6 Exponential Functions and Sequences

SEE the Big Idea

Fibonacci and Flowers *(p. 323)*

Soup Kitchen *(p. 318)*

Plutonium Decay *(p. 307)*

Coyote Population *(p. 297)*

Jellyfish *(p. 281)*

Mathematical Thinking: *Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.*

Maintaining Mathematical Proficiency

Using Order of Operations (6.7.A)

Example 1 Evaluate $10^2 \div (30 \div 3) - 4(3 - 9) + 5^1$.

First:	Parentheses	$10^2 \div (30 \div 3) - 4(3 - 9) + 5^1 = 10^2 \div 10 - 4(-6) + 5^1$
Second:	Exponents	$= 100 \div 10 - 4(-6) + 5$
Third:	Multiplication and Division (from left to right)	$= 10 + 24 + 5$
Fourth:	Addition and Subtraction (from left to right)	$= 39$

Evaluate the expression.

1. $12\left(\dfrac{14}{2}\right) - 3^3 + 15 - 9^2$ **2.** $5^2 \cdot 8 \div 2^2 + 20 \cdot 3 - 4$ **3.** $-7 + 16 \div 2^4 + (10 - 4^2)$

Finding Square Roots (A.11.A)

Example 2 Find $-\sqrt{81}$.

▶ $-\sqrt{81}$ represents the negative square root. Because $9^2 = 81$, $-\sqrt{81} = -\sqrt{9^2} = -9$.

Find the square root(s).

4. $\sqrt{64}$ **5.** $-\sqrt{4}$ **6.** $-\sqrt{25}$ **7.** $\pm\sqrt{121}$

Writing Equations for Arithmetic Sequences (A.12.D)

Example 3 Write an equation for the nth term of the arithmetic sequence 5, 15, 25, 35, ...

The first term is 5, and the common difference is 10.

$a_n = a_1 + (n - 1)d$	Equation for an arithmetic sequence
$a_n = 5 + (n - 1)(10)$	Substitute 5 for a_1 and 10 for d.
$a_n = 10n - 5$	Simplify.

Write an equation for the nth term of the arithmetic sequence.

8. 12, 14, 16, 18, . . .   **9.** 6, 3, 0, −3, . . .   **10.** 22, 15, 8, 1, . . .

11. ABSTRACT REASONING Recall that a perfect square is a number with integers as its square roots. Is the product of two perfect squares always a perfect square? Is the quotient of two perfect squares always a perfect square? Explain your reasoning.

Mathematical Thinking

Mathematically proficient students use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying the solution, and evaluating the problem-solving process and the reasonableness of the solution. **(A.1.B)**

Problem-Solving Strategies

### Core Concept

Finding a Pattern

When solving a real-life problem, look for a pattern in the data. The pattern could include repeating items, numbers, or events. After you find the pattern, describe it and use it to solve the problem.

EXAMPLE 1 Using a Problem-Solving Strategy

The volumes of seven chambers of a chambered nautilus are given. Find the volume of Chamber 10.

Chamber 7: 1.207 cm^3

Chamber 6: 1.135 cm^3

Chamber 5: 1.068 cm^3

Chamber 4: 1.005 cm^3

Chamber 3: 0.945 cm^3

Chamber 2: 0.889 cm^3

Chamber 1: 0.836 cm^3

SOLUTION

To find a pattern, try dividing each volume by the volume of the previous chamber.
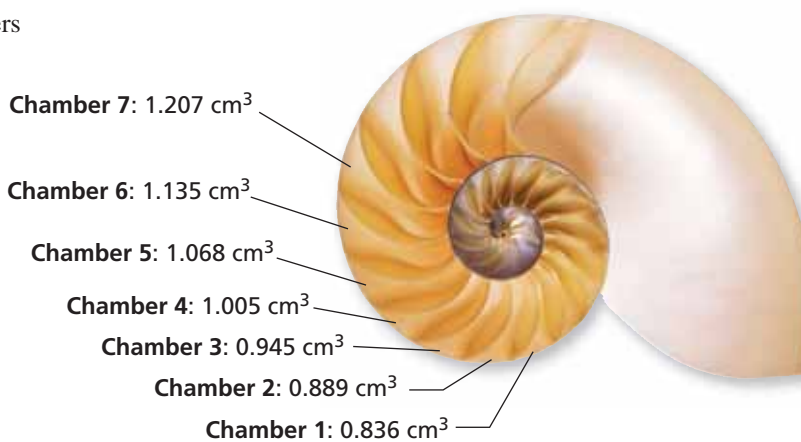
$$\frac{0.889}{0.836} \approx 1.063 \qquad \frac{0.945}{0.889} \approx 1.063 \qquad \frac{1.005}{0.945} \approx 1.063$$

$$\frac{1.068}{1.005} \approx 1.063 \qquad \frac{1.135}{1.068} \approx 1.063 \qquad \frac{1.207}{1.135} \approx 1.063$$

From this, you can see that the volume of each chamber is about 6.3% greater than the volume of the previous chamber. To find the volume of Chamber 10, multiply the volume of Chamber 7 by 1.063 three times.

$$1.207(1.063) \approx \mathbf{1.283} \qquad 1.283(1.063) \approx \mathbf{1.364} \qquad 1.364(1.063) \approx \mathbf{1.450}$$
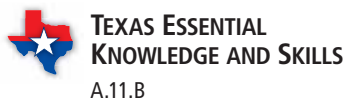
volume of Chamber 8   volume of Chamber 9   volume of Chamber 10

▶ The volume of Chamber 10 is about 1.450 cubic centimeters.

Monitoring Progress

1. A rabbit population over 8 consecutive years is given by 50, 80, 128, 205, 328, 524, 839, 1342. Find the population in the tenth year.

2. The sums of the numbers in the first eight rows of Pascal's Triangle are 1, 2, 4, 8, 16, 32, 64, 128. Find the sum of the numbers in the tenth row.

6.1 Properties of Exponents

Essential Question How can you write general rules involving properties of exponents?

EXPLORATION 1 **Writing Rules for Properties of Exponents**

Work with a partner.

> **ANALYZING MATHEMATICAL RELATIONSHIPS**
>
> To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results in writing general rules.

a. What happens when you multiply two powers with the same base? Write the product of the two powers as a single power. Then write a *general rule* for finding the product of two powers with the same base.

i. $(2^2)(2^3) = $ ⬜ **ii.** $(4^1)(4^5) = $ ⬜

iii. $(5^3)(5^5) = $ ⬜ **iv.** $(x^2)(x^6) = $ ⬜

b. What happens when you divide two powers with the same base? Write the quotient of the two powers as a single power. Then write a *general rule* for finding the quotient of two powers with the same base.

i. $\dfrac{4^3}{4^2} = $ ⬜ **ii.** $\dfrac{2^5}{2^2} = $ ⬜

iii. $\dfrac{x^6}{x^3} = $ ⬜ **iv.** $\dfrac{3^4}{3^4} = $ ⬜

c. What happens when you find a power of a power? Write the expression as a single power. Then write a *general rule* for finding a power of a power.

i. $(2^2)^4 = $ ⬜ **ii.** $(7^3)^2 = $ ⬜

iii. $(y^3)^3 = $ ⬜ **iv.** $(x^4)^2 = $ ⬜

d. What happens when you find a power of a product? Write the expression as the product of two powers. Then write a *general rule* for finding a power of a product.

i. $(2 \cdot 5)^2 = $ ⬜ **ii.** $(5 \cdot 4)^3 = $ ⬜

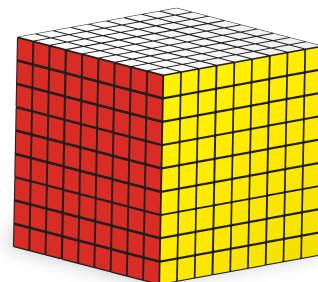**iii.** $(6a)^2 = $ ⬜ **iv.** $(3x)^2 = $ ⬜

e. What happens when you find a power of a quotient? Write the expression as the quotient of two powers. Then write a *general rule* for finding a power of a quotient.

i. $\left(\dfrac{2}{3}\right)^2 = $ ⬜ **ii.** $\left(\dfrac{4}{3}\right)^3 = $ ⬜

iii. $\left(\dfrac{x}{2}\right)^3 = $ ⬜ **iv.** $\left(\dfrac{a}{b}\right)^4 = $ ⬜

Communicate Your Answer

2. How can you write general rules involving properties of exponents?

3. There are 3^3 small cubes in the cube below. Write an expression for the number of small cubes in the large cube at the right.

What You Will Learn

▶ Use zero and negative exponents.

▶ Use the properties of exponents.

▶ Solve real-life problems involving exponents.

Core Vocabulary

Previous
power
exponent
base
scientific notation

Using Zero and Negative Exponents

Core Concept

Zero Exponent

Words For any nonzero number a, $a^0 = 1$. The power 0^0 is undefined.

Numbers $4^0 = 1$ **Algebra** $a^0 = 1$, where $a \neq 0$

Negative Exponents

Words For any integer n and any nonzero number a, a^{-n} is the reciprocal of a^n.

Numbers $4^{-2} = \dfrac{1}{4^2}$ **Algebra** $a^{-n} = \dfrac{1}{a^n}$, where $a \neq 0$

EXAMPLE 1 **Using Zero and Negative Exponents**

Evaluate each expression.

a. 6.7^0 **b.** $(-2)^{-4}$

SOLUTION

a. $6.7^0 = 1$ Definition of zero exponent

b. $(-2)^{-4} = \dfrac{1}{(-2)^4}$ Definition of negative exponent

$= \dfrac{1}{16}$ Simplify.

EXAMPLE 2 **Simplifying an Expression**

Simplify the expression $\dfrac{4x^0}{y^{-3}}$. Write your answer using only positive exponents.

SOLUTION

$\dfrac{4x^0}{y^{-3}} = 4x^0 y^3$ Definition of negative exponent

$= 4y^3$ Definition of zero exponent

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Evaluate the expression.

1. $(-9)^0$ **2.** 3^{-3} **3.** $\dfrac{-5^0}{2^{-2}}$

4. Simplify the expression $\dfrac{3^{-2}x^{-5}}{y^0}$. Write your answer using only positive exponents.

Using the Properties of Exponents

Core Concept

Product of Powers Property

Let a be a real number, and let m and n be integers.

Words To multiply powers with the same base, add their exponents.

Numbers $4^6 \cdot 4^3 = 4^{6+3} = 4^9$ **Algebra** $a^m \cdot a^n = a^{m+n}$

Quotient of Powers Property

Let a be a nonzero real number, and let m and n be integers.

Words To divide powers with the same base, subtract their exponents.

Numbers $\dfrac{4^6}{4^3} = 4^{6-3} = 4^3$ **Algebra** $\dfrac{a^m}{a^n} = a^{m-n}$, where $a \neq 0$

Power of a Power Property

Let a be a real number, and let m and n be integers.

Words To find a power of a power, multiply the exponents.

Numbers $(4^6)^3 = 4^{6 \cdot 3} = 4^{18}$ **Algebra** $(a^m)^n = a^{mn}$

EXAMPLE 3 **Using Properties of Exponents**

Simplify each expression. Write your answer using only positive exponents.

a. $3^2 \cdot 3^6$
b. $\dfrac{(-4)^2}{(-4)^7}$
c. $(z^4)^{-3}$

SOLUTION

a. $3^2 \cdot 3^6 = 3^{2+6}$ *Product of Powers Property*

$\quad\quad\quad\quad = 3^8 = 6561$ *Simplify.*

b. $\dfrac{(-4)^2}{(-4)^7} = (-4)^{2-7}$ *Quotient of Powers Property*

$\quad\quad\quad\quad = (-4)^{-5}$ *Simplify.*

$\quad\quad\quad\quad = \dfrac{1}{(-4)^5} = -\dfrac{1}{1024}$ *Definition of negative exponent*

c. $(z^4)^{-3} = z^{4 \cdot (-3)}$ *Power of a Power Property*

$\quad\quad\quad\quad = z^{-12}$ *Simplify.*

$\quad\quad\quad\quad = \dfrac{1}{z^{12}}$ *Definition of negative exponent*

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Simplify the expression. Write your answer using only positive exponents.

5. $10^4 \cdot 10^{-6}$
6. $x^9 \cdot x^{-9}$
7. $\dfrac{-5^8}{-5^4}$

8. $\dfrac{y^6}{y^7}$
9. $(6^{-2})^{-1}$
10. $(w^{12})^5$

⑤ Core Concept

Power of a Product Property

Let a and b be real numbers, and let m be an integer.

Words To find a power of a product, find the power of each factor and multiply.

Numbers $(3 \cdot 2)^5 = 3^5 \cdot 2^5$ **Algebra** $(ab)^m = a^m b^m$

Power of a Quotient Property

Let a and b be real numbers with $b \neq 0$, and let m be an integer.

Words To find the power of a quotient, find the power of the numerator and the power of the denominator and divide.

Numbers $\left(\dfrac{3}{2}\right)^5 = \dfrac{3^5}{2^5}$ **Algebra** $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$, where $b \neq 0$

EXAMPLE 4   **Using Properties of Exponents**

Simplify each expression. Write your answer using only positive exponents.

a. $(-1.5y)^2$ **b.** $\left(\dfrac{a}{-10}\right)^3$ **c.** $\left(\dfrac{3d}{2}\right)^4$ **d.** $\left(\dfrac{2x}{3}\right)^{-5}$

SOLUTION

a. $(-1.5y)^2 = (-1.5)^2 \cdot y^2$ Power of a Product Property

$\qquad\qquad\quad = 2.25y^2$ Simplify.

b. $\left(\dfrac{a}{-10}\right)^3 = \dfrac{a^3}{(-10)^3}$ Power of a Quotient Property

$\qquad\qquad\quad = -\dfrac{a^3}{1000}$ Simplify.

ANOTHER WAY

Because the exponent is negative, you could find the reciprocal of the base first. Then simplify.

$\left(\dfrac{2x}{3}\right)^{-5} = \left(\dfrac{3}{2x}\right)^5 = \dfrac{243}{32x^5}$

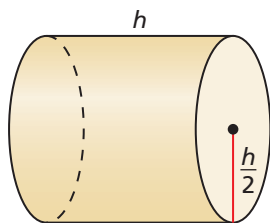**c.** $\left(\dfrac{3d}{2}\right)^4 = \dfrac{(3d)^4}{2^4}$ Power of a Quotient Property

$\qquad\qquad\quad = \dfrac{3^4 d^4}{2^4}$ Power of a Product Property

$\qquad\qquad\quad = \dfrac{81d^4}{16}$ Simplify.

d. $\left(\dfrac{2x}{3}\right)^{-5} = \dfrac{(2x)^{-5}}{3^{-5}}$ Power of a Quotient Property

$\qquad\qquad\quad = \dfrac{3^5}{(2x)^5}$ Definition of negative exponent

$\qquad\qquad\quad = \dfrac{3^5}{2^5 x^5}$ Power of a Product Property

$\qquad\qquad\quad = \dfrac{243}{32x^5}$ Simplify.

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Simplify the expression. Write your answer using only positive exponents.

11. $(10y)^{-3}$ **12.** $\left(-\dfrac{4}{n}\right)^5$ **13.** $\left(\dfrac{1}{2k^2}\right)^5$ **14.** $\left(\dfrac{6c}{7}\right)^{-2}$

Solving Real-Life Problems

EXAMPLE 5   **Simplifying a Real-Life Expression**

Which of the expressions shown represent the volume of the cylinder, where r is the radius and h is the height?

SOLUTION

$$V = \pi r^2 h \qquad \qquad \text{Formula for the volume of a cylinder}$$

$$= \pi \left(\frac{h}{2}\right)^2 (h) \qquad \text{Substitute } \frac{h}{2} \text{ for } r.$$

$$= \pi \left(\frac{h^2}{2^2}\right)(h) \qquad \text{Power of a Quotient Property}$$

$$= \frac{\pi h^3}{4} \qquad \qquad \text{Simplify.}$$

h

$\frac{h}{2}$

Volume = ?

$2\pi r^3 \qquad \pi h^3 2^{-2} \qquad \pi h 4^{-1}$

$\dfrac{\pi h^2}{4} \qquad \dfrac{\pi h^3}{4} \qquad \dfrac{\pi h^3}{2}$

Any expression equivalent to $\dfrac{\pi h^3}{4}$ represents the volume of the cylinder.

- You can use the properties of exponents to write $\pi h^3 2^{-2}$ as $\dfrac{\pi h^3}{4}$.

- Note $h = 2r$. When you substitute $2r$ for h in $\dfrac{\pi h^3}{4}$, you can write $\dfrac{\pi (2r)^3}{4}$ as $2\pi r^3$.

- None of the other expressions are equivalent to $\dfrac{\pi h^3}{4}$.

▶ The expressions $2\pi r^3$, $\pi h^3 2^{-2}$, and $\dfrac{\pi h^3}{4}$ represent the volume of the cylinder.

EXAMPLE 6   **Solving a Real-Life Problem**

A jellyfish emits about 1.25×10^8 particles of light, or photons, in 6.25×10^{-4} second. How many photons does the jellyfish emit each second? Write your answer in scientific notation and in standard form.

SOLUTION

Divide to find the unit rate.

$$\frac{1.25 \times 10^8}{6.25 \times 10^{-4}} \qquad \begin{array}{l}\text{photons}\\\text{seconds}\end{array} \qquad \text{Write the rate.}$$

$$= \frac{1.25}{6.25} \times \frac{10^8}{10^{-4}} \qquad \text{Rewrite.}$$

$$= 0.2 \times 10^{12} \qquad \text{Simplify.}$$

$$= 2 \times 10^{11} \qquad \text{Write in scientific notation.}$$

▶ The jellyfish emits 2×10^{11}, or 200,000,000,000 photons per second.

REMEMBER

A number is written in scientific notation when it is of the form $a \times 10^b$, where $1 \le a < 10$ and b is an integer.

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15. Write two expressions that represent the area of a base of the cylinder in Example 5.

16. It takes the Sun about 2.3×10^8 years to orbit the center of the Milky Way. It takes Pluto about 2.5×10^2 years to orbit the Sun. How many times does Pluto orbit the Sun while the Sun completes one orbit around the center of the Milky Way? Write your answer in scientific notation.

Vocabulary and Core Concept Check

1. **VOCABULARY** Which definitions or properties would you use to simplify the expression $(4^8 \cdot 4^{-4})^{-2}$? Explain.

2. **WRITING** Explain when and how to use the Power of a Product Property.

3. **WRITING** Explain when and how to use the Quotient of Powers Property.

4. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

| Simplify $3^3 \cdot 3^6$. | Simplify 3^{3+6}. | Simplify $3^6 \cdot {}^3$. | Simplify $3^6 \cdot 3^3$. |

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, evaluate the expression.
(See Example 1.)

