principal (initial amount)
r = annual interest rate (in decimal form)
t = time (in years)
n = number of times interest is compounded per year

EXAMPLE 5 Writing a Function

You deposit \$100 in a savings account that earns 6% annual interest compounded monthly. Write a function that represents the balance after t years.

SOLUTION

$$y = P\left(1 + \frac{r}{n}\right)^{nt}$$ Write the compound interest formula.

$$= 100\left(1 + \frac{0.06}{12}\right)^{12t}$$ Substitute 100 for P, 0.06 for r, and 12 for n.

$$= 100(1.005)^{12t}$$ Simplify.

EXAMPLE 6 Solving a Real-Life Problem

The table shows the balance of a money market account over time.

a. Write a function that represents the balance after t years.

b. Graph the functions from part (a) and from Example 5 in the same coordinate plane. Compare the account balances.

Year, t	Balance
0	\$100
1	\$110
2	\$121
3	\$133.10
4	\$146.41
5	\$161.05

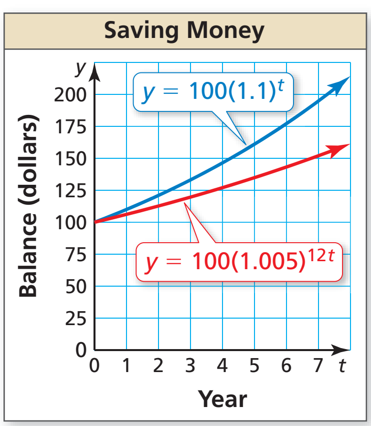
SOLUTION

a. From the table, you know the initial balance is \$100, and it increases 10% each year. So, $P = 100$ and $r = 0.1$.

$$y = P(1 + r)^t$$ Write the compound interest formula when $n = 1$.

$$= 100(1 + 0.1)^t$$ Substitute 100 for P and 0.1 for r.

$$= 100(1.1)^t$$ Add.



b. The money market account earns 10% interest each year, and the savings account earns 6% interest each year. So, the balance of the money market account increases faster.

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7. You deposit \$500 in a savings account that earns 9% annual interest compounded monthly. Write and graph a function that represents the balance y (in dollars) after t years.

EXAMPLE 7 Solving a Real-Life Problem

The value of a car is \$21,500. It loses 12% of its value every year. (a) Write a function that represents the value y (in dollars) of the car after t years. (b) Find the approximate monthly percent decrease in value. (c) Graph the function from part (a). Identify and interpret any asymptotes of the graph. (d) Estimate the value of the car after 6 years.

SOLUTION

1. **Understand the Problem** You know the value of the car and its annual percent decrease in value. You are asked to write a function that represents the value of the car over time and approximate the monthly percent decrease in value. Then graph the function and use the graph to estimate the value of the car in the future.
2. **Make a Plan** Use the initial amount and the annual percent decrease in value to write an exponential decay function. Rewrite the function using the properties of exponents to approximate the monthly percent decrease (rate of decay). Then graph the original function and use the graph to estimate the y-value when the t-value is 6.
3. **Solve the Problem**

STUDY TIP
In real life, the percent decrease in value of an asset is called the *depreciation rate*.

a. The initial value is \$21,500, and the rate of decay is 12%, or 0.12.

$y = a(1 - r)^t$	Write the exponential decay function.
$= 21{,}500(1 - 0.12)^t$	Substitute 21,500 for *a* and 0.12 for *r*.
$= 21{,}500(0.88)^t$	Subtract.

▶ The value of the car can be represented by $y = 21{,}500(0.88)^t$.

b. Use the fact that $t = \frac{1}{12}(12t)$ and the properties of exponents to rewrite the function in a form that reveals the monthly rate of decay.

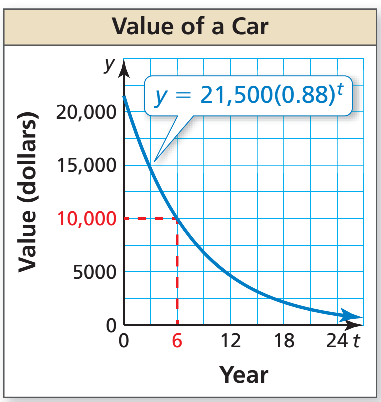
$y = 21{,}500(0.88)^t$	Write the original function.
$= 21{,}500(0.88)^{(1/12)(12t)}$	Rewrite the exponent.
$= 21{,}500(0.88^{1/12})^{12t}$	Power of a Power Property
$\approx 21{,}500(0.989)^{12t}$	Evaluate the power.

Use the decay factor $1 - r \approx 0.989$ to find the rate of decay $r \approx 0.011$.

▶ So, the monthly percent decrease is about 1.1%.

c. You can see that the graph approaches, but never intersects, the t-axis.

▶ So, the graph has an asymptote at $t = 0$. This makes sense because the car will never have a value of \$0.



d. From the graph, you can see that the y-value is about 10,000 when $t = 6$.

▶ So, the value of the car is about \$10,000 after 6 years.

4. **Look Back** To check that the monthly percent decrease is reasonable, multiply it by 12 to see if it is close in value to the annual percent decrease of 12%.

$1.1\% \times 12 = 13.2\%$	13.2% is close to 12%, so 1.1% is reasonable.

When you evaluate $y = 21{,}500(0.88)^t$ for $t = 6$, you get about \$9985. So, \$10,000 is a reasonable estimation.

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8. **WHAT IF?** The car loses 9% of its value every year. (a) Write a function that represents the value y (in dollars) of the car after t years. (b) Find the approximate monthly percent decrease in value. (c) Graph the function from part (a). Estimate the value of the car after 12 years. Round your answer to the nearest thousand.

6.4 Exercises

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Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** In the exponential growth function $y = a(1 + r)^t$, the quantity r is called the ________.

2. **VOCABULARY** What is the decay factor in the exponential decay function $y = a(1 - r)^t$?

3. **VOCABULARY** Compare exponential growth and exponential decay.

4. **WRITING** When does the function $y = ab^x$ represent exponential growth? exponential decay?

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, identify the initial amount a and the rate of growth r (as a percent) of the exponential function. Evaluate the function when $t = 5$. Round your answer to the nearest tenth.

5. $y = 350(1 + 0.75)^t$

6. $y = 10(1 + 0.4)^t$

7. $y = 25(1.2)^t$

8. $y = 12(1.05)^t$

9. $f(t) = 1500(1.074)^t$

10. $h(t) = 175(1.028)^t$

11. $g(t) = 6.72(2)^t$

12. $p(t) = 1.8^t$

In Exercises 13–16, write a function that represents the situation.

13. Sales of \$10,000 increase by 65% each year.

14. Your starting annual salary of \$35,000 increases by 4% each year.

15. A population of 210,000 increases by 12.5% each year.

16. An item costs \$4.50, and its price increases by 3.5% each year.

17. **MODELING WITH MATHEMATICS** The population of a city has been increasing by 2% annually. The sign shown is from the year 2000. *(See Example 1.)*

    a. Write an exponential growth function that represents the population t years after 2000.

    b. What will the population be in 2020? Round your answer to the nearest thousand.

18. **MODELING WITH MATHEMATICS** A young channel catfish weighs about 0.1 pound. During the next 8 weeks, its weight increases by about 23% each week.

    a. Write an exponential growth function that represents the weight of the catfish after t weeks during the 8-week period.

    b. About how much will the catfish weigh after 4 weeks? Round your answer to the nearest thousandth.

In Exercises 19–26, identify the initial amount a and the rate of decay r (as a percent) of the exponential function. Evaluate the function when $t = 3$. Round your answer to the nearest tenth.

19. $y = 575(1 - 0.6)^t$

20. $y = 8(1 - 0.15)^t$

21. $g(t) = 240(0.75)^t$

22. $f(t) = 475(0.5)^t$

23. $w(t) = 700(0.995)^t$

24. $h(t) = 1250(0.865)^t$

25. $y = \left(\frac{7}{8}\right)^t$

26. $y = 0.5\left(\frac{3}{4}\right)^t$

In Exercises 27–30, write a function that represents the situation.

27. A population of 100,000 decreases by 2% each year.

28. A \$900 sound system decreases in value by 9% each year.

29. A stock valued at \$100 decreases in value by 9.5% each year.

30. A company profit of $20,000 decreases by 13.4% each year.

31. **ERROR ANALYSIS** The growth rate of a bacterial culture is 150% each hour. Initially, there are 10 bacteria. Describe and correct the error in finding the number of bacteria in the culture after 8 hours.

✗

$b(t) = 10(1.5)^t$

$b(8) = 10(1.5)^8 \approx 256.3$

After 8 hours, there are about 256 bacteria in the culture.