5. $(-7)^0$

6. 4^0

7. 5^{-4}

8. $(-2)^{-5}$

9. $\dfrac{2^{-4}}{4^0}$

10. $\dfrac{5^{-1}}{-9^0}$

11. $\dfrac{-3^{-3}}{6^{-2}}$

12. $\dfrac{(-8)^{-2}}{3^{-4}}$

In Exercises 13–22, simplify the expression. Write your answer using only positive exponents. *(See Example 2.)*

13. x^{-7}

14. y^0

15. $9x^0y^{-3}$

16. $15c^{-8}d^0$

17. $\dfrac{2^{-2}m^{-3}}{n^0}$

18. $\dfrac{10^0r^{-11}s}{3^2}$

19. $\dfrac{4^{-3}a^0}{b^{-7}}$

20. $\dfrac{p^{-8}}{7^{-2}q^{-9}}$

21. $\dfrac{2^2y^{-6}}{8^{-1}z^0x^{-7}}$

22. $\dfrac{13x^{-5}y^0}{5^{-3}z^{-10}}$

In Exercises 23–32, simplify the expression. Write your answer using only positive exponents. *(See Example 3.)*

23. $\dfrac{5^6}{5^2}$

24. $\dfrac{(-6)^8}{(-6)^5}$

25. $(-9)^2 \cdot (-9)^2$

26. $4^{-5} \cdot 4^5$

27. $(p^6)^4$

28. $(s^{-5})^3$

29. $6^{-8} \cdot 6^5$

30. $-7 \cdot (-7)^{-4}$

31. $\dfrac{x^5}{x^4} \cdot x$

32. $\dfrac{z^8 \cdot z^2}{z^5}$

33. **USING PROPERTIES** A microscope magnifies an object 10^5 times. The length of an object is 10^{-7} meter. What is its magnified length?

34. **USING PROPERTIES** The area of the rectangular computer chip is $112a^3b^2$ square microns. What is the length?

width = $8ab$ microns

ERROR ANALYSIS In Exercises 35 and 36, describe and correct the error in simplifying the expression.

35.
✗ $\quad 2^4 \cdot 2^5 = (2 \cdot 2)^{4+5}$
$\qquad\qquad = 4^9$

36.
✗ $\quad \dfrac{x^5 \cdot x^3}{x^4} = \dfrac{x^8}{x^4}$
$\qquad\qquad = x^{8/4}$
$\qquad\qquad = x^2$

In Exercises 37–44, simplify the expression. Write your answer using only positive exponents. *(See Example 4.)*
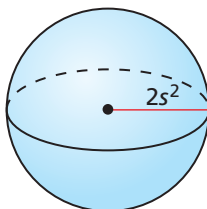
37. $(-5z)^3$

38. $(4x)^{-4}$

39. $\left(\dfrac{6}{n}\right)^{-2}$

40. $\left(\dfrac{-t}{3}\right)^2$

41. $(3s^8)^{-5}$

42. $(-5p^3)^3$

43. $\left(-\dfrac{w^3}{6}\right)^{-2}$

44. $\left(\dfrac{1}{2r^6}\right)^{-6}$

45. USING PROPERTIES
Which of the expressions represent the volume of the sphere? Explain. *(See Example 5.)*

$2s^2$

Ⓐ $\left(\dfrac{3s^2}{2^4\pi s^8}\right)^{-1}$

Ⓑ $(2^5\pi s^6)(3^{-1})$

Ⓒ $\dfrac{32\pi s^6}{3}$

Ⓓ $(2s)^5 \cdot \dfrac{\pi s}{3}$

Ⓔ $\left(\dfrac{3\pi s^6}{32}\right)^{-1}$

Ⓕ $\dfrac{32}{3}\pi s^5$

46. MODELING WITH MATHEMATICS Diffusion is the movement of molecules from one location to another. The time t (in seconds) it takes molecules to diffuse a distance of x centimeters is given by $t = \dfrac{x^2}{2D}$, where D is the diffusion coefficient. The diffusion coefficient for a drop of ink in water is about 10^{-5} square centimeters per second. How long will it take the ink to diffuse 1 micrometer (10^{-4} centimeter)?

In Exercises 47–50, simplify the expression. Write your answer using only positive exponents.

47. $\left(\dfrac{2x^{-2}y^3}{3xy^{-4}}\right)^4$

48. $\left(\dfrac{4s^5t^{-7}}{-2s^{-2}t^4}\right)^3$

49. $\left(\dfrac{3m^{-5}n^2}{4m^{-2}n^0}\right)^2 \cdot \left(\dfrac{mn^4}{9n}\right)^2$

50. $\left(\dfrac{3x^3y^0}{x^{-2}}\right)^4 \cdot \left(\dfrac{y^2x^{-4}}{5xy^{-8}}\right)^3$

In Exercises 51–54, evaluate the expression. Write your answer in scientific notation and standard form.

51. $(3 \times 10^2)(1.5 \times 10^{-5})$

52. $(6.1 \times 10^{-3})(8 \times 10^9)$

**53.** $\dfrac{(6.4 \times 10^7)}{(1.6 \times 10^5)}$

54. $\dfrac{(3.9 \times 10^{-5})}{(7.8 \times 10^{-8})}$

55. PROBLEM SOLVING In 2012, on average, about 9.46×10^{-1} pound of potatoes was produced for every 2.3×10^{-5} acre harvested. How many pounds of potatoes on average were produced for each acre harvested? Write your answer in scientific notation and in standard form. *(See Example 6.)*

56. PROBLEM SOLVING The speed of light is approximately 3×10^5 kilometers per second. How long does it take sunlight to reach Jupiter? Write your answer in scientific notation and in standard from.

OUR SOLAR SYSTEM
Average Distance: 7.8×10^8 kilometers

57. MATHEMATICAL CONNECTIONS Consider Cube A and Cube B.

A
$2x$
B
$6x$
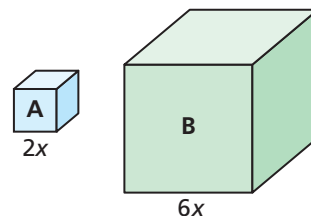
a. Which property of exponents should you use to simplify an expression for the volume of each cube?

b. How can you use the Power of a Quotient Property to find how many times greater the volume of Cube B is than the volume of Cube A?

58. PROBLEM SOLVING A byte is a unit used to measure a computer's memory. The table shows the numbers of bytes in several units of measure.

Unit	kilobyte	megabyte	gigabyte	terabyte
Number of bytes	2^{10}	2^{20}	2^{30}	2^{40}

a. How many kilobytes are in 1 terabyte?

b. How many megabytes are in 16 gigabytes?

c. Another unit used to measure a computer's memory is a bit. There are 8 bits in a byte. How can you convert the number of bytes in each unit of measure given in the table to bits? Can you still use a base of 2? Explain.

**REWRITING EXPRESSIONS** In Exercises 59–62, rewrite the expression as a power of a product.

59. $8a^3b^3$

60. $16r^2s^2$

61. $64w^{18}z^{12}$

62. $81x^4y^8$

63. **USING STRUCTURE** The probability of rolling a 6 on a number cube is $\frac{1}{6}$. The probability of rolling a 6 twice in a row is $\left(\frac{1}{6}\right)^2 = \frac{1}{36}$.

   a. Write an expression that represents the probability of rolling a 6 n times in a row.

   b. What is the probability of rolling a 6 four times in a row?

   c. What is the probability of flipping heads on a coin five times in a row? Explain.

64. **HOW DO YOU SEE IT?** The shaded part of Figure n represents the portion of a piece of paper visible after folding the paper in half n times.

Figure 1   Figure 2

Figure 3   Figure 4
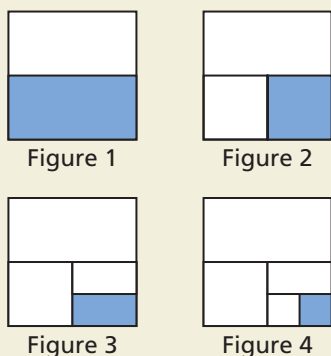
   a. What fraction of the original piece of paper is each shaded part?

   b. Rewrite each fraction from part (a) in the form 2^x.

65. **REASONING** Find x and y when $\dfrac{b^x}{b^y} = b^9$ and $\dfrac{b^x \cdot b^2}{b^{3y}} = b^{13}$. Explain how you found your answer.

66. **THOUGHT PROVOKING** Write expressions for r and h so that the volume of the cone can be represented by the expression $27\pi x^8$. Find r and h.
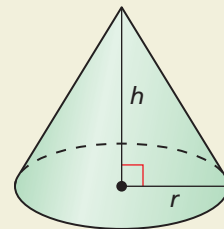
67. **MAKING AN ARGUMENT** One of the smallest plant seeds comes from an orchid, and one of the largest plant seeds comes from a double coconut palm. A seed from an orchid has a mass of 10^{-6} gram. The mass of a seed from a double coconut palm is 10^{10} times the mass of the seed from the orchid. Your friend says that the seed from the double coconut palm has a mass of about 1 kilogram. Is your friend correct? Explain.

68. **CRITICAL THINKING** Your school is conducting a survey. Students can answer the questions in either part with "agree" or "disagree."

Part 1: 13 questions		
Part 2: 10 questions		
Part 1: Classroom	Agree	Disagree
1. I come prepared for class.	○	○
2. I enjoy my assignments.	○	○

   a. What power of 2 represents the number of different ways that a student can answer all the questions in Part 1?

   b. What power of 2 represents the number of different ways that a student can answer all the questions on the entire survey?

   c. The survey changes, and students can now answer "agree," "disagree," or "no opinion." How does this affect your answers in parts (a) and (b)?

69. **ABSTRACT REASONING** Compare the values of a^n and a^{-n} when $n < 0$, when $n = 0$, and when $n > 0$ for (a) $a > 1$ and (b) $0 < a < 1$. Explain your reasoning.

Maintaining Mathematical Proficiency
Reviewing what you learned in previous grades and lessons

Find the square root(s). *(Skills Review Handbook)*

70. $\sqrt{25}$

71. $-\sqrt{100}$

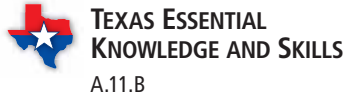**72.** $\pm\sqrt{\dfrac{1}{64}}$

Classify the real number in as many ways as possible. *(Skills Review Handbook)*

73. 12

74. $\dfrac{65}{9}$

75. $\dfrac{\pi}{4}$

6.2 Radicals and Rational Exponents

Essential Question How can you write and evaluate an nth root of a number?

Recall that you cube a number as follows.

3rd power

$$2^3 = 2 \cdot 2 \cdot 2 = 8 \qquad \text{2 cubed is 8.}$$

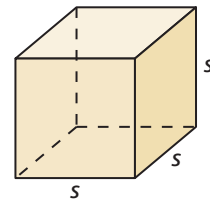To "undo" cubing a number, take the cube root of the number.

Symbol for cube root is $\sqrt[3]{}$.

$$\sqrt[3]{8} = \sqrt[3]{2^3} = 2 \qquad \text{The cube root of 8 is 2.}$$

EXPLORATION 1 Finding Cube Roots

Work with a partner. Use a cube root symbol to write the side length of each cube. Then find the cube root. Check your answers by multiplying. Which cube is the largest? Which two cubes are the same size? Explain your reasoning.

a. Volume = 27 ft³   **b.** Volume = 125 cm³   **c.** Volume = 3375 in.³

d. Volume = 3.375 m³   **e.** Volume = 1 yd³   **f.** Volume = $\dfrac{125}{8}$ mm³

JUSTIFYING THE SOLUTION

To be proficient in math, you need to justify your conclusions and communicate them to others.

EXPLORATION 2 Estimating nth Roots

Work with a partner. Estimate each positive nth root. Then match each nth root with the point on the number line. Justify your answers.

a. $\sqrt[4]{25}$ **b.** $\sqrt{0.5}$ **c.** $\sqrt[5]{2.5}$

d. $\sqrt[3]{65}$ **e.** $\sqrt[3]{55}$ **f.** $\sqrt[6]{20{,}000}$

Communicate Your Answer

3. How can you write and evaluate an nth root of a number?

4. The body mass m (in kilograms) of a dinosaur that walked on two feet can be modeled by

$$m = (0.00016)C^{2.73}$$

where C is the circumference (in millimeters) of the dinosaur's femur. The mass of a *Tyrannosaurus rex* was 4000 kilograms. Use a calculator to approximate the circumference of its femur.

6.2 Lesson

Core Vocabulary

*n*th root of *a*, p. 286
radical, p. 286
index of a radical, p. 286

Previous
square root

What You Will Learn

▶ Find *n*th roots.

▶ Evaluate expressions with rational exponents.

▶ Solve real-life problems involving rational exponents.

Finding *n*th Roots

You can extend the concept of a square root to other types of roots. For example, 2 is a cube root of 8 because $2^3 = 8$, and 3 is a fourth root of 81 because $3^4 = 81$. In general, for an integer *n* greater than 1, if $b^n = a$, then *b* is an **nth root of *a***. An *n*th root of *a* is written as $\sqrt[n]{a}$, where the expression $\sqrt[n]{a}$ is called a **radical** and *n* is the **index** of the radical.

You can also write an *n*th root of *a* as a power of *a*. If you assume the Power of a Power Property applies to rational exponents, then the following is true.

$$(a^{1/2})^2 = a^{(1/2) \cdot 2} = a^1 = a$$
$$(a^{1/3})^3 = a^{(1/3) \cdot 3} = a^1 = a$$
$$(a^{1/4})^4 = a^{(1/4) \cdot 4} = a^1 = a$$

Because $a^{1/2}$ is a number whose square is *a*, you can write $\sqrt{a} = a^{1/2}$. Similarly, $\sqrt[3]{a} = a^{1/3}$ and $\sqrt[4]{a} = a^{1/4}$. In general, $\sqrt[n]{a} = a^{1/n}$ for any integer *n* greater than 1.

READING

$\pm \sqrt[n]{a}$ represents both the positive and negative *n*th roots of *a*.

🄖 Core Concept

Real *n*th Roots of *a*

Let *n* be an integer greater than 1, and let *a* be a real number.

- If *n* is odd, then *a* has one real *n*th root: $\sqrt[n]{a} = a^{1/n}$

- If *n* is even and $a > 0$, then *a* has two real *n*th roots: $\pm \sqrt[n]{a} = \pm a^{1/n}$

- If *n* is even and $a = 0$, then *a* has one real *n*th root: $\sqrt[n]{0} = 0$

- If *n* is even and $a < 0$, then *a* has no real *n*th roots.

The *n*th roots of a number may be real numbers or *imaginary numbers*. You will study imaginary numbers in a future course.

EXAMPLE 1 Finding *n*th Roots

Find the indicated real *n*th root(s) of *a*.

a. $n = 3, a = -27$ **b.** $n = 4, a = 16$

SOLUTION

a. The index $n = 3$ is odd, so -27 has one real cube root. Because $(-3)^3 = -27$, the cube root of -27 is $\sqrt[3]{-27} = -3$, or $(-27)^{1/3} = -3$.

b. The index $n = 4$ is even, and $a > 0$. So, 16 has two real fourth roots. Because $2^4 = 16$ and $(-2)^4 = 16$, the fourth roots of 16 are $\pm\sqrt[4]{16} = \pm 2$, or $\pm 16^{1/4} = \pm 2$.

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Find the indicated real *n*th root(s) of *a*.

    1. $n = 3, a = -125$ **2.** $n = 6, a = 64$

Evaluating Expressions with Rational Exponents

Recall that the radical $\sqrt{a}$ indicates the positive square root of a. Similarly, an nth root of a, $\sqrt[n]{a}$, with an *even* index indicates the positive nth root of a.

REMEMBER

The expression under the radical sign is the radicand.

EXAMPLE 2   **Evaluating nth Root Expressions**

Evaluate each expression.

a. $\sqrt[3]{-8}$ b. $-\sqrt[3]{8}$ c. $16^{1/4}$ d. $(-16)^{1/4}$

SOLUTION

a. $\sqrt[3]{-8} = \sqrt[3]{(-2) \cdot (-2) \cdot (-2)}$ Rewrite the expression showing factors.

       $= -2$ Evaluate the cube root.

b. $-\sqrt[3]{8} = -\left(\sqrt[3]{2 \cdot 2 \cdot 2}\right)$ Rewrite the expression showing factors.

       $= -(2)$ Evaluate the cube root.

       $= -2$ Simplify.

c. $16^{1/4} = \sqrt[4]{16}$ Rewrite the expression in radical form.

       $= \sqrt[4]{2 \cdot 2 \cdot 2 \cdot 2}$ Rewrite the expression showing factors.

       $= 2$ Evaluate the fourth root.

d. $(-16)^{1/4}$ is not a real number because there is no real number that can be multiplied by itself four times to produce -16.

A rational exponent does not have to be of the form $1/n$. Other rational numbers such as 3/2 can also be used as exponents. You can use the properties of exponents to evaluate or simplify expressions involving rational exponents.

STUDY TIP

You can rewrite $27^{2/3}$ as $27^{(1/3) \cdot 2}$ and then use the Power of a Power Property to show that

$27^{(1/3) \cdot 2} = (27^{1/3})^2$.

🅖 Core Concept

Rational Exponents

Let $a^{1/n}$ be an nth root of a, and let m be a positive integer.

Algebra $a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$

Numbers $27^{2/3} = (27^{1/3})^2 = \left(\sqrt[3]{27}\right)^2$

EXAMPLE 3   **Evaluating Expressions with Rational Exponents**

Evaluate (a) $16^{3/4}$ and (b) $27^{4/3}$.

SOLUTION

a. $16^{3/4} = (16^{1/4})^3$ Rational exponents   b. $27^{4/3} = (27^{1/3})^4$

       $= 2^3$ Evaluate the nth root. $= 3^4$

       $= 8$ Evaluate the power. $= 81$

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Evaluate the expression.

3. $\sqrt[3]{-125}$ **4.** $(-64)^{2/3}$ **5.** $9^{5/2}$ **6.** $256^{3/4}$

EXAMPLE 4 **Simplifying Expressions with Rational Exponents**

a. $27^{4/3} \cdot 27^{1/3} = 27^{4/3 + 1/3}$ Product of Powers Property

$\qquad\qquad\qquad = 27^{5/3}$ Simplify.

$\qquad\qquad\qquad = (27^{1/3})^5$ Definition of rational exponent

$\qquad\qquad\qquad = 3^5$ Evaluate the nth root.

$\qquad\qquad\qquad = 243$ Evaluate the power.

b. $x^{1/2} \div x^{3/5} = x^{1/2 - 3/5}$ Quotient of Powers Property

$\qquad\qquad\quad = x^{5/10 - 6/10}$ Rewrite fractions with common denominator.

$\qquad\qquad\quad = x^{-1/10}$ Simplify.