32. **ERROR ANALYSIS** You purchase a car in 2010 for $25,000. The value of the car decreases by 14% annually. Describe and correct the error in finding the value of the car in 2015.

✗

$v(t) = 25{,}000(1.14)^t$

$v(4) = 25{,}000(1.14)^5 \approx 48{,}135$

The value of the car in 2015 is about $48,000.

In Exercises 33–38, determine whether the table represents an *exponential growth function*, an *exponential decay function*, or *neither*. Explain. *(See Example 2.)*

33.

x	*y*
−1	50
0	10
1	2
2	0.4

34.

x	*y*
0	32
1	28
2	24
3	20

35.

x	*y*
0	35
1	29
2	23
3	17

36.

x	*y*
1	17
2	51
3	153
4	459

37.

x	*y*
5	2
10	8
15	32
20	128

38.

x	*y*
3	432
5	72
7	12
9	2

39. **ANALYZING RELATIONSHIPS** The table shows the value of a camper t years after it is purchased.

t	Value
1	$37,000
2	$29,600
3	$23,680
4	$18,944

a. Determine whether the table represents an exponential growth function, an exponential decay function, or neither.

b. What is the value of the camper after 5 years?

40. **ANALYZING RELATIONSHIPS** The table shows the total numbers of visitors to a website t days after it is online.

t	42	43	44	45
Visitors	11,000	12,100	13,310	14,641

a. Determine whether the table represents an exponential growth function, an exponential decay function, or neither.

b. How many people will have visited the website after it is online 47 days?

In Exercises 41–44, determine whether the function represents *exponential growth* or *exponential decay*. Identify the initial amount and interpret the growth factor or decay factor. *(See Example 3.)*

41. The function $y = 22(0.94)^t$ represents the population y (in thousands of people) of a town after t years.

42. The function $y = 13(1.2)^x$ represents the number y of customers at a store x days after the store first opens.

43. The function $y = 2^n$ represents the number y of sprints required n days since the start of a training routine.

44. The function $y = 900(0.85)^d$ represents the number y of students who are healthy d days after a flu outbreak.

In Exercises 45–52, rewrite the function in the form $f(x) = ab^x$ to determine whether it represents *exponential growth* or *exponential decay*. *(See Example 4.)*

45. $f(x) = (0.9)^{x-4}$

46. $f(x) = (1.4)^{x+8}$

47. $f(x) = 2(1.06)^{9x}$

48. $f(x) = 5(0.82)^{x/5}$

49. $f(x) = (1.45)^{x/2}$

50. $f(x) = 0.4(1.16)^{x-1}$

51. $f(x) = 4(0.55)^{x+3}$

52. $f(x) = (0.88)^{4x}$

In Exercises 53–56, write a function that represents the balance after t years. *(See Example 5.)*

53. \$2000 deposit that earns 5% annual interest compounded quarterly

54. \$1400 deposit that earns 10% annual interest compounded semiannually

55. \$6200 deposit that earns 8.4% annual interest compounded monthly

56. \$3500 deposit that earns 9.2% annual interest compounded quarterly

57. **PROBLEM SOLVING** The cross-sectional area of a tree 4.5 feet from the ground is called its *basal area*. The table shows the basal areas (in square inches) of Tree A over time. *(See Example 6.)*

Year, t	0	1	2	3	4
Basal area, A	120	132	145.2	159.7	175.7

Growth rate: 6%

Initial basal area: 154 in.^{2}

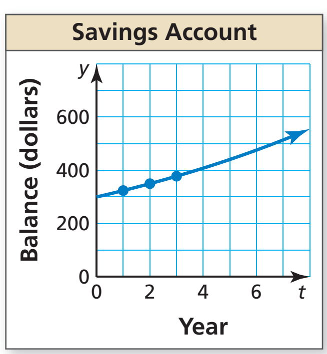
a. Write functions that represent the basal areas of the trees after t years.

b. Graph the functions from part (a) in the same coordinate plane. Compare the basal areas.

58. **PROBLEM SOLVING** You deposit \$300 into an investment account that earns 12% annual interest compounded quarterly. The graph shows the balance of a savings account over time.

a. Write functions that represent the balances of the accounts after t years.

b. Graph the functions from part (a) in the same coordinate plane. Compare the account balances.