$\qquad\qquad\quad = \dfrac{1}{x^{1/10}}$ Definition of negative exponent

Solving Real-Life Problems

EXAMPLE 5   **Solving a Real-Life Problem**

Volume = 113 cubic feet

The radius r of a sphere is given by the equation $r = \left(\dfrac{3V}{4\pi}\right)^{1/3}$, where V is the volume of the sphere. Find the radius of the beach ball to the nearest foot. Use 3.14 for π.

SOLUTION

1. **Understand the Problem** You know the equation that represents the radius of a sphere in terms of its volume. You are asked to find the radius for a given volume.

2. **Make a Plan** Substitute the given volume into the equation. Then evaluate to find the radius.

3. **Solve the Problem**

$r = \left(\dfrac{3V}{4\pi}\right)^{1/3}$ Write the equation.

$\ = \left(\dfrac{3(113)}{4(3.14)}\right)^{1/3}$ Substitute 113 for V and 3.14 for π.

$\ = \left(\dfrac{339}{12.56}\right)^{1/3}$ Multiply.

$\ \approx 3$ Use a calculator.

▶ The radius of the beach ball is about 3 feet.

4. **Look Back** To check that your answer is reasonable, compare the size of the ball to the size of the woman pushing the ball. The ball appears to be slightly taller than the woman. The average height of a woman is between 5 and 6 feet. So, a radius of 3 feet, or height of 6 feet, seems reasonable for the beach ball.

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Simplify the expression. Write your answer using only positive exponents.

7. $12^{5/2} \div 12^{1/2}$ **8.** $16^{-3/4} \cdot 16^{1/4}$ **9.** $2y^{2/3} \cdot y^{5/6}$

10. WHAT IF? In Example 5, the volume of the beach ball is 17,000 cubic inches. Find the radius to the nearest inch. Use 3.14 for π.

Vocabulary and Core Concept Check

1. **WRITING** Explain how to evaluate $81^{1/4}$.

2. **WHICH ONE DOESN'T BELONG?** Which expression does *not* belong with the other three? Explain your reasoning.

$$\left(\sqrt[3]{27}\right)^2 \qquad 27^{2/3} \qquad 3^2 \qquad \left(\sqrt[2]{27}\right)^3$$

Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, rewrite the expression in rational exponent form.

3. $\sqrt{10}$

4. $\sqrt[5]{34}$

In Exercises 5 and 6, rewrite the expression in radical form.

5. $15^{1/3}$

6. $140^{1/8}$

In Exercises 7–10, find the indicated real *n*th root(s) of *a*. *(See Example 1.)*

7. $n = 2, a = 36$

8. $n = 4, a = 81$

9. $n = 3, a = 1000$

10. $n = 9, a = -512$

MATHEMATICAL CONNECTIONS In Exercises 11 and 12, find the dimensions of the cube. Check your answer.

11. Volume = 64 in.³
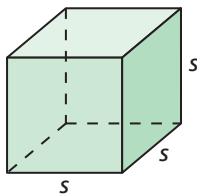
12. Volume = 216 cm³
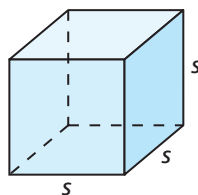
In Exercises 13–18, evaluate the expression. *(See Example 2.)*

13. $\sqrt[4]{256}$

14. $\sqrt[3]{-216}$

15. $\sqrt[3]{-343}$

16. $-\sqrt[5]{1024}$

17. $128^{1/7}$

18. $(-64)^{1/2}$

In Exercises 19 and 20, rewrite the expression in rational exponent form.

19. $\left(\sqrt[5]{8}\right)^4$

20. $\left(\sqrt[5]{-21}\right)^6$

In Exercises 21 and 22, rewrite the expression in radical form.

21. $(-4)^{2/7}$

22. $9^{5/2}$

In Exercises 23–28, evaluate the expression. *(See Example 3.)*

23. $32^{3/5}$

24. $125^{2/3}$

25. $(-36)^{3/2}$

26. $(-243)^{2/5}$

27. $(-128)^{5/7}$

28. $343^{4/3}$

29. **ERROR ANALYSIS** Describe and correct the error in rewriting the expression in rational exponent form.

$$✗ \qquad \left(\sqrt[3]{2}\right)^4 = 2^{3/4}$$

30. **ERROR ANALYSIS** Describe and correct the error in evaluating the expression.

$$✗ \qquad (-81)^{3/4} = [(-81)^{1/4}]^3$$
$$= (-3)^3$$
$$= -27$$

In Exercises 31–34, evaluate the expression.

31. $\left(\dfrac{1}{1000}\right)^{1/3}$

32. $\left(\dfrac{1}{64}\right)^{1/6}$

33. $(27)^{-2/3}$

34. $(9)^{-5/2}$

35. PROBLEM SOLVING A math club is having a bake sale. Find the area of the bake sale sign.

$$4^{1/2} \text{ ft}$$

Math Club Bake Sale this Saturday

$$\sqrt[6]{729} \text{ ft}$$

36. PROBLEM SOLVING The volume of a cube-shaped box is 27^5 cubic millimeters. Find the length of one side of the box.

In Exercises 37–44, simplify the expression. Write your answer using only positive exponents. *(See Example 4.)*

37. $5^{-3/2} \cdot 5^{7/2}$

38. $2^{5/3} \div 2^{-4/3}$

39. $125^3 \div 125^{8/3}$

40. $\left(\frac{1}{16}\right)^{1/2} \cdot \left(\frac{1}{16}\right)^{3/4}$

41. $d^{3/4} \div d^{1/6}$

42. $z^{-7/5} \cdot z^{2/3}$

43. $4n^{1/8} \cdot n^{3/2}$

44. $a^{-3/4} \div 7a^{-1/12}$

45. MODELING WITH MATHEMATICS The radius r of the base of a cone is given by the equation

$$r = \left(\frac{3V}{\pi h}\right)^{1/2}$$
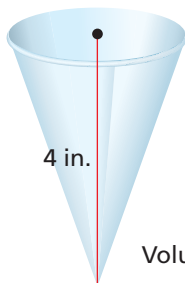
where V is the volume of the cone and h is the height of the cone. Find the radius of the paper cup to the nearest inch. Use 3.14 for π. *(See Example 5.)*

4 in.

Volume = 5 in.³

46. MODELING WITH MATHEMATICS The volume of a sphere is given by the equation $V = \dfrac{1}{6\sqrt{\pi}} S^{3/2}$, where S is the surface area of the sphere. Find the volume of a sphere, to the nearest cubic meter, that has a surface area of 60 square meters. Use 3.14 for π.

47. WRITING Explain how to write $(\sqrt[n]{a})^m$ in rational exponent form.

48. HOW DO YOU SEE IT? Write an expression in rational exponent form that represents the side length of the square.

Area = x in.²
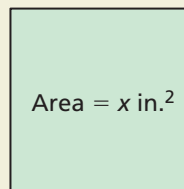
49. REASONING For what values of x is $x = x^{1/5}$?

50. MAKING AN ARGUMENT Your friend says that for a real number a and a positive integer n, the value of $\sqrt[n]{a}$ is always positive and the value of $-\sqrt[n]{a}$ is always negative. Is your friend correct? Explain.

In Exercises 51–54, simplify the expression.

51. $(y^{1/6})^3 \cdot \sqrt{x}$

52. $(y \cdot y^{1/3})^{3/2}$

53. $x \cdot \sqrt[3]{y^6} + y^2 \cdot \sqrt[3]{x^3}$

54. $(x^{1/3} \cdot y^{1/2})^9 \cdot \sqrt{y}$

55. PROBLEM SOLVING The formula for the volume of a regular dodecahedron is $V \approx 7.66\,\ell^3$, where ℓ is the length of an edge. The volume of the dodecahedron is 20 cubic feet. Estimate the edge length.

ℓ
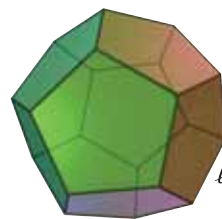
56. THOUGHT PROVOKING Find a formula (for instance, from geometry or physics) that contains a radical. Rewrite the formula using rational exponents.

ABSTRACT REASONING In Exercises 57–62, let x be a nonnegative real number. Determine whether the statement is *always*, *sometimes*, or *never* true. Justify your answer.

57. $(x^{1/3})^3 = x$

58. $x^{1/3} = x^{-3}$

59. $x^{1/3} = \sqrt[3]{x}$

60. $x^{1/3} = x^3$

61. $\dfrac{x^{2/3}}{x^{1/3}} = \sqrt[3]{x}$

62. $x = x^{1/3} \cdot x^3$

─ **Maintaining Mathematical Proficiency** Reviewing what you learned in previous grades and lessons

Evaluate the function when $x = -3, 0,$ and 8. *(Section 3.3)*

63. $f(x) = 2x - 10$

64. $w(x) = -5x - 1$

65. $h(x) = 13 - x$

66. $g(x) = 8x + 16$

6.3   Exponential Functions

Essential Question   What are some of the characteristics of the graph of an exponential function?

EXPLORATION 1   **Exploring an Exponential Function**

Work with a partner. Copy and complete each table for the *exponential function* $y = 16(2)^x$. In each table, what do you notice about the values of x? What do you notice about the values of y?

x	$y = 16(2)^x$
0	
1	
2	
3	
4	
5	

x	$y = 16(2)^x$
0	
2	
4	
6	
8	
10	

JUSTIFYING THE SOLUTION

To be proficient in math, you need to justify your conclusions and communicate them to others.

EXPLORATION 2   **Exploring an Exponential Function**

Work with a partner. Repeat Exploration 1 for the exponential function $y = 16\left(\frac{1}{2}\right)^x$. Do you think the statement below is true for *any* exponential function? Justify your answer.

"As the independent variable x changes by a constant amount, the dependent variable y is multiplied by a constant factor."

EXPLORATION 3   **Graphing Exponential Functions**

Work with a partner. Sketch the graphs of the functions given in Explorations 1 and 2. How are the graphs similar? How are they different?

Communicate Your Answer

4. What are some of the characteristics of the graph of an exponential function?

5. Sketch the graph of each exponential function. Does each graph have the characteristics you described in Question 4? Explain your reasoning.

   a. $y = 2^x$
   b. $y = 2(3)^x$
   c. $y = 3(1.5)^x$
   d. $y = \left(\frac{1}{2}\right)^x$
   e. $y = 3\left(\frac{1}{2}\right)^x$
   f. $y = 2\left(\frac{3}{4}\right)^x$

What You Will Learn

▶ Identify and evaluate exponential functions.

▶ Graph exponential functions.

▶ Solve real-life problems involving exponential functions.

Core Vocabulary

exponential function, *p. 292*
asymptote, *p. 293*

Previous
independent variable
dependent variable
parent function

Identifying and Evaluating Exponential Functions

An **exponential function** is a nonlinear function of the form $y = ab^x$, where $a \neq 0$, $b \neq 1$, and $b > 0$. As the independent variable x changes by a constant amount, the dependent variable y is multiplied by a constant factor, which means consecutive y-values form a constant ratio.

EXAMPLE 1 Identifying Functions

Does each table represent a *linear* or an *exponential* function? Explain.

a.

x	0	1	2	3
y	2	4	6	8

b.

x	0	1	2	3
y	4	8	16	32

SOLUTION

STUDY TIP

In Example 1b, consecutive y-values form a constant ratio.

$$\frac{8}{4} = 2, \frac{16}{8} = 2, \frac{32}{16} = 2$$

a.

+1 +1 +1

x	0	1	2	3
y	2	4	6	8

+2 +2 +2

▶ As x increases by 1, y increases by 2. The rate of change is constant. So, the function is linear.

b.

+1 +1 +1

x	0	1	2	3
y	4	8	16	32

×2 ×2 ×2

▶ As x increases by 1, y is multiplied by 2. So, the function is exponential.

EXAMPLE 2 Evaluating Exponential Functions

Evaluate each function for the given value of x.

a. $y = -2(5)^x$; $x = 3$

b. $y = 3(0.5)^x$; $x = -2$

SOLUTION

a. $y = -2(5)^x$ Write the function.

$\quad = -2(5)^3$ Substitute for x.

$\quad = -2(125)$ Evaluate the power.

$\quad = -250$ Multiply.

b. $y = 3(0.5)^x$

$\quad = 3(0.5)^{-2}$

$\quad = 3(4)$

$\quad = 12$

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Does the table represent a *linear* or an *exponential* function? Explain.

1.

x	0	1	2	3
y	8	4	2	1

2.

x	−4	0	4	8
y	1	0	−1	−2

Evaluate the function when $x = -2, 0,$ and $\frac{1}{2}$.
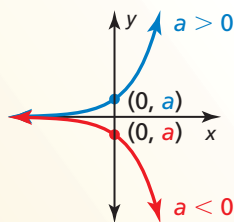
3. $y = 2(9)^x$

4. $y = 1.5(2)^x$

Graphing Exponential Functions

The graph of a function $y = ab^x$ is a vertical stretch or shrink by a factor of $|a|$ of the graph of the parent function $y = b^x$. When $a < 0$, the graph is also reflected in the x-axis. The y-intercept of the graph of $y = ab^x$ is a.

Core Concept

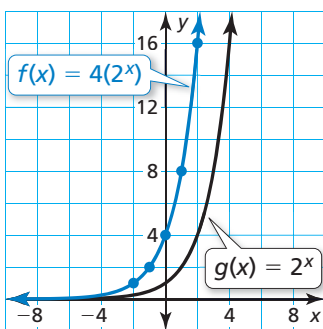**Graphing $y = ab^x$ When $b > 1$ Graphing $y = ab^x$ When $0 < b < 1$**

The x-axis is an *asymptote* of the graph of $y = ab^x$. An **asymptote** is a line that a graph approaches but never intersects.
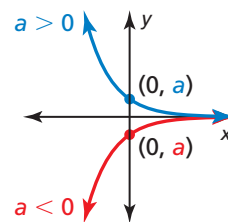
EXAMPLE 3   Graphing $y = ab^x$ When $b > 1$

Graph $f(x) = 4(2)^x$. Compare the graph to the graph of the parent function. Identify the y-intercepts and asymptotes of the graphs. Describe the domain and range of f.
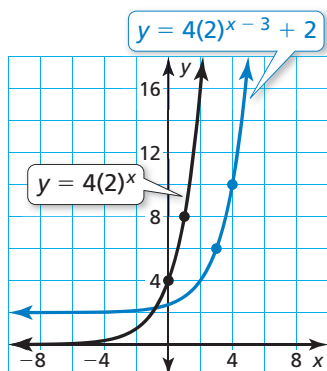
SOLUTION

Step 1 Make a table of values.

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.

x	−2	−1	0	1	2
f(x)	1	2	4	8	16

▶ The parent function is $g(x) = 2^x$. The graph of f is a vertical stretch by a factor of 4 of the graph of g. The y-intercept of the graph of f, 4, is above the y-intercept of the graph of g, 1. The x-axis is an asymptote of both the graphs of f and g. From the graph of f, you can see that the domain is all real numbers and the range is $y > 0$.

EXAMPLE 4   Graphing $y = ab^x$ When $0 < b < 1$

Graph $f(x) = -\left(\frac{1}{2}\right)^x$. Compare the graph to the graph of the parent function. Identify the y-intercepts and asymptotes of the graphs. Describe the domain and range of f.

SOLUTION

Step 1 Make a table of values.

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.

x	−2	−1	0	1	2
f(x)	−4	−2	−1	$-\frac{1}{2}$	$-\frac{1}{4}$
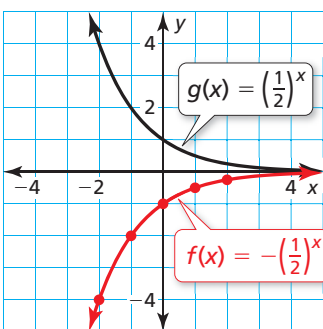
▶ The parent function is $g(x) = \left(\frac{1}{2}\right)^x$. The graph of f is a reflection in the x-axis of the graph of g. The y-intercept of the graph of f, -1, is below the y-intercept of the graph of g, 1. The x-axis is an asymptote of both the graphs of f and g. From the graph of f, you can see that the domain is all real numbers and the range is $y < 0$.

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Graph the function. Compare the graph to the graph of the parent function. Identify the y-intercepts and asymptotes of the graphs. Describe the domain and range of f.

5. $f(x) = -2(4)^x$

6. $f(x) = 2\left(\frac{1}{4}\right)^x$

To graph a function of the form $y = ab^{x - h} + k$, begin by graphing $y = ab^x$. Then translate the graph horizontally h units and vertically k units.

EXAMPLE 5 Graphing $y = ab^{x - h} + k$

Graph $y = 4(2)^{x - 3} + 2$. Identify the asymptote. Describe the domain and range.

SOLUTION

$y = 4(2)^{x - 3} + 2$

$y = 4(2)^x$

Step 1 Graph $y = 4(2)^x$. This is the same function that is in Example 3, which passes through $(0, 4)$ and $(1, 8)$.

Step 2 Translate the graph 3 units right and 2 units up. The graph passes through $(3, 6)$ and $(4, 10)$.

Notice that the graph approaches the line $y = 2$ but does not intersect it.

▶ So, the graph has an asymptote at $y = 2$. From the graph, you can see that the domain is all real numbers and the range is $y > 2$.

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Graph the function. Identify the asymptote. Describe the domain and range.

7. $y = -2(3)^{x + 2} - 1$

8. $f(x) = (0.25)^x + 3$

Solving Real-Life Problems

For an exponential function of the form $y = ab^x$, the y-values change by a factor of b as x increases by 1. You can use this fact to write an exponential function when you know the y-intercept, a.

EXAMPLE 6 Modeling with Mathematics
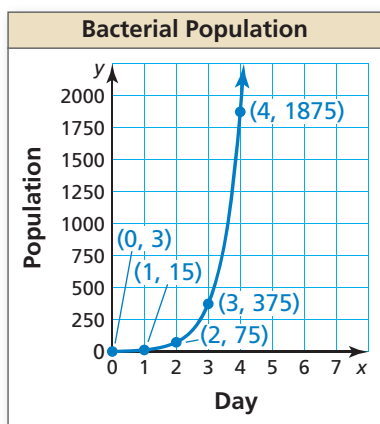
The graph represents a bacterial population y after x days.

Bacterial Population

(4, 1875)
(0, 3)
(1, 15)
(3, 375)
(2, 75)

Day

a. Write an exponential function that represents the population. Identify and interpret the y-intercept.

b. Describe the domain and range of the function.

c. Find the population after 5 days.

SOLUTION

a. Use the graph to make a table of values. The y-intercept is 3. The y-values increase by a factor of 5 as x increases by 1.

	+1	+1	+1	+1
x	0	1	2	3	4
y	3	15	75	375	1875
	×5	×5	×5	×5

▶ So, the population can be modeled by $y = 3(5)^x$. The y-intercept of 3 means that the initial bacterial population is 3.

b. From the graph, you can see that the domain is $x \geq 0$ and the range is $y \geq 3$. Note that $x \geq 0$ because the number of days cannot be negative.

c. Substitute 5 for x in the function $y = 3(5)^x$.

▶ After 5 days, there are $y = 3(5)^5 = 3(3125) = 9375$ bacteria.

You learned to use the *linear regression* feature of a graphing calculator to find an equation of the line of best fit in Section 4.6. Similarly, you can use exponential regression to find an exponential function that fits a data set.

Writing an Exponential Function Using Technology

The table shows the time x (in minutes) since a cup of hot coffee was poured and the temperature y (in degrees Fahrenheit) of the coffee. (a) Use a graphing calculator to find an exponential function that fits the data. Then plot the data and graph the function in the same viewing window. (b) Predict the temperature of the coffee 10 minutes after it is poured.

Time, x	Temperature, y
0	175
1	156
2	142
3	127
4	113
5	101
6	94
7	84
8	75

SOLUTION

a. Step 1   Enter the data from the table into two lists.

L1	L2	L3
0	175	------
1	156	
2	142	
3	127	
4	113	
5	101	
6	94	

L1(1)=0

Step 2   Use the *exponential regression* feature. The values in the equation can be rounded to obtain $y = 174(0.9)^x$.

```
ExpReg
y=a*b^x
a=173.9522643
b=.9003179174
r²=.9987302754
r=-.999364936
```

Step 3   Enter the equation $y = 174(0.9)^x$ into the calculator. Then plot the data and graph the equation in the same viewing window.
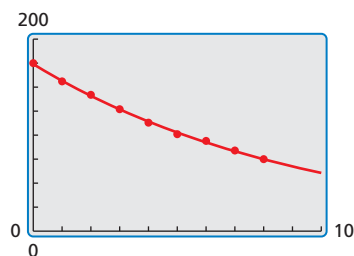
b. After 10 minutes, the temperature of the coffee will be about
$y = 174(0.9)^{10} \approx 60.7$ degrees Fahrenheit.

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9. A bacterial population y after x days can be represented by an exponential function whose graph passes through (0, 100) and (1, 200). (a) Write a function that represents the population. Identify and interpret the y-intercept. (b) Find the population after 6 days. (c) Does this bacterial population grow faster than the bacterial population in Example 6? Explain.

10. In Example 7, (a) identify and interpret the correlation coefficient and (b) predict the temperature of the coffee 15 minutes after it is poured.

Vocabulary and Core Concept Check

1. **OPEN-ENDED** Sketch an increasing exponential function whose graph has a *y*-intercept of 2.

2. **REASONING** Why is *a* the *y*-intercept of the graph of the function $y = ab^x$?

3. **WRITING** Compare the graph of $y = 2(5)^x$ with the graph of $y = 5^x$.

4. **WHICH ONE DOESN'T BELONG?** Which equation does *not* belong with the other three? Explain your reasoning.

$$y = 3^x \qquad f(x) = 2(4)^x \qquad f(x) = (-3)^x \qquad y = 5(3)^x$$

Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, determine whether the equation represents an exponential function. Explain.

5. $y = 4(7)^x$

6. $y = -6x$

7. $y = 2x^3$

8. $y = -3^x$

9. $y = 9(-5)^x$

10. $y = \frac{1}{2}(1)^x$

In Exercises 11–14, determine whether the table represents a *linear* or an *exponential* function. Explain. *(See Example 1.)*

11.

x	y
1	−2
2	0
3	2
4	4

12.

x	y
1	6
2	12
3	24
4	48

13.

x	−1	0	1	2	3
y	0.25	1	4	16	64

14.

x	−3	0	3	6	9
y	10	1	−8	−17	−26

In Exercises 15–20, evaluate the function for the given value of *x*. *(See Example 2.)*

15. $y = 3^x$; $x = 2$

16. $f(x) = 3(2)^x$; $x = -1$

17. $y = -4(5)^x$; $x = 2$

18. $f(x) = 0.5^x$; $x = -3$

19. $f(x) = \frac{1}{3}(6)^x$; $x = 3$

20. $y = \frac{1}{4}(4)^x$; $x = \frac{3}{2}$

USING STRUCTURE In Exercises 21–24, match the function with its graph.

21. $f(x) = 2(0.5)^x$

22. $y = -2(0.5)^x$

23. $y = 2(2)^x$

24. $f(x) = -2(2)^x$

A.

B.

C.

D.
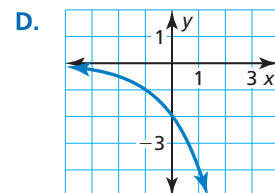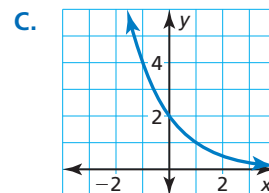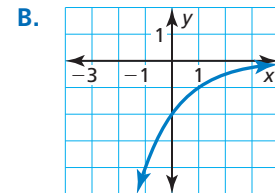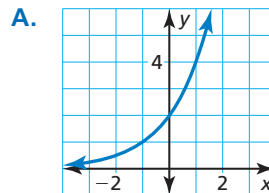
In Exercises 25–30, graph the function. Compare the graph to the graph of the parent function. Identify the *y*-intercepts and asymptotes of the graphs. Describe the domain and range of *f*. *(See Examples 3 and 4.)*

25. $f(x) = 3(0.5)^x$

26. $f(x) = -4^x$

27. $f(x) = -2(7)^x$

28. $f(x) = 6\left(\frac{1}{3}\right)^x$

29. $f(x) = \frac{1}{2}(8)^x$
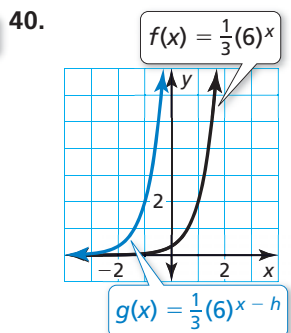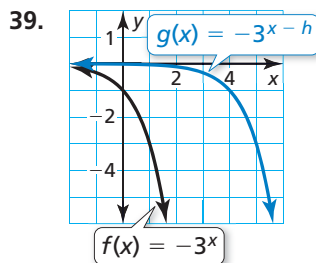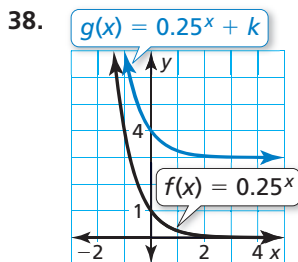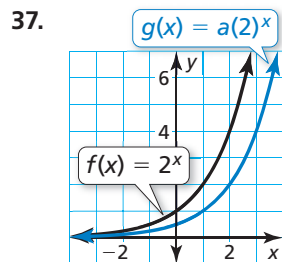
30. $f(x) = \frac{3}{2}(0.25)^x$

In Exercises 31–36, graph the function. Identify the asymptote. Describe the domain and range. *(See Example 5.)*

31. $f(x) = 3^x - 1$

32. $f(x) = 4^{x+3}$

33. $y = 5^{x-2} + 7$

34. $y = -\left(\frac{1}{2}\right)^{x+1} - 3$

35. $y = -8(0.75)^{x+2} - 2$

36. $f(x) = 3(6)^{x-1} - 5$

**In Exercises 37–40, compare the graphs. Find the value of h, k, or a.**

37.

$g(x) = a(2)^x$

$f(x) = 2^x$

38.

$g(x) = 0.25^x + k$

$f(x) = 0.25^x$

39.

$g(x) = -3^{x-h}$

$f(x) = -3^x$

40.

$f(x) = \frac{1}{3}(6)^x$

$g(x) = \frac{1}{3}(6)^{x-h}$

41. MODELING WITH MATHEMATICS The graph represents the amount of area visible y on an online map after you zoom out x times. *(See Example 6.)*

Online Map

(0, 20)
(1, 80)
(2, 320)
(3, 1280)

Zoom outs / Area visible (square miles)
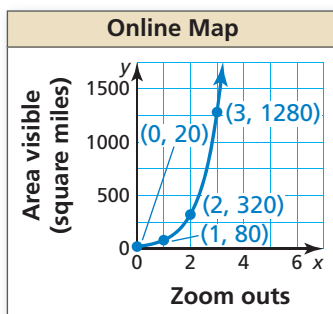
a. Write an exponential function that represents the area. Identify and interpret the y-intercept.

b. Describe the domain and range of the function.

c. Find the area visible after you zoom out 10 times.

42. MODELING WITH MATHEMATICS The graph represents the value y of a boat after x years.

Value of a Boat

(0, 30)
(1, 24)
(2, 19.2)
(3, 15.36)

Year / Value (thousands of dollars)
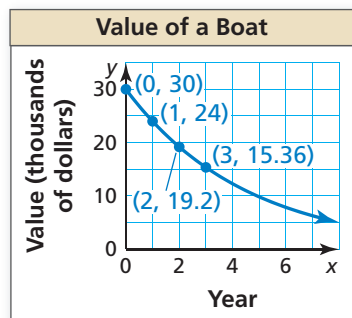
a. Write an exponential function that represents the value. Identify and interpret the y-intercept.

b. Describe the domain and range of the function.

c. Find the value of the boat after 8 years.

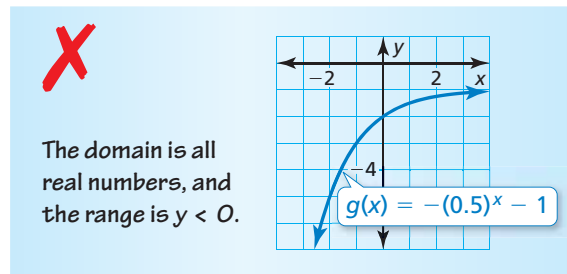**43. ERROR ANALYSIS** Describe and correct the error in evaluating the function.

$g(x) = 6(0.5)^x; x = -2$
$g(-2) = 6(0.5)^{-2}$
$= 3^{-2}$
$= \frac{1}{9}$

44. ERROR ANALYSIS Describe and correct the error in finding the domain and range of the function.

The domain is all real numbers, and the range is $y < 0$.

$g(x) = -(0.5)^x - 1$

45. MODELING WITH MATHEMATICS The table shows the number y of views an online video receives after being online for x days. *(See Example 7.)*

Day, x	0	1	2	3	4	5
Views, y	12	68	613	3996	27,810	205,017

a. Use a graphing calculator to find an exponential function that fits the data. Then plot the data and graph the function in the same viewing window.

b. Predict the number of views the video receives after being online for 7 days.

46. MODELING WITH MATHEMATICS The table shows the coyote population y in a national park after t decades.
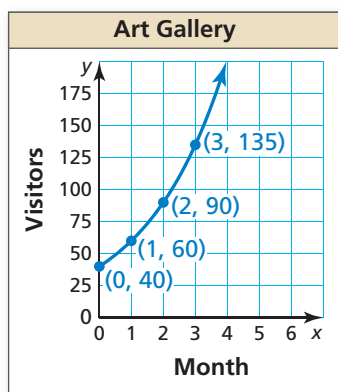
Decade, t	0	1	2	3	4
Population, y	15	26	41	72	123

a. Use a graphing calculator to find an exponential function that fits the data. Then plot the data and graph the function in the same viewing window.

b. Predict the coyote population after 60 years.

47. MODELING WITH MATHEMATICS The graph represents the number y of visitors to a new art gallery after x months.

Art Gallery

- **a.** Write an exponential function that represents this situation.
- **b.** Approximate the number of visitors after 5 months.

48. PROBLEM SOLVING A sales report shows that 3300 gas grills were purchased from a chain of hardware stores last year. The store expects grill sales to increase 6% each year. About how many grills does the store expect to sell in Year 6? Use an equation to justify your answer.

49. WRITING Graph the function $f(x) = -2^x$. Then graph $g(x) = -2^x - 3$. How are the y-intercept, domain, and range affected by the translation?

50. MAKING AN ARGUMENT Your friend says that the table represents an exponential function because y is multiplied by a constant factor. Is your friend correct? Explain.

x	0	1	3	6
y	2	10	50	250

51. WRITING Describe the effect of a on the graph of $y = a \cdot 2^x$ when a is positive and when a is negative.

52. OPEN-ENDED Write a function whose graph is a horizontal translation of the graph of $h(x) = 4^x$.

53. USING STRUCTURE The graph of g is a translation 4 units up and 3 units right of the graph of $f(x) = 5^x$. Write an equation for g.

54. HOW DO YOU SEE IT? The exponential function $y = V(x)$ represents the projected value of a stock x weeks after a corporation loses an important legal battle. The graph of the function is shown.
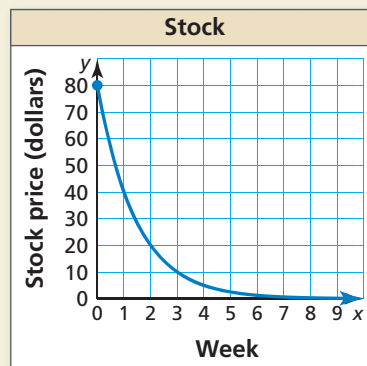
Stock

- **a.** After how many weeks will the stock be worth $20?
- **b.** Describe the change in the stock price from Week 1 to Week 3.

55. USING GRAPHS The graph represents the exponential function f. Find $f(7)$.
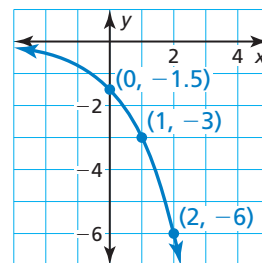
56. THOUGHT PROVOKING Write a function of the form $y = ab^x$ that represents a real-life population. Explain the meaning of each of the constants a and b in the real-life context.

57. REASONING Let $f(x) = ab^x$. Show that when x is increased by a constant k, the quotient $\dfrac{f(x + k)}{f(x)}$ is always the same regardless of the value of x.

58. PROBLEM SOLVING A function g models a relationship in which the dependent variable is multiplied by 4 for every 2 units the independent variable increases. The value of the function at 0 is 5. Write an equation that represents the function.

59. PROBLEM SOLVING Write an exponential function f so that the slope from the point $(0, f(0))$ to the point $(2, f(2))$ is equal to 12.

Maintaining Mathematical Proficiency
Reviewing what you learned in previous grades and lessons

Write the percent as a decimal. *(Skills Review Handbook)*

60. 4% **61.** 35% **62.** 128% **63.** 250%

Essential Question

What are some of the characteristics of exponential growth and exponential decay functions?

APPLYING MATHEMATICS

To be proficient in math, you need to apply the mathematics you know to solve problems arising in everyday life.

EXPLORATION 1 Predicting a Future Event

Work with a partner. It is estimated, that in 1782, there were about 100,000 nesting pairs of bald eagles in the United States. By the 1960s, this number had dropped to about 500 nesting pairs. In 1967, the bald eagle was declared an endangered species in the United States. With protection, the nesting pair population began to increase. Finally, in 2007, the bald eagle was removed from the list of endangered and threatened species.

Describe the pattern shown in the graph. Is it exponential growth? Assume the pattern continues. When will the population return to that of the late 1700s? Explain your reasoning.

Bald Eagle Nesting Pairs in Lower 48 States
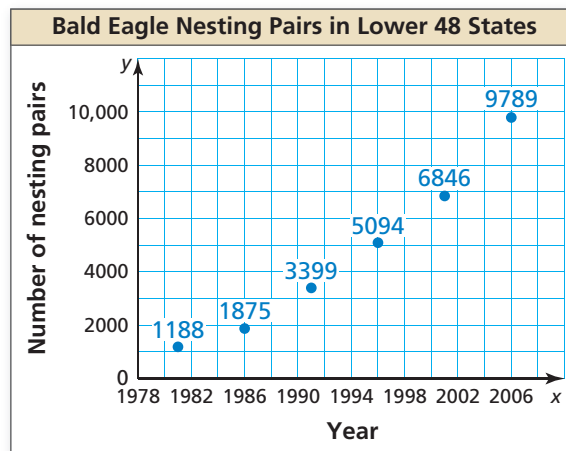
EXPLORATION 2 Describing a Decay Pattern

Work with a partner. A forensic pathologist was called to estimate the time of death of a person. At midnight, the body temperature was 80.5°F and the room temperature was a constant 60°F. One hour later, the body temperature was 78.5°F.

a. By what percent did the difference between the body temperature and the room temperature drop during the hour?

b. Assume that the original body temperature was 98.6°F. Use the percent decrease found in part (a) to make a table showing the decreases in body temperature. Use the table to estimate the time of death.

Communicate Your Answer

3. What are some of the characteristics of exponential growth and exponential decay functions?

4. Use the Internet or some other reference to find an example of each type of function. Your examples should be different than those given in Explorations 1 and 2.

a. exponential growth

b. exponential decay

Core Vocabulary

exponential growth, *p. 300*
exponential growth function,
    p. 300
exponential decay, *p. 301*
exponential decay function,
    p. 301
compound interest, *p. 303*

What You Will Learn

▶ Use and identify exponential growth and decay functions.
▶ Interpret and rewrite exponential growth and decay functions.
▶ Solve real-life problems involving exponential growth and decay.

Exponential Growth and Decay Functions

Exponential growth occurs when a quantity increases by the same factor over equal intervals of time.

Core Concept

Exponential Growth Functions

A function of the form $y = a(1 + r)^t$, where $a > 0$ and $r > 0$, is an **exponential growth function**.

initial amount   rate of growth (in decimal form)

final amount → $y = a(1 + r)^t$ ← time

growth factor

STUDY TIP

Notice that an exponential growth function is of the form $y = ab^x$, where b is replaced by $1 + r$ and x is replaced by t.
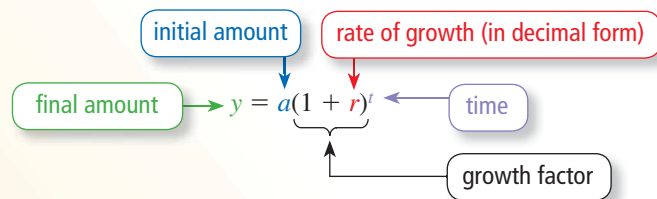
EXAMPLE 1 **Using an Exponential Growth Function**

The inaugural attendance of an annual music festival is 150,000. The attendance y increases by 8% each year.

a. Write an exponential growth function that represents the attendance after t years.

b. How many people will attend the festival in the fifth year? Round your answer to the nearest thousand.

SOLUTION

a. The initial amount is 150,000, and the rate of growth is 8%, or 0.08.

$y = a(1 + r)^t$ Write the exponential growth function.

$= 150,000(1 + 0.08)^t$ Substitute 150,000 for *a* and 0.08 for *r*.

$= 150,000(1.08)^t$ Add.

▶ The festival attendance can be represented by $y = 150,000(1.08)^t$.

b. The value $t = 4$ represents the fifth year because $t = 0$ represents the first year.