59. **PROBLEM SOLVING** A city has a population of 25,000. The population is expected to increase by 5.5% annually for the next decade. *(See Example 7.)*

a. Write a function that represents the population y after t years.

b. Find the approximate monthly percent increase in population.

c. Graph the function from part (a). Estimate the population after 4 years.

60. **PROBLEM SOLVING** Plutonium-238 is a material that generates steady heat due to decay and is used in power systems for some spacecraft. The function $y = a(0.5)^{t/x}$ represents the amount y of a substance remaining after t years, where a is the initial amount and x is the length of the half-life (in years).

a. A scientist is studying a 3-gram sample. Write a function that represents the amount y of plutonium-238 after t years.

b. What is the yearly percent decrease of plutonium-238?

c. Graph the function from part (a). Identify and interpret any asymptotes of the graph.

d. Estimate the amount remaining after 12 years.

61. **COMPARING FUNCTIONS** The three given functions describe the amount y of ibuprofen (in milligrams) in a person's bloodstream t hours after taking the dosage.

$$y \approx 800(0.71)^t$$
$$y \approx 800(0.9943)^{60t}$$
$$y \approx 800(0.843)^{2t}$$

a. Show that these expressions are approximately equivalent.

b. Describe the information given by each of the functions.

62. COMBINING FUNCTIONS You deposit \$9000 in a savings account that earns 3.6% annual interest compounded monthly. You also save \$40 per month in a safe at home. Write a function $C(t) = b(t) + h(t)$, where $b(t)$ represents the balance of your savings account and $h(t)$ represents the amount in your safe after t years. What does $C(t)$ represent?

63. NUMBER SENSE During a flu epidemic, the number of sick people triples every week. What is the growth rate as a percent? Explain your reasoning.

64. HOW DO YOU SEE IT? Match each situation with its graph. Explain your reasoning.

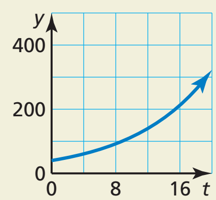
a. A bacterial population doubles each hour.

b. The value of a computer decreases by 18% each year.

c. A deposit earns 11% annual interest compounded yearly.

d. A radioactive element decays 5.5% each year.

A.



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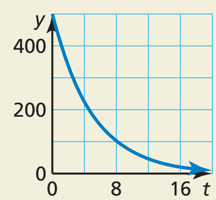

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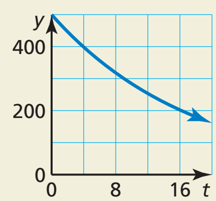
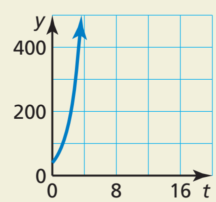


D.

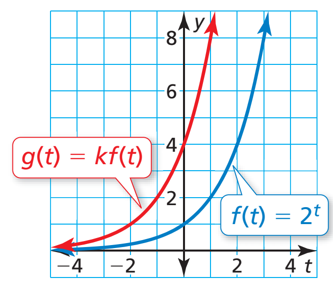


65. WRITING Give an example of an equation in the form $y = ab^x$ that does not represent an exponential growth function or an exponential decay function. Explain your reasoning.

66. THOUGHT PROVOKING Describe two account options into which you can deposit \$1000 and earn compound interest. Write a function that represents the balance of each account after t years. Which account would you rather use? Explain your reasoning.

67. MAKING AN ARGUMENT A store is having a sale on sweaters. On the first day, the prices of the sweaters are reduced by 20%. The prices will be reduced another 20% each day until the sweaters are sold. Your friend says the sweaters will be free on the fifth day. Is your friend correct? Explain.

68. COMPARING FUNCTIONS The graphs of f and g are shown.



a. Explain why f is an exponential growth function. Identify the rate of growth.

b. Describe the transformation from the graph of f to the graph of g. Determine the value of k.

c. The graph of g is the same as the graph of $h(t) = f(t + r)$. Use properties of exponents to find the value of r.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution. *(Section 1.3)*

69. $8x + 12 = 4x$

70. $5 - t = 7t + 21$

71. $6(r - 2) = 2r + 8$

Determine whether the sequence is arithmetic. If so, find the common difference. *(Section 4.7)*

72. $-20, -26, -32, -38, \ldots$

73. $9, 18, 36, 72, \ldots$

74. $-5, -8, -12, -17, \ldots$

75. $10, 20, 30, 40, \ldots$