$y = 150,000(1.08)^t$ Write the exponential growth function.

$= 150,000(1.08)^4$ Substitute 4 for *t*.

$\approx 204,073$ Use a calculator.

▶ About 204,000 people will attend the festival in the fifth year.

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1. A website has 500,000 members in 2010. The number y of members increases by 15% each year. (a) Write an exponential growth function that represents the website membership t years after 2010. (b) How many members will there be in 2016? Round your answer to the nearest ten thousand.

Exponential decay occurs when a quantity decreases by the same factor over equal intervals of time.
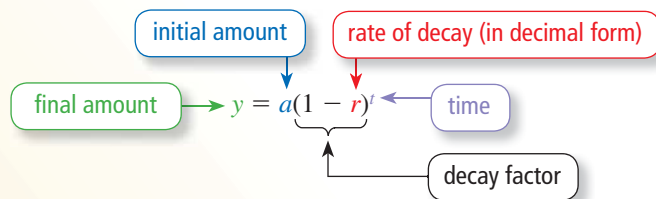
🌀 Core Concept

Exponential Decay Functions

A function of the form $y = a(1 - r)^t$, where $a > 0$ and $0 < r < 1$, is an **exponential decay function**.

initial amount | rate of decay (in decimal form)

final amount $\longrightarrow y = a(1 - r)^t \longleftarrow$ time

decay factor

STUDY TIP

Notice that an exponential decay function is of the form $y = ab^x$, where b is replaced by $1 - r$ and x is replaced by t.

For exponential growth, the value inside the parentheses is greater than 1 because r is added to 1. For exponential decay, the value inside the parentheses is less than 1 because r is subtracted from 1.

EXAMPLE 2 Identifying Exponential Growth and Decay

Determine whether each table represents an *exponential growth function*, an *exponential decay function*, or *neither*.

a.

x	y
0	270
1	90
2	30
3	10

b.

x	0	1	2	3
y	5	10	20	40

SOLUTION

a.

x	y
0	270
1	90
2	30
3	10

$+1$ ... $\times \frac{1}{3}$

▶ As x increases by 1, y is multiplied by $\frac{1}{3}$. So, the table represents an exponential decay function.

b.

$+1$ $+1$ $+1$

x	0	1	2	3
y	5	10	20	40

$\times 2$ $\times 2$ $\times 2$

▶ As x increases by 1, y is multiplied by 2. So, the table represents an exponential growth function.

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Determine whether the table represents an *exponential growth function*, an *exponential decay function*, or *neither*. **Explain.**

2.

x	0	1	2	3
y	64	16	4	1

3.

x	1	3	5	7
y	4	11	18	25

Interpreting and Rewriting Exponential Functions

EXAMPLE 3   **Interpreting Exponential Functions**

Determine whether each function represents *exponential growth* or *exponential decay*. Identify the initial amount and interpret the growth factor or decay factor.

a. The function $y = 0.5(1.07)^t$ represents the value y (in dollars) of a baseball card t years after it is issued.

b. The function $y = 128(0.5)^x$ represents the number y of players left in a video game tournament after x rounds.

STUDY TIP

You can rewrite exponential expressions and functions using the properties of exponents. Changing the form of an exponential function can reveal important attributes of the function.

SOLUTION

a. The function is of the form $y = a(1 + r)^t$, where $1 + r > 1$, so it represents exponential growth. The initial amount is \$0.50, and the growth factor of 1.07 means that the value of the baseball card increases by 7% each year.

b. The function is of the form $y = a(1 - r)^t$, where $1 - r < 1$, so it represents exponential decay. The initial amount is 128 players, and the decay factor of 0.5 means that 50% of the players are left after each round.

EXAMPLE 4   **Rewriting Exponential Functions**

Rewrite each function in the form $f(x) = ab^x$ to determine whether it represents *exponential growth* or *exponential decay*.

a. $f(x) = 100(0.96)^{x/4}$ **b.** $f(x) = (1.1)^{x-3}$

SOLUTION

a. $f(x) = 100(0.96)^{x/4}$ Write the function.

$ = 100(0.96^{1/4})^x$ Power of a Power Property

$ \approx 100(0.99)^x$ Evaluate the power.

▶ So, the function represents exponential decay.

b. $f(x) = (1.1)^{x-3}$ Write the function.

$ = \dfrac{(1.1)^x}{(1.1)^3}$ Quotient of Powers Property

$ \approx 0.75(1.1)^x$ Evaluate the power and simplify.

▶ So, the function represents exponential growth.

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4. The function $y = 10(1.12)^n$ represents the average game attendance y (in thousands of people) of a professional baseball team in its nth season. Determine whether the function represents *exponential growth* or *exponential decay*. Identify the initial amount and interpret the growth factor or decay factor.

Rewrite the function in the form $f(x) = ab^x$ to determine whether it represents *exponential growth* or *exponential decay*.

5. $f(x) = 3(1.02)^{10x}$ **6.** $f(x) = (0.95)^{x+2}$

Solving Real-Life Problems

Exponential growth functions are used in real-life situations involving *compound interest*. Although interest earned is expressed as an *annual* rate, the interest is usually compounded more frequently than once per year. So, the formula $y = a(1 + r)^t$ must be modified for compound interest problems.

G Core Concept

Compound Interest

Compound interest is the interest earned on the principal *and* on previously earned interest. The balance y of an account earning compound interest is

$$y = P\left(1 + \frac{r}{n}\right)^{nt}.$$

P = principal (initial amount)
r = annual interest rate (in decimal form)
t = time (in years)
n = number of times interest is compounded per year

EXAMPLE 5   Writing a Function

You deposit $100 in a savings account that earns 6% annual interest compounded monthly. Write a function that represents the balance after t years.

SOLUTION

$$y = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Write the compound interest formula.}$$

$$= 100\left(1 + \frac{0.06}{12}\right)^{12t} \qquad \text{Substitute 100 for } P, 0.06 \text{ for } r, \text{ and 12 for } n.$$

$$= 100(1.005)^{12t} \qquad \text{Simplify.}$$

EXAMPLE 6   Solving a Real-Life Problem

The table shows the balance of a money market account over time.

a. Write a function that represents the balance after t years.

b. Graph the functions from part (a) and from Example 5 in the same coordinate plane. Compare the account balances.

Year, t	Balance
0	$100
1	$110
2	$121
3	$133.10
4	$146.41
5	$161.05

SOLUTION

a. From the table, you know the initial balance is $100, and it increases 10% each year. So, $P = 100$ and $r = 0.1$.

$$y = P(1 + r)^t \qquad \text{Write the compound interest formula when } n = 1.$$

$$= 100(1 + 0.1)^t \qquad \text{Substitute 100 for } P \text{ and 0.1 for } r.$$

$$= 100(1.1)^t \qquad \text{Add.}$$

b. The money market account earns 10% interest each year, and the savings account earns 6% interest each year. So, the balance of the money market account increases faster.

Saving Money

$y = 100(1.1)^t$

$y = 100(1.005)^{12t}$

Balance (dollars) / Year
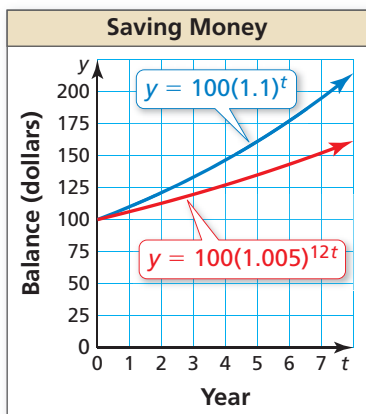
Monitoring Progress ◄)) Help in English and Spanish at *BigIdeasMath.com*

7. You deposit $500 in a savings account that earns 9% annual interest compounded monthly. Write and graph a function that represents the balance y (in dollars) after t years.

EXAMPLE 7 **Solving a Real-Life Problem**

The value of a car is $21,500. It loses 12% of its value every year. (a) Write a function that represents the value y (in dollars) of the car after t years. (b) Find the approximate monthly percent decrease in value. (c) Graph the function from part (a). Identify and interpret any asymptotes of the graph. (d) Estimate the value of the car after 6 years.

SOLUTION

1. **Understand the Problem** You know the value of the car and its annual percent decrease in value. You are asked to write a function that represents the value of the car over time and approximate the monthly percent decrease in value. Then graph the function and use the graph to estimate the value of the car in the future.

2. **Make a Plan** Use the initial amount and the annual percent decrease in value to write an exponential decay function. Rewrite the function using the properties of exponents to approximate the monthly percent decrease (rate of decay). Then graph the original function and use the graph to estimate the y-value when the t-value is 6.

3. **Solve the Problem**

   a. The initial value is $21,500, and the rate of decay is 12%, or 0.12.

   $$y = a(1 - r)^t \qquad \text{Write the exponential decay function.}$$
   $$= 21{,}500(1 - 0.12)^t \qquad \text{Substitute 21,500 for } a \text{ and 0.12 for } r.$$
   $$= 21{,}500(0.88)^t \qquad \text{Subtract.}$$

   ▶ The value of the car can be represented by $y = 21{,}500(0.88)^t$.

   b. Use the fact that $t = \frac{1}{12}(12t)$ and the properties of exponents to rewrite the function in a form that reveals the monthly rate of decay.

   $$y = 21{,}500(0.88)^t \qquad \text{Write the original function.}$$
   $$= 21{,}500(0.88)^{(1/12)(12t)} \qquad \text{Rewrite the exponent.}$$
   $$= 21{,}500(0.88^{1/12})^{12t} \qquad \text{Power of a Power Property}$$
   $$\approx 21{,}500(0.989)^{12t} \qquad \text{Evaluate the power.}$$

   Use the decay factor $1 - r \approx 0.989$ to find the rate of decay $r \approx 0.011$.

   ▶ So, the monthly percent decrease is about 1.1%.

   c. You can see that the graph approaches, but never intersects, the t-axis.

   ▶ So, the graph has an asymptote at $t = 0$. This makes sense because the car will never have a value of $0.

   d. From the graph, you can see that the y-value is about 10,000 when $t = 6$.

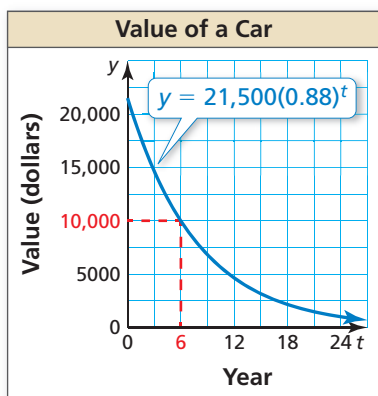   ▶ So, the value of the car is about $10,000 after 6 years.

4. **Look Back** To check that the monthly percent decrease is reasonable, multiply it by 12 to see if it is close in value to the annual percent decrease of 12%.

   $$1.1\% \times 12 = 13.2\% \qquad \text{13.2\% is close to 12\%, so 1.1\% is reasonable.}$$

   When you evaluate $y = 21{,}500(0.88)^t$ for $t = 6$, you get about $9985. So, $10,000 is a reasonable estimation.

Value of a Car

$y = 21{,}500(0.88)^t$

Value (dollars) vs Year

Monitoring Progress ◀)) Help in English and Spanish at *BigIdeasMath.com*

8. **WHAT IF?** The car loses 9% of its value every year. (a) Write a function that represents the value y (in dollars) of the car after t years. (b) Find the approximate monthly percent decrease in value. (c) Graph the function from part (a). Estimate the value of the car after 12 years. Round your answer to the nearest thousand.

STUDY TIP

In real life, the percent decrease in value of an asset is called the *depreciation rate*.

x
x

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** In the exponential growth function $y = a(1 + r)^t$, the quantity r is called the _____.

2. **VOCABULARY** What is the decay factor in the exponential decay function $y = a(1 - r)^t$?

3. **VOCABULARY** Compare exponential growth and exponential decay.

4. **WRITING** When does the function $y = ab^x$ represent exponential growth? exponential decay?

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, identify the initial amount a and the rate of growth r (as a percent) of the exponential function. Evaluate the function when $t = 5$. Round your answer to the nearest tenth.

5. $y = 350(1 + 0.75)^t$

6. $y = 10(1 + 0.4)^t$

7. $y = 25(1.2)^t$

8. $y = 12(1.05)^t$

9. $f(t) = 1500(1.074)^t$

10. $h(t) = 175(1.028)^t$

11. $g(t) = 6.72(2)^t$

12. $p(t) = 1.8^t$

In Exercises 13–16, write a function that represents the situation.

13. Sales of $10,000 increase by 65% each year.

14. Your starting annual salary of $35,000 increases by 4% each year.

15. A population of 210,000 increases by 12.5% each year.

16. An item costs $4.50, and its price increases by 3.5% each year.

17. **MODELING WITH MATHEMATICS** The population of a city has been increasing by 2% annually. The sign shown is from the year 2000. *(See Example 1.)*

    a. Write an exponential growth function that represents the population t years after 2000.

    CITY LIMIT
    BROOKFIELD
    POP. **315,000**

    b. What will the population be in 2020? Round your answer to the nearest thousand.

18. **MODELING WITH MATHEMATICS** A young channel catfish weighs about 0.1 pound. During the next 8 weeks, its weight increases by about 23% each week.

    a. Write an exponential growth function that represents the weight of the catfish after t weeks during the 8-week period.

    b. About how much will the catfish weigh after 4 weeks? Round your answer to the nearest thousandth.

In Exercises 19–26, identify the initial amount a and the rate of decay r (as a percent) of the exponential function. Evaluate the function when $t = 3$. Round your answer to the nearest tenth.

19. $y = 575(1 - 0.6)^t$

20. $y = 8(1 - 0.15)^t$

21. $g(t) = 240(0.75)^t$

22. $f(t) = 475(0.5)^t$

23. $w(t) = 700(0.995)^t$

24. $h(t) = 1250(0.865)^t$

25. $y = \left(\frac{7}{8}\right)^t$

26. $y = 0.5\left(\frac{3}{4}\right)^t$

In Exercises 27–30, write a function that represents the situation.

27. A population of 100,000 decreases by 2% each year.

28. A $900 sound system decreases in value by 9% each year.

29. A stock valued at $100 decreases in value by 9.5% each year.

30. A company profit of $20,000 decreases by 13.4% each year.

31. **ERROR ANALYSIS** The growth rate of a bacterial culture is 150% each hour. Initially, there are 10 bacteria. Describe and correct the error in finding the number of bacteria in the culture after 8 hours.

> ✗ $b(t) = 10(1.5)^t$
> $b(8) = 10(1.5)^8 \approx 256.3$
>
> After 8 hours, there are about 256 bacteria in the culture.

32. **ERROR ANALYSIS** You purchase a car in 2010 for $25,000. The value of the car decreases by 14% annually. Describe and correct the error in finding the value of the car in 2015.

> ✗ $v(t) = 25,000(1.14)^t$
> $v(4) = 25,000(1.14)^5 \approx 48,135$
>
> The value of the car in 2015 is about $48,000.

In Exercises 33–38, determine whether the table represents an *exponential growth function*, an *exponential decay function*, or *neither*. Explain. *(See Example 2.)*

33.

x	y
−1	50
0	10
1	2
2	0.4

34.

x	y
0	32
1	28
2	24
3	20

35.

x	y
0	35
1	29
2	23
3	17

36.

x	y
1	17
2	51
3	153
4	459

37.

x	y
5	2
10	8
15	32
20	128

38.

x	y
3	432
5	72
7	12
9	2

39. **ANALYZING RELATIONSHIPS** The table shows the value of a camper t years after it is purchased.

a. Determine whether the table represents an exponential growth function, an exponential decay function, or neither.

b. What is the value of the camper after 5 years?

t	Value
1	$37,000
2	$29,600
3	$23,680
4	$18,944

40. **ANALYZING RELATIONSHIPS** The table shows the total numbers of visitors to a website t days after it is online.

t	42	43	44	45
Visitors	11,000	12,100	13,310	14,641

a. Determine whether the table represents an exponential growth function, an exponential decay function, or neither.

b. How many people will have visited the website after it is online 47 days?

In Exercises 41–44, determine whether the function represents *exponential growth* or *exponential decay*. Identify the initial amount and interpret the growth factor or decay factor. *(See Example 3.)*

41. The function $y = 22(0.94)^t$ represents the population y (in thousands of people) of a town after t years.

42. The function $y = 13(1.2)^x$ represents the number y of customers at a store x days after the store first opens.

43. The function $y = 2^n$ represents the number y of sprints required n days since the start of a training routine.

44. The function $y = 900(0.85)^d$ represents the number y of students who are healthy d days after a flu outbreak.

In Exercises 45–52, rewrite the function in the form $f(x) = ab^x$ to determine whether it represents *exponential growth* or *exponential decay*. *(See Example 4.)*

45. $f(x) = (0.9)^{x-4}$

46. $f(x) = (1.4)^{x+8}$

47. $f(x) = 2(1.06)^{9x}$

48. $f(x) = 5(0.82)^{x/5}$

49. $f(x) = (1.45)^{x/2}$

50. $f(x) = 0.4(1.16)^{x-1}$

51. $f(x) = 4(0.55)^{x+3}$

52. $f(x) = (0.88)^{4x}$

In Exercises 53–56, write a function that represents the balance after t years. *(See Example 5.)*

53. $2000 deposit that earns 5% annual interest compounded quarterly

54. $1400 deposit that earns 10% annual interest compounded semiannually

55. $6200 deposit that earns 8.4% annual interest compounded monthly

56. $3500 deposit that earns 9.2% annual interest compounded quarterly

57. PROBLEM SOLVING The cross-sectional area of a tree 4.5 feet from the ground is called its *basal area*. The table shows the basal areas (in square inches) of Tree A over time. *(See Example 6.)*

Year, t	0	1	2	3	4
Basal area, A	120	132	145.2	159.7	175.7

Tree B
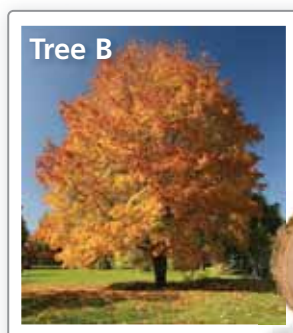
Growth rate: 6%

Initial basal area: 154 in.^{2}

a. Write functions that represent the basal areas of the trees after t years.

b. Graph the functions from part (a) in the same coordinate plane. Compare the basal areas.

58. PROBLEM SOLVING You deposit $300 into an investment account that earns 12% annual interest compounded quarterly. The graph shows the balance of a savings account over time.

a. Write functions that represent the balances of the accounts after t years.

b. Graph the functions from part (a) in the same coordinate plane. Compare the account balances.

Savings Account

(graph: Balance (dollars) vs. Year)
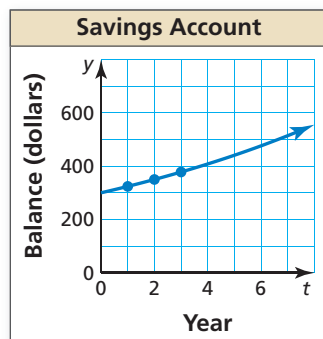
59. PROBLEM SOLVING A city has a population of 25,000. The population is expected to increase by 5.5% annually for the next decade. *(See Example 7.)*

a. Write a function that represents the population y after t years.

b. Find the approximate monthly percent increase in population.

c. Graph the function from part (a). Estimate the population after 4 years.

60. PROBLEM SOLVING Plutonium-238 is a material that generates steady heat due to decay and is used in power systems for some spacecraft. The function $y = a(0.5)^{t/x}$ represents the amount y of a substance remaining after t years, where a is the initial amount and x is the length of the half-life (in years).

Plutonium-238
Half-life ≈ 88 years

a. A scientist is studying a 3-gram sample. Write a function that represents the amount y of plutonium-238 after t years.

b. What is the yearly percent decrease of plutonium-238?

c. Graph the function from part (a). Identify and interpret any asymptotes of the graph.

d. Estimate the amount remaining after 12 years.

61. COMPARING FUNCTIONS The three given functions describe the amount y of ibuprofen (in milligrams) in a person's bloodstream t hours after taking the dosage.

$$y \approx 800(0.71)^t$$
$$y \approx 800(0.9943)^{60t}$$
$$y \approx 800(0.843)^{2t}$$

a. Show that these expressions are approximately equivalent.

b. Describe the information given by each of the functions.

62. COMBINING FUNCTIONS You deposit $9000 in a savings account that earns 3.6% annual interest compounded monthly. You also save $40 per month in a safe at home. Write a function $C(t) = b(t) + h(t)$, where $b(t)$ represents the balance of your savings account and $h(t)$ represents the amount in your safe after t years. What does $C(t)$ represent?

63. NUMBER SENSE During a flu epidemic, the number of sick people triples every week. What is the growth rate as a percent? Explain your reasoning.

64. HOW DO YOU SEE IT? Match each situation with its graph. Explain your reasoning.

   a. A bacterial population doubles each hour.

   b. The value of a computer decreases by 18% each year.

   c. A deposit earns 11% annual interest compounded yearly.

   d. A radioactive element decays 5.5% each year.
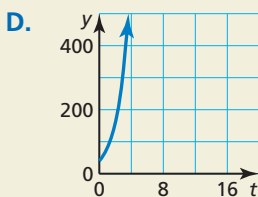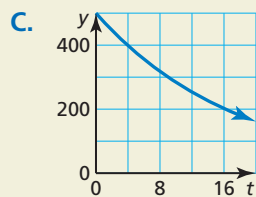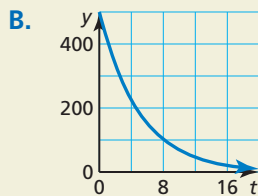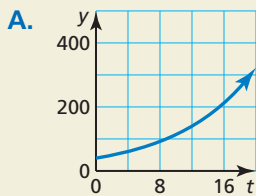
65. WRITING Give an example of an equation in the form $y = ab^x$ that does not represent an exponential growth function or an exponential decay function. Explain your reasoning.

66. THOUGHT PROVOKING Describe two account options into which you can deposit $1000 and earn compound interest. Write a function that represents the balance of each account after t years. Which account would you rather use? Explain your reasoning.

67. MAKING AN ARGUMENT A store is having a sale on sweaters. On the first day, the prices of the sweaters are reduced by 20%. The prices will be reduced another 20% each day until the sweaters are sold. Your friend says the sweaters will be free on the fifth day. Is your friend correct? Explain.

68. COMPARING FUNCTIONS The graphs of f and g are shown.
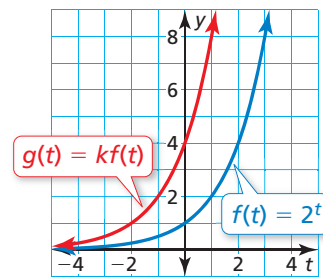
   a. Explain why f is an exponential growth function. Identify the rate of growth.

   b. Describe the transformation from the graph of f to the graph of g. Determine the value of k.

   c. The graph of g is the same as the graph of $h(t) = f(t + r)$. Use properties of exponents to find the value of r.

Maintaining Mathematical Proficiency
Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution. *(Section 1.3)*

69. $8x + 12 = 4x$

70. $5 - t = 7t + 21$

71. $6(r - 2) = 2r + 8$

Determine whether the sequence is arithmetic. If so, find the common difference. *(Section 4.7)*

72. $-20, -26, -32, -38, \ldots$

73. $9, 18, 36, 72, \ldots$

74. $-5, -8, -12, -17, \ldots$

75. $10, 20, 30, 40, \ldots$

6.1–6.4 What Did You Learn?

Core Vocabulary

nth root of a, *p. 286*
radical, *p. 286*
index of a radical, *p. 286*
exponential function, *p. 292*
asymptote, *p. 293*

exponential growth, *p. 300*
exponential growth function, *p. 300*
exponential decay, *p. 301*
exponential decay function, *p. 301*
compound interest, *p. 303*

Core Concepts

Section 6.1

Zero Exponent, *p. 278*
Negative Exponents, *p. 278*
Product of Powers Property, *p. 279*
Quotient of Powers Property, *p. 279*

Power of a Power Property, *p. 279*
Power of a Product Property, *p. 280*
Power of a Quotient Property, *p. 280*

Section 6.2

Real nth Roots of a, *p. 286*

Rational Exponents, *p. 287*

Section 6.3

Graphing $y = ab^x$ When $b > 1$, *p. 293*

Graphing $y = ab^x$ When $0 < b < 1$, *p. 293*

Section 6.4

Exponential Growth Functions, *p. 300*
Exponential Decay Functions, *p. 301*

Compound Interest, *p. 303*

Mathematical Thinking

1. How did you apply what you know to simplify the complicated situation in Exercise 56 on page 283?

2. How can you use previously established results to construct an argument in Exercise 50 on page 290?

3. How is the form of the function you wrote in Exercise 62 on page 308 related to the forms of other types of functions you have learned about in this course?

------------------------------ Study Skills ------------------------------

Analyzing Your Errors

Misreading Directions

- **What Happens:** You incorrectly read or do not understand directions.

- **How to Avoid This Error:** Read the instructions for exercises at least twice and make sure you understand what they mean. Make this a habit and use it when taking tests.

Simplify the expression. Write your answer using only positive exponents. *(Section 6.1)*

1. $3^2 \cdot 3^4$

2. $(k^4)^{-3}$

3. $\left(\dfrac{4r^2}{3s^5}\right)^3$

4. $\left(\dfrac{2x^0}{4x^{-2}y^4}\right)^2$

Evaluate the expression. *(Section 6.2)*

5. $\sqrt[3]{27}$

6. $\left(\dfrac{1}{16}\right)^{1/4}$

7. $512^{2/3}$

8. $\left(\sqrt{4}\right)^5$

Graph the function. Identify the asymptote. Describe the domain and range. *(Section 6.3)*

9. $y = 5^x$

10. $y = -2\left(\dfrac{1}{6}\right)^x$

11. $y = 6(2)^{x-4} - 1$

Determine whether the table represents an *exponential growth function*, an *exponential decay function*, or *neither*. Explain. *(Section 6.4)*

12.

x	0	1	2	3
y	7	21	63	189

13.

x	1	2	3	4
y	14,641	1331	121	11

Rewrite the function in the form $f(x) = ab^x$ to determine whether it represents *exponential growth* or *exponential decay*. *(Section 6.4)*

14. $f(x) = 3(1.6)^{x-1}$

15. $f(x) = \dfrac{1}{3}(0.96)^{4x}$

16. $f(x) = 80\left(\dfrac{4}{5}\right)^{x/2}$

17. The table shows several units of mass. *(Section 6.1)*

Unit of mass	kilogram	hectogram	dekagram	decigram	centigram	milligram	microgram	nanogram
Mass (in grams)	10^3	10^2	10^1	10^{-1}	10^{-2}	10^{-3}	10^{-6}	10^{-9}

    a. How many times larger is a kilogram than a nanogram? Write your answer using only positive exponents.

    b. How many times smaller is a milligram than a hectogram? Write your answer using only positive exponents.

    c. Which is greater, 10,000 milligrams or 1000 decigrams? Explain your reasoning.

18. You store blankets in a cedar chest. What is the volume of the cedar chest? *(Section 6.2)*

$243^{1/5}$ ft

$16^{3/4}$ ft

$\sqrt[6]{64}$ ft

19. The function $f(t) = 5(4)^t$ represents the number of frogs in a pond after t years. *(Section 6.3 and Section 6.4)*

    a. Does the function represent *exponential growth* or *exponential decay*? Explain.

    b. Identify the initial amount and interpret the growth factor or decay factor.

    c. Graph the function. Describe the domain and range.

    d. What is the approximate monthly percent change?

    e. How many frogs are in the pond after 4 years?

6.5 Geometric Sequences
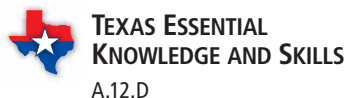
## Essential Question How can you use a geometric sequence to describe a pattern?

In a **geometric sequence**, the ratio between each pair of consecutive terms is the same. This ratio is called the **common ratio**.

EXPLORATION 1 Describing Calculator Patterns

Work with a partner. Enter the keystrokes on a calculator and record the results in the table. Describe the pattern.

a. Step 1   2   =

   Step 2   ×   2   =

    Step 3   ×   2   =

    Step 4   ×   2   =

    Step 5   ×   2   =

b. Step 1   6   4   =

    Step 2   ×   .   5   =

    Step 3   ×   .   5   =

    Step 4   ×   .   5   =

    Step 5   ×   .   5   =

Step	1	2	3	4	5
Calculator display					

Step	1	2	3	4	5
Calculator display					

c. Use a calculator to make your own sequence. Start with any number and multiply by 3 each time. Record your results in the table.

Step	1	2	3	4	5
Calculator display					

d. Part (a) involves a geometric sequence with a common ratio of 2. What is the common ratio in part (b)? part (c)?

EXPLORATION 2 Folding a Sheet of Paper

Work with a partner. A sheet of paper is about 0.1 millimeter thick.

a. How thick will it be when you fold it in half once? twice? three times?

b. What is the greatest number of times you can fold a piece of paper in half? How thick is the result?

c. Do you agree with the statement below? Explain your reasoning.

"If it were possible to fold the paper in half 15 times, it would be taller than you."

ANALYZING MATHEMATICAL RELATIONSHIPS

To be proficient in math, you need to notice when calculations are repeated and look both for general methods and for shortcuts.

Communicate Your Answer

3. How can you use a geometric sequence to describe a pattern?

4. Give an example of a geometric sequence from real life other than paper folding.

What You Will Learn

▶ Identify geometric sequences.

▶ Extend and graph geometric sequences.

▶ Write geometric sequences as functions.

Core Vocabulary

geometric sequence, *p. 312*
common ratio, *p. 312*

Previous
arithmetic sequence
common difference

Identifying Geometric Sequences

Core Concept

Geometric Sequence

In a **geometric sequence**, the ratio between each pair of consecutive terms is the same. This ratio is called the **common ratio**. Each term is found by multiplying the previous term by the common ratio.

$$1, \quad 5, \quad 25, \quad 125, \ldots \quad \text{Terms of a geometric sequence}$$

$$\times 5 \quad \times 5 \quad \times 5 \longleftarrow \boxed{\text{common ratio}}$$

EXAMPLE 1   **Identifying Geometric Sequences**

Decide whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

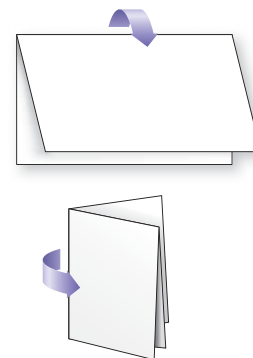**a.** 120, 60, 30, 15, . . .   **b.** 2, 6, 11, 17, . . .

SOLUTION

a. Find the ratio between each pair of consecutive terms.

120   60   30   15

$\dfrac{60}{120} = \dfrac{1}{2} \quad \dfrac{30}{60} = \dfrac{1}{2} \quad \dfrac{15}{30} = \dfrac{1}{2}$ The ratios are the same. The common ratio is $\frac{1}{2}$.

▶ So, the sequence is geometric.

b. Find the ratio between each pair of consecutive terms.

2   6   11   17

$\dfrac{6}{2} = 3 \quad \dfrac{11}{6} = 1\dfrac{5}{6} \quad \dfrac{17}{11} = 1\dfrac{6}{11}$ There is no common ratio, so the sequence is *not* geometric.
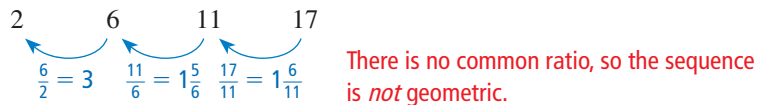
Find the difference between each pair of consecutive terms.

2   6   11   17

$6 - 2 = 4 \quad 11 - 6 = 5 \quad 17 - 11 = 6$ There is no common difference, so the sequence is *not* arithmetic.
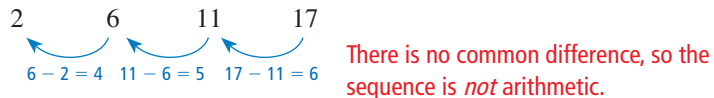
▶ So, the sequence is *neither* geometric nor arithmetic.

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Decide whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

1. 5, 1, −3, −7, . . .   **2.** 1024, 128, 16, 2, . . .   **3.** 2, 6, 10, 16, . . .

Extending and Graphing Geometric Sequences

EXAMPLE 2 **Extending Geometric Sequences**

Write the next three terms of each geometric sequence.

a. 3, 6, 12, 24, . . .   **b.** 64, −16, 4, −1, . . .

SOLUTION

Use tables to organize the terms and extend each sequence.

a.

Position	1	2	3	4	5	6	7
Term	3	6	12	24	48	96	192

× 2   × 2   × 2   × 2   × 2   × 2

Each term is twice the previous term. So, the common ratio is 2.

Multiply a term by 2 to find the next term.

▶ The next three terms are 48, 96, and 192.

b.

Position	1	2	3	4	5	6	7
Term	64	−16	4	−1	$\frac{1}{4}$	$-\frac{1}{16}$	$\frac{1}{64}$

$\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$

Multiply a term by $-\frac{1}{4}$ to find the next term.

ANALYZING MATHEMATICAL RELATIONSHIPS

When the terms of a geometric sequence alternate between positive and negative terms, or vice versa, the common ratio is negative.

▶ The next three terms are $\frac{1}{4}$, $-\frac{1}{16}$, and $\frac{1}{64}$.

EXAMPLE 3 **Graphing a Geometric Sequence**

Graph the geometric sequence 32, 16, 8, 4, 2, ... What do you notice?

SOLUTION

Make a table. Then plot the ordered pairs (n, a_n).

STUDY TIP

The points of any geometric sequence with a *positive* common ratio lie on an exponential curve.

Position, n	1	2	3	4	5
Term, a_n	32	16	8	4	2
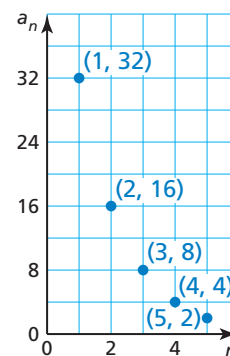
▶ The points appear to lie on an exponential curve.

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Write the next three terms of the geometric sequence. Then graph the sequence.

4. 1, 3, 9, 27, . . .   **5.** 2500, 500, 100, 20, . . .

6. 80, −40, 20, −10, . . .   **7.** −2, 4, −8, 16, . . .

Writing Geometric Sequences as Functions

Because consecutive terms of a geometric sequence have a common ratio, you can use the first term a_1 and the common ratio r to write an exponential function that describes a geometric sequence. Let $a_1 = 1$ and $r = 5$.

Position, n	Term, a_n	Written using a_1 and r	Numbers
1	first term, a_1	a_1	1
2	second term, a_2	$a_1 r$	$1 \cdot 5 = 5$
3	third term, a_3	$a_1 r^2$	$1 \cdot 5^2 = 25$
4	fourth term, a_4	$a_1 r^3$	$1 \cdot 5^3 = 125$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	nth term, a_n	$a_1 r^{n-1}$	$1 \cdot 5^{n-1}$

🅖 Core Concept

Equation for a Geometric Sequence

Let a_n be the nth term of a geometric sequence with first term a_1 and common ratio r. The nth term is given by

$$a_n = a_1 r^{n-1}.$$

EXAMPLE 4 **Finding the nth Term of a Geometric Sequence**

Write an equation for the nth term of the geometric sequence 2, 12, 72, 432, ... Then find a_{10}.

SOLUTION

The first term is 2, and the common ratio is 6.

$a_n = a_1 r^{n-1}$	Equation for a geometric sequence
$a_n = 2(6)^{n-1}$	Substitute 2 for a_1 and 6 for r.

Use the equation to find the 10th term.

$a_n = 2(6)^{n-1}$	Write the equation.
$a_{10} = 2(6)^{10-1}$	Substitute 10 for n.
$= 20{,}155{,}392$	Simplify.

▶ The 10th term of the geometric sequence is 20,155,392.

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Write an equation for the nth term of the geometric sequence. Then find a_7.

8. 1, −5, 25, −125, . . .

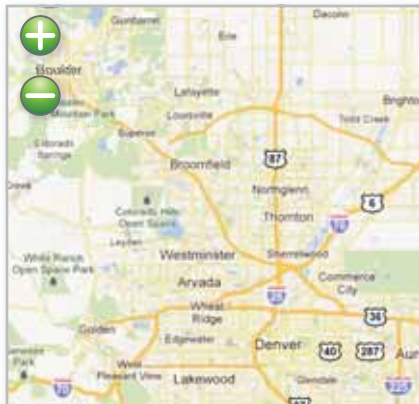**9.** 13, 26, 52, 104, . . .

10. 432, 72, 12, 2, . . .

11. 4, 10, 25, 62.5, . . .

You can rewrite the equation for a geometric sequence with first term a_1 and common ratio r in function notation by replacing a_n with $f(n)$.

$$f(n) = a_1 r^{n-1}$$

The domain of the function is the set of positive integers.

EXAMPLE 5 **Modeling with Mathematics**

Clicking the *zoom-out* button on a mapping website doubles the side length of the square map. After how many clicks on the *zoom-out* button is the side length of the map 640 miles?

Zoom-out clicks	1	2	3
Map side length (miles)	5	10	20

SOLUTION

1. **Understand the Problem** You know that the side length of the square map doubles after each click on the *zoom-out* button. So, the side lengths of the map represent the terms of a geometric sequence. You need to find the number of clicks it takes for the side length of the map to be 640 miles.

2. **Make a Plan** Begin by writing a function f for the nth term of the geometric sequence. Then find the value of n for which $f(n) = 640$.

3. **Solve the Problem** The first term is 5, and the common ratio is 2.

$$f(n) = a_1 r^{n-1} \qquad \text{Function for a geometric sequence}$$

$$f(n) = 5(2)^{n-1} \qquad \text{Substitute 5 for } a_1 \text{ and 2 for } r.$$

The function $f(n) = 5(2)^{n-1}$ represents the geometric sequence. Use this function to find the value of n for which $f(n) = 640$. So, use the equation $640 = 5(2)^{n-1}$ to write a system of equations.

$$y = 5(2)^{n-1} \qquad \text{Equation 1}$$

$$y = 640 \qquad \text{Equation 2}$$

Then use a graphing calculator to graph the equations and find the point of intersection. The point of intersection is (8, 640).
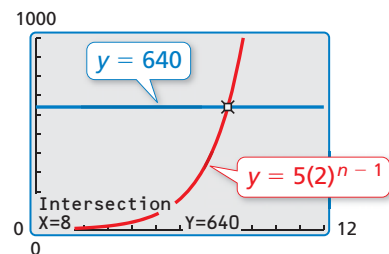
▶ So, after eight clicks, the side length of the map is 640 miles.

4. **Look Back** You can use the *table* feature of a graphing calculator to find the value of n for which $f(n) = 640$.

X	Y1	Y2
3	20	640
4	40	640
5	80	640
6	160	640
7	320	640
8	640	640
9	1280	640

X=8

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12. **WHAT IF?** After how many clicks on the *zoom-out* button is the side length of the map 2560 miles?

Vocabulary and Core Concept Check

1. **WRITING** Compare the two sequences.

$$2, 4, 6, 8, 10, \ldots \qquad 2, 4, 8, 16, 32, \ldots$$

2. **CRITICAL THINKING** Why do the points of a geometric sequence lie on an exponential curve only when the common ratio is positive?

Monitoring Progress and Modeling with Mathematics

**In Exercises 3–8, find the common ratio of the geometric sequence.**

3. $4, 12, 36, 108, \ldots$

4. $36, 6, 1, \frac{1}{6}, \ldots$

5. $\frac{3}{8}, -3, 24, -192, \ldots$

6. $0.1, 1, 10, 100, \ldots$

7. $128, 96, 72, 54, \ldots$

8. $-162, 54, -18, 6, \ldots$

In Exercises 9–14, determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning. *(See Example 1.)*

9. $-8, 0, 8, 16, \ldots$

10. $-1, 4, -7, 10, \ldots$

11. $9, 14, 20, 27, \ldots$

12. $\frac{3}{49}, \frac{3}{7}, 3, 21, \ldots$

13. $192, 24, 3, \frac{3}{8}, \ldots$

14. $-25, -18, -11, -4, \ldots$

In Exercises 15–18, determine whether the graph represents an *arithmetic sequence*, a *geometric sequence*, or *neither*. Explain your reasoning.

15.

16.

17.

18.
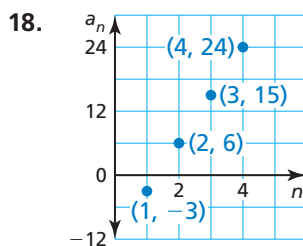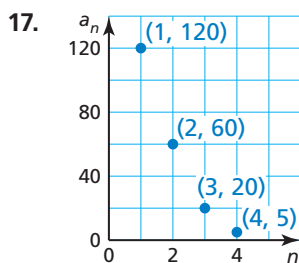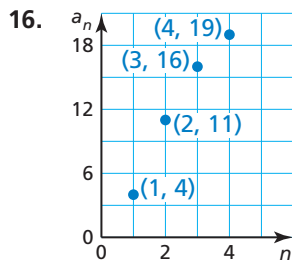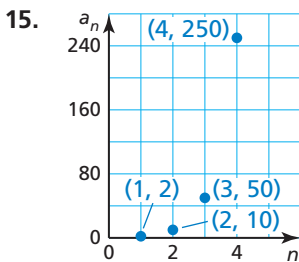
In Exercises 19–24, write the next three terms of the geometric sequence. Then graph the sequence. *(See Examples 2 and 3.)*

19. $5, 20, 80, 320, \ldots$

20. $-3, 12, -48, 192, \ldots$

21. $81, -27, 9, -3, \ldots$

22. $-375, -75, -15, -3, \ldots$

23. $32, 8, 2, \frac{1}{2}, \ldots$

24. $\frac{16}{9}, \frac{8}{3}, 4, 6, \ldots$

In Exercises 25–32, write an equation for the *n*th term of the geometric sequence. Then find a_6. *(See Example 4.)*

25. $2, 8, 32, 128, \ldots$

26. $0.6, -3, 15, -75, \ldots$

27. $-\frac{1}{8}, -\frac{1}{4}, -\frac{1}{2}, -1, \ldots$

28. $0.1, 0.9, 8.1, 72.9, \ldots$

29.

n	1	2	3	4
a_n	7640	764	76.4	7.64

30.

n	1	2	3	4
a_n	-192	48	-12	3

31.

32.
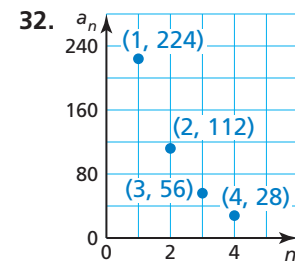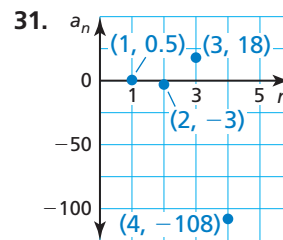
33. **PROBLEM SOLVING** A badminton tournament begins with 128 teams. After the first round, 64 teams remain. After the second round, 32 teams remain. How many teams remain after the third, fourth, and fifth rounds?

34. PROBLEM SOLVING A graphing calculator screen displays an area of 96 square units. After you zoom out once, the area is 384 square units. After you zoom out a second time, the area is 1536 square units. What is the screen area after you zoom out four times?
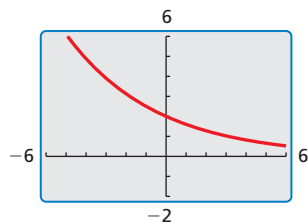
35. ERROR ANALYSIS Describe and correct the error in writing the next three terms of the geometric sequence.

$$-8, \quad 4, \quad -2, \quad 1, \ldots$$
$$\times(-2) \quad \times(-2) \quad \times(-2)$$

The next three terms are -2, 4, and -8.

36. ERROR ANALYSIS Describe and correct the error in writing an equation for the nth term of the geometric sequence.

$$-2, -12, -72, -432, \ldots$$

The first term is -2, and the common ratio is -6.

$$a_n = a_1 r^{n-1}$$
$$a_n = -2(-6)^{n-1}$$

37. MODELING WITH MATHEMATICS The distance (in millimeters) traveled by a swinging pendulum decreases after each swing, as shown in the table. *(See Example 5.)*

Swing	1	2	3
Distance (in millimeters)	625	500	400
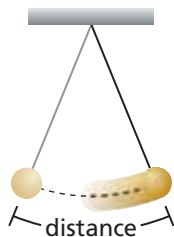
distance

a. Write a function that represents the distance the pendulum swings on its nth swing.

b. After how many swings is the distance 256 millimeters?

38. MODELING WITH MATHEMATICS You start a chain email and send it to six friends. The next day, each of your friends forwards the email to six people. The process continues for a few days.

a. Write a function that represents the number of people who have received the email after n days.

b. After how many days will 1296 people have received the email?

MATHEMATICAL CONNECTIONS In Exercises 39 and 40, (a) write a function that represents the sequence of figures and (b) describe the 10th figure in the sequence.

39.

40.
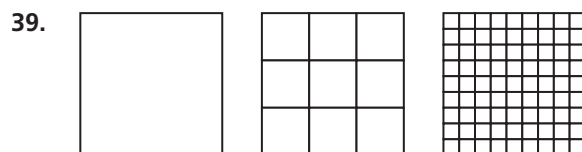
41. REASONING Write a sequence that represents the number of teams that have been eliminated after n rounds of the badminton tournament in Exercise 33. Determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

42. REASONING Write a sequence that represents the perimeter of the graphing calculator screen in Exercise 34 after you zoom out n times. Determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

43. WRITING Compare the graphs of arithmetic sequences to the graphs of geometric sequences.

44. MAKING AN ARGUMENT You are given two consecutive terms of a sequence.

$$\ldots, -8, 0, \ldots$$

Your friend says that the sequence is not geometric. A classmate says that is impossible to know given only two terms. Who is correct? Explain.

45. CRITICAL THINKING Is the sequence shown an *arithmetic* sequence? a *geometric* sequence? Explain your reasoning.

$$3, 3, 3, 3, \ldots$$

47. REASONING What is the 9th term of the geometric sequence where $a_3 = 81$ and $r = 3$?

48. OPEN-ENDED Write a sequence that has a pattern but is not arithmetic or geometric. Describe the pattern.

49. ATTENDING TO PRECISION Are the terms of a geometric sequence independent or dependent? Explain your reasoning.

50. DRAWING CONCLUSIONS A college student makes a deal with her parents to live at home instead of living on campus. She will pay her parents $0.01 for the first day of the month, $0.02 for the second day, $0.04 for the third day, and so on.
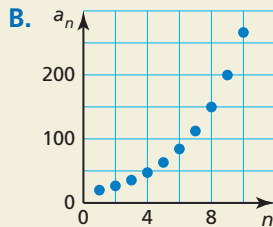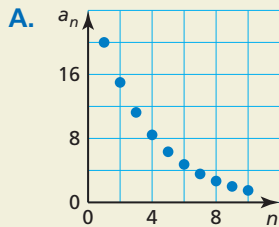
a. Write an equation that represents the nth term of the geometric sequence.

b. What will she pay on the 25th day?

c. Did the student make a good choice or should she have chosen to live on campus? Explain.

51. REPEATED REASONING A soup kitchen makes 16 gallons of soup. Each day, a quarter of the soup is served and the rest is saved for the next day.

a. Write the first five terms of the sequence of the number of fluid ounces of soup left each day.

b. Write an equation that represents the nth term of the sequence.

c. When is all the soup gone? Explain.

53. OPEN-ENDED Write a geometric sequence in which $a_2 < a_1 < a_3$.

54. NUMBER SENSE Write an equation that represents the nth term of each geometric sequence shown.

n	1	2	3	4
a_n	2	6	18	54

n	1	2	3	4
b_n	1	5	25	125

a. Do the terms $a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots$ form a geometric sequence? If so, how does the common ratio relate to the common ratios of the sequences above?

b. Do the terms $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ form a geometric sequence? If so, how does the common ratio relate to the common ratios of the sequences above?

Maintaining Mathematical Proficiency
Reviewing what you learned in previous grades and lessons

Use residuals to determine whether the model is a good fit for the data in the table. Explain. *(Section 4.6)*

55. $y = 3x - 8$

x	0	1	2	3	4	5	6
y	-10	-2	-1	2	1	7	10

56. $y = -5x + 1$

x	-3	-2	-1	0	1	2	3
y	6	4	6	1	2	-4	-3

6.6 Recursively Defined Sequences
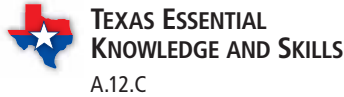
Essential Question How can you define a sequence recursively?

A **recursive rule** gives the beginning term(s) of a sequence and a *recursive equation* that tells how a_n is related to one or more preceding terms.

EXPLORATION 1 Describing a Pattern

Work with a partner. Consider a hypothetical population of rabbits. Start with one breeding pair. After each month, each breeding pair produces another breeding pair. The total number of rabbits each month follows the exponential pattern 2, 4, 8, 16, 32, ... Now suppose that in the first month after each pair is born, the pair is too young to reproduce. Each pair produces another pair after it is 2 months old. Find the total number of pairs in months 6, 7, and 8.

ANALYZING MATHEMATICAL RELATIONSHIPS

To be proficient in math, you need to look closely to discern a pattern or structure.

EXPLORATION 2 Using a Recursive Equation

Work with a partner. Consider the following recursive equation.

$$a_n = a_{n-1} + a_{n-2}$$

Each term in the sequence is the sum of the two preceding terms.

Copy and complete the table. Compare the results with the sequence of the number of pairs in Exploration 1.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1						

Communicate Your Answer

3. How can you define a sequence recursively?

4. Use the Internet or some other reference to determine the mathematician who first described the sequences in Explorations 1 and 2.

What You Will Learn

▶ Write terms of recursively defined sequences.

▶ Write recursive rules for sequences.

▶ Translate between recursive rules and explicit rules.

▶ Write recursive rules for special sequences.

Writing Terms of Recursively Defined Sequences

So far in this book, you have defined arithmetic and geometric sequences *explicitly*. An **explicit rule** gives a_n as a function of the term's position number n in the sequence. For example, an explicit rule for the arithmetic sequence 3, 5, 7, 9, . . . is $a_n = 3 + 2(n - 1)$, or $a_n = 2n + 1$.

Now, you will define arithmetic and geometric sequences *recursively*. A **recursive rule** gives the beginning term(s) of a sequence and a *recursive equation* that tells how a_n is related to one or more preceding terms.

⑤ Core Concept

Recursive Equation for an Arithmetic Sequence

$a_n = a_{n-1} + d$, where d is the common difference

Recursive Equation for a Geometric Sequence

$a_n = r \cdot a_{n-1}$, where r is the common ratio

EXAMPLE 1 **Writing Terms of Recursively Defined Sequences**

Write the first six terms of each sequence. Then graph each sequence.

a. $a_1 = 2, a_n = a_{n-1} + 3$　　　　　**b.** $a_1 = 1, a_n = 3a_{n-1}$

SOLUTION

You are given the first term. Use the recursive equation to find the next five terms.

a. $a_1 = 2$

$a_2 = a_1 + 3 = 2 + 3 = 5$

$a_3 = a_2 + 3 = 5 + 3 = 8$

$a_4 = a_3 + 3 = 8 + 3 = 11$

$a_5 = a_4 + 3 = 11 + 3 = 14$

$a_6 = a_5 + 3 = 14 + 3 = 17$

b. $a_1 = 1$

$a_2 = 3a_1 = 3(1) = 3$

$a_3 = 3a_2 = 3(3) = 9$

$a_4 = 3a_3 = 3(9) = 27$

$a_5 = 3a_4 = 3(27) = 81$

$a_6 = 3a_5 = 3(81) = 243$

STUDY TIP

A sequence is a discrete function. So, the points on the graph are not connected.
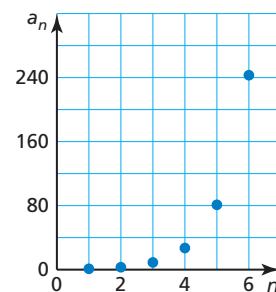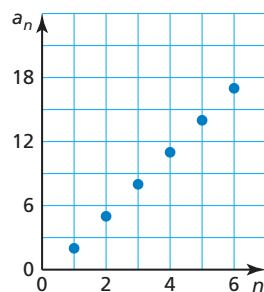
Write the first six terms of the sequence. Then graph the sequence.

1. $a_1 = 0, a_n = a_{n-1} - 8$

2. $a_1 = -7.5, a_n = a_{n-1} + 2.5$

3. $a_1 = -36, a_n = \frac{1}{2}a_{n-1}$

4. $a_1 = 0.7, a_n = 10a_{n-1}$

Writing Recursive Rules

EXAMPLE 2   **Writing Recursive Rules**

Write a recursive rule for each sequence.

a. $-30, -18, -6, 6, 18, \ldots$

b. $500, 100, 20, 4, 0.8, \ldots$

SOLUTION

Use a table to organize the terms and find the pattern.

a.

Position, n	1	2	3	4	5
Term, a_n	-30	-18	-6	6	18

$+12 \quad +12 \quad +12 \quad +12$

COMMON ERROR

When writing a recursive rule for a sequence, you need to write both the beginning term(s) and the recursive equation.

The sequence is arithmetic, with first term $a_1 = -30$ and common difference $d = 12$.

$a_n = a_{n-1} + d$ Recursive equation for an arithmetic sequence

$a_n = a_{n-1} + 12$ Substitute 12 for d.

▶ So, a recursive rule for the sequence is $a_1 = -30, a_n = a_{n-1} + 12$.

b.

Position, n	1	2	3	4	5
Term, a_n	500	100	20	4	0.8

$\times \frac{1}{5} \quad \times \frac{1}{5} \quad \times \frac{1}{5} \quad \times \frac{1}{5}$

The sequence is geometric, with first term $a_1 = 500$ and common ratio $r = \frac{1}{5}$.

$a_n = r \cdot a_{n-1}$ Recursive equation for a geometric sequence

$a_n = \frac{1}{5}a_{n-1}$ Substitute $\frac{1}{5}$ for r.

▶ So, a recursive rule for the sequence is $a_1 = 500, a_n = \frac{1}{5}a_{n-1}$.

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Write a recursive rule for the sequence.

5. $8, 3, -2, -7, -12, \ldots$

6. $1.3, 2.6, 3.9, 5.2, 6.5, \ldots$

7. $4, 20, 100, 500, 2500, \ldots$

8. $128, -32, 8, -2, 0.5, \ldots$

9. Write a recursive rule for the height of the sunflower over time.

1 month: 2 feet   2 months: 3.5 feet   3 months: 5 feet   4 months: 6.5 feet

Translating between Recursive and Explicit Rules

EXAMPLE 3   **Translating from Recursive Rules to Explicit Rules**

Write an explicit rule for each recursive rule.

a. $a_1 = 25, a_n = a_{n-1} - 10$ **b.** $a_1 = 19.6, a_n = -0.5a_{n-1}$

SOLUTION

a. The recursive rule represents an arithmetic sequence, with first term $a_1 = 25$ and common difference $d = -10$.

$a_n = a_1 + (n-1)d$ Explicit rule for an arithmetic sequence

$a_n = 25 + (n-1)(-10)$ Substitute 25 for a_1 and -10 for d.

$a_n = -10n + 35$ Simplify.

▶ An explicit rule for the sequence is $a_n = -10n + 35$.

b. The recursive rule represents a geometric sequence, with first term $a_1 = 19.6$ and common ratio $r = -0.5$.

$a_n = a_1 r^{n-1}$ Explicit rule for a geometric sequence

$a_n = 19.6(-0.5)^{n-1}$ Substitute 19.6 for a_1 and -0.5 for r.

▶ An explicit rule for the sequence is $a_n = 19.6(-0.5)^{n-1}$.

EXAMPLE 4   **Translating from Explicit Rules to Recursive Rules**

Write a recursive rule for each explicit rule.

a. $a_n = -2n + 3$ **b.** $a_n = -3(2)^{n-1}$

SOLUTION

a. The explicit rule represents an arithmetic sequence, with first term $a_1 = -2(1) + 3 = 1$ and common difference $d = -2$.

$a_n = a_{n-1} + d$ Recursive equation for an arithmetic sequence

$a_n = a_{n-1} + (-2)$ Substitute -2 for d.

▶ So, a recursive rule for the sequence is $a_1 = 1, a_n = a_{n-1} - 2$.

b. The explicit rule represents a geometric sequence, with first term $a_1 = -3$ and common ratio $r = 2$.

$a_n = r \cdot a_{n-1}$ Recursive equation for a geometric sequence

$a_n = 2a_{n-1}$ Substitute 2 for r.

▶ So, a recursive rule for the sequence is $a_1 = -3, a_n = 2a_{n-1}$.

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Write an explicit rule for the recursive rule.

10. $a_1 = -45, a_n = a_{n-1} + 20$ **11.** $a_1 = 13, a_n = -3a_{n-1}$

Write a recursive rule for the explicit rule.

12. $a_n = -n + 1$ **13.** $a_n = -2.5(4)^{n-1}$

Writing Recursive Rules for Special Sequences

You can write recursive rules for sequences that are neither arithmetic nor geometric. One way is to look for patterns in the sums of consecutive terms.

EXAMPLE 5 **Writing Recursive Rules for Other Sequences**

Use the sequence shown.

$$1, 1, 2, 3, 5, 8, \ldots$$

a. Write a recursive rule for the sequence.

b. Write the next three terms of the sequence.

SOLUTION

a. Find the difference and ratio between each pair of consecutive terms.

1   1   2   3

$1 - 1 = 0$ $2 - 1 = 1$ $3 - 2 = 1$

There is no common difference, so the sequence is *not* arithmetic.

1   1   2   3

$\frac{1}{1} = 1$ $\frac{2}{1} = 2$ $\frac{3}{2} = 1\frac{1}{2}$

There is no common ratio, so the sequence is *not* geometric.

The sequence in Example 5 is called the *Fibonacci sequence*. This pattern is naturally occurring in many objects, such as flowers.

Find the sum of each pair of consecutive terms.

$a_1 + a_2 = 1 + 1 = 2$	2 is the third term.
$a_2 + a_3 = 1 + 2 = 3$	3 is the fourth term.
$a_3 + a_4 = 2 + 3 = 5$	5 is the fifth term.
$a_4 + a_5 = 3 + 5 = 8$	8 is the sixth term.

Beginning with the third term, each term is the sum of the two previous terms. A recursive equation for the sequence is $a_n = a_{n-2} + a_{n-1}$.

▶ So, a recursive rule for the sequence is $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-2} + a_{n-1}$.

b. Use the recursive equation $a_n = a_{n-2} + a_{n-1}$ to find the next three terms.

$a_7 = a_5 + a_6$	$a_8 = a_6 + a_7$	$a_9 = a_7 + a_8$
$= 5 + 8$	$= 8 + 13$	$= 13 + 21$
$= 13$	$= 21$	$= 34$

▶ The next three terms are 13, 21, and 34.

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Write a recursive rule for the sequence. Then write the next three terms of the sequence.

14. $5, 6, 11, 17, 28, \ldots$

15. $-3, -4, -7, -11, -18, \ldots$

16. $1, 1, 0, -1, -1, 0, 1, 1, \ldots$

17. $4, 3, 1, 2, -1, 3, -4, \ldots$

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A recursive rule gives the beginning term(s) of a sequence and a(n) _____ that tells how a_n is related to one or more preceding terms.

2. **WHICH ONE DOESN'T BELONG?** Which rule does *not* belong with the other three? Explain your reasoning.

$$a_1 = -1, a_n = 5a_{n-1} \qquad a_n = 6n - 2 \qquad a_1 = -3, a_n = a_{n-1} + 1 \qquad a_1 = 9, a_n = 4a_{n-1}$$

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, determine whether the recursive rule represents an *arithmetic sequence* or a *geometric sequence*.

3. $a_1 = 2, a_n = 7a_{n-1}$

4. $a_1 = 18, a_n = a_{n-1} + 1$

5. $a_1 = 5, a_n = a_{n-1} - 4$

6. $a_1 = 3, a_n = -6a_{n-1}$

In Exercises 7–12, write the first six terms of the sequence. Then graph the sequence. *(See Example 1.)*

7. $a_1 = 0, a_n = a_{n-1} + 2$

8. $a_1 = 10, a_n = a_{n-1} - 5$

9. $a_1 = 2, a_n = 3a_{n-1}$

10. $a_1 = 8, a_n = 1.5a_{n-1}$

11. $a_1 = 80, a_n = -\frac{1}{2}a_{n-1}$

12. $a_1 = -7, a_n = -4a_{n-1}$

In Exercises 13–20, write a recursive rule for the sequence. *(See Example 2.)*

13.

n	1	2	3	4
a_n	7	16	25	34

14.

n	1	2	3	4
a_n	8	24	72	216

15. 243, 81, 27, 9, 3, . . .

16. 3, 11, 19, 27, 35, . . .

17. 0, −3, −6, −9, −12, . . .

18. 5, −20, 80, −320, 1280, . . .

19.

20.
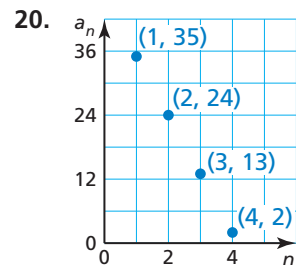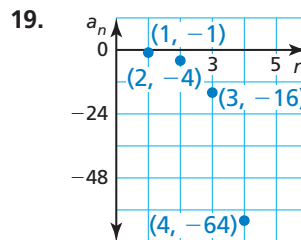
21. **MODELING WITH MATHEMATICS** Write a recursive rule for the number of bacterial cells over time.
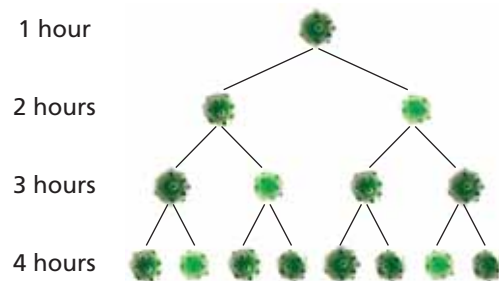
22. **MODELING WITH MATHEMATICS** Write a recursive rule for the length of the deer antler over time.

1 day: $4\frac{1}{2}$ in. 2 days: $4\frac{3}{4}$ in. 3 days: 5 in. 4 days: $5\frac{1}{4}$ in.

In Exercises 23–28, write an explicit rule for the recursive rule. *(See Example 3.)*

23. $a_1 = -3, a_n = a_{n-1} + 3$

24. $a_1 = 8, a_n = a_{n-1} - 12$

25. $a_1 = 16, a_n = 0.5a_{n-1}$

26. $a_1 = -2, a_n = 9a_{n-1}$

27. $a_1 = 4, a_n = a_{n-1} + 17$

28. $a_1 = 5, a_n = -5a_{n-1}$

In Exercises 29–34, write a recursive rule for the explicit rule. *(See Example 4.)*

29. $a_n = 7(3)^{n-1}$

30. $a_n = -4n + 2$

31. $a_n = 1.5n + 3$

32. $a_n = 6n - 20$

33. $a_n = (-5)^{n-1}$

34. $a_n = -81\left(\dfrac{2}{3}\right)^{n-1}$

In Exercises 35–38, graph the first four terms of the sequence with the given description. Write a recursive rule and an explicit rule for the sequence.

35. The first term of a sequence is 5. Each term of the sequence is 15 more than the preceding term.

36. The first term of a sequence is 16. Each term of the sequence is half the preceding term.

37. The first term of a sequence is -1. Each term of the sequence is -3 times the preceding term.

38. The first term of a sequence is 19. Each term of the sequence is 13 less than the preceding term.

In Exercises 39–44, write a recursive rule for the sequence. Then write the next two terms of the sequence. *(See Example 5.)*

39. $1, 3, 4, 7, 11, \ldots$

40. $10, 9, 1, 8, -7, 15, \ldots$

41. $2, 4, 2, -2, -4, -2, \ldots$

42. $6, 1, 7, 8, 15, 23, \ldots$

43.

44.
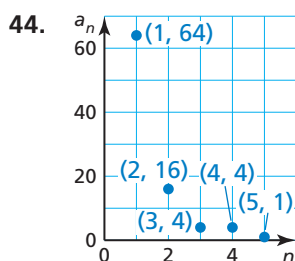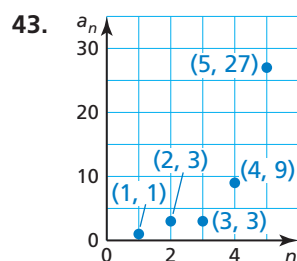
45. **ERROR ANALYSIS** Describe and correct the error in writing an explicit rule for the recursive rule $a_1 = 6$, $a_n = a_{n-1} - 12$.

$$a_n = a_1 + (n-1)d$$
$$a_n = 6 + (n-1)(12)$$
$$a_n = 6 + 12n - 12$$
$$a_n = -6 + 12n$$

46. **ERROR ANALYSIS** Describe and correct the error in writing a recursive rule for the sequence $2, 4, 6, 10, 16, \ldots$.

$2, \quad 4, \quad 6, \ldots$
$\quad +2 \quad +2$

The sequence is arithmetic, with first term $a_1 = 2$ and common difference $d = 2$.

$$a_n = a_{n-1} + d$$
$$a_1 = 2, a_n = a_{n-1} + 2$$

In Exercises 47–51, the function f represents a sequence. Find the 2nd, 5th, and 10th terms of the sequence.

47. $f(1) = 3, f(n) = f(n-1) + 7$

48. $f(1) = -1, f(n) = 6f(n-1)$

49. $f(1) = 8, f(n) = -f(n-1)$

50. $f(1) = 4, f(2) = 5, f(n) = f(n-2) + f(n-1)$

51. $f(1) = 10, f(2) = 15, f(n) = f(n-1) - f(n-2)$

52. **MODELING WITH MATHEMATICS** The X-ray shows the lengths (in centimeters) of bones in a human hand.
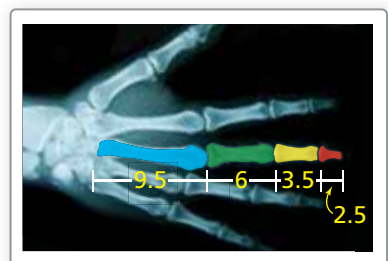
a. Write a recursive rule for the lengths of the bones.

b. Measure the lengths of different sections of your hand. Can the lengths be represented by a recursively defined sequence? Explain.

53. USING TOOLS You can use a spreadsheet to generate the terms of a sequence.

A2 ▼	=	=A1+2	
	A	B	C
1	3		
2	5		
3			
4			

a. To generate the terms of the sequence $a_1 = 3$, $a_n = a_{n-1} + 2$, enter the value of a_1, 3, into cell A1. Then enter "=A1+2" into cell A2, as shown. Use the *fill down* feature to generate the first 10 terms of the sequence.

b. Use a spreadsheet to generate the first 10 terms of the sequence $a_1 = 3$, $a_n = 4a_{n-1}$. (*Hint:* Enter "=4*A1" into cell A2.)

c. Use a spreadsheet to generate the first 10 terms of the sequence $a_1 = 4$, $a_2 = 7$, $a_n = a_{n-1} - a_{n-2}$. (*Hint:* Enter "=A2-A1" into cell A3.)

54. HOW DO YOU SEE IT? Consider Squares 1–6 in the diagram.
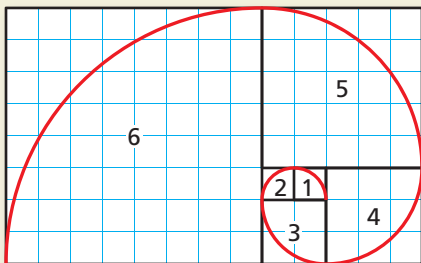
a. Write a sequence in which each term a_n is the side length of square n.

b. What is the name of this sequence? What is the next term of this sequence?

c. Use the term in part (b) to add another square to the diagram and extend the spiral.

55. REASONING Write the first 5 terms of the sequence $a_1 = 5$, $a_n = 3a_{n-1} + 4$. Determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

56. THOUGHT PROVOKING Describe the pattern for the numbers in Pascal's Triangle, shown below. Write a recursive rule that gives the mth number in the nth row.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

57. REASONING The explicit rule $a_n = a_1 + (n-1)d$ defines an arithmetic sequence.

a. Explain why $a_{n-1} = a_1 + [(n-1) - 1]d$.

b. Justify each step in showing that a recursive equation for the sequence is $a_n = a_{n-1} + d$.

$a_n = a_1 + (n-1)d$

$\quad = a_1 + [(n-1) + 0]d$

$\quad = a_1 + [(n-1) - 1 + 1]d$

$\quad = a_1 + [((n-1) - 1) + 1]d$

$\quad = a_1 + [(n-1) - 1]d + d$

$\quad = a_{n-1} + d$

58. MAKING AN ARGUMENT Your friend claims that the sequence

$$-5, 5, -5, 5, -5, \ldots$$

cannot be represented by a recursive rule. Is your friend correct? Explain.

59. PROBLEM SOLVING Write a recursive rule for the sequence.

$$3, 7, 15, 31, 63, \ldots$$

Maintaining Mathematical Proficiency
Reviewing what you learned in previous grades and lessons

Simplify the expression. *(Skills Review Handbook)*

60. $5x + 12x$ **61.** $9 - 6y - 14$ **62.** $2d - 7 - 8d$ **63.** $3 - 3m + 11m$

Write a linear function f with the given values. *(Section 4.2)*

64. $f(2) = 6, f(-1) = -3$ **65.** $f(-2) = 0, f(6) = -4$

66. $f(-3) = 5, f(-1) = 5$ **67.** $f(3) = -1, f(-4) = -15$

Core Vocabulary

geometric sequence, *p. 312*
common ratio, *p. 312*

explicit rule, *p. 320*
recursive rule, *p. 320*

Core Concepts

Section 6.5

Geometric Sequence, *p. 312*
Equation for a Geometric Sequence, *p. 314*

Section 6.6

Recursive Equation for an Arithmetic Sequence, *p. 320*
Recursive Equation for a Geometric Sequence, *p. 320*

Mathematical Thinking

1. Explain how writing a function in Exercise 39 part (a) on page 317 created a shortcut for answering part (b).

2. How did you choose an appropriate tool in Exercise 52 part (b) on page 325?

Performance Task

The New Car

There is so much more to buying a new car than the purchase price. Interest rates, depreciation, and inflation are all factors. So, what is the real cost of your new car?

To explore the answer to this question and more, go to ***BigIdeasMath.com***.

6 Chapter Review

6.1 Properties of Exponents (pp. 277–284)

Simplify $\left(\dfrac{x}{4}\right)^{-4}$. Write your answer using only positive exponents.

$$\left(\frac{x}{4}\right)^{-4} = \frac{x^{-4}}{4^{-4}}$$ Power of a Quotient Property

$$= \frac{4^4}{x^4}$$ Definition of negative exponent

$$= \frac{256}{x^4}$$ Simplify.

Simplify the expression. Write your answer using only positive exponents.

1. $y^3 \cdot y^{-5}$ **2.** $\dfrac{x^4}{x^7}$ **3.** $(x^0 y^2)^3$ **4.** $\left(\dfrac{2x^2}{5y^4}\right)^{-2}$

6.2 Radicals and Rational Exponents (pp. 285–290)

Evaluate $512^{1/3}$.

$$512^{1/3} = \sqrt[3]{512}$$ Rewrite the expression in radical form.

$$= \sqrt[3]{8 \cdot 8 \cdot 8}$$ Rewrite the expression showing factors.

$$= 8$$ Evaluate the cube root.

Evaluate the expression.

5. $\sqrt[3]{8}$ **6.** $\sqrt[5]{-243}$ **7.** $625^{3/4}$ **8.** $(-25)^{1/2}$

6.3 Exponential Functions (pp. 291–298)

Graph $f(x) = 9(3)^x$. Compare the graph to the graph of the parent function. Identify the y-intercepts and asymptotes of the graphs. Describe the domain and range of f.

Step 1 Make a table of values.

x	−2	−1	0	1	2
f(x)	1	3	9	27	81

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.

▶ The parent function is $g(x) = 3^x$. The graph of f is a vertical stretch by a factor of 9 of the graph of g. The y-intercept of the graph of f, 9, is above the y-intercept of the graph of g, 1. The x-axis is an asymptote of both the graphs of f and g. From the graph of f, you can see that the domain is all real numbers and the range is $y > 0$.

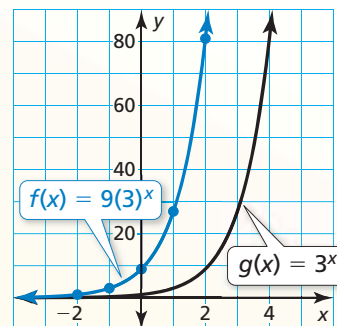**Graph the function. Compare the graph to the graph of the parent function. Identify the**
y-intercepts and asymptotes of the graphs. Describe the domain and range of f.

9. $f(x) = -4\left(\frac{1}{4}\right)^x$

10. $f(x) = 2(3)^x$

11. Graph $f(x) = 2^{x-4} - 3$. Identify the asymptote. Describe the domain and range.

12. Use the table shown. (a) Use a graphing calculator to find an exponential function that fits the data. Then plot the data and graph the function in the same viewing window. (b) Identify and interpret the correlation coefficient.

x	0	1	2	3	4
y	200	96	57	25	10

6.4 Exponential Growth and Decay (pp. 299–308)

The function $f(x) = 75(0.5)^{x/10}$ represents the amount (in milligrams) of a radioactive substance remaining after x days. Rewrite the function in the form $f(x) = ab^x$ to determine whether it represents *exponential growth* or *exponential decay*. Identify the initial amount and interpret the growth factor or decay factor.

$f(x) = 75(0.5)^{x/10}$ Write the function.

$= 75(0.5^{1/10})^x$ Power of a Power Property

$\approx 75(0.93)^x$ Evaluate the power.

The function is of the form $y = a(1 - r)^t$, where $1 - r < 1$, so it represents exponential decay. The initial amount is 75 milligrams, and the decay factor of 0.93 means that about 93% of the radioactive substance remains after each day.

Determine whether the table represents an *exponential growth function*, an *exponential decay*
***function*, or *neither*. Explain.**

13.

x	0	1	2	3
y	3	6	12	24

14.

x	1	2	3	4
y	162	108	72	48

15. The function $y = 1.75(1.02)^t$ represents the value y (in dollars) of a share of stock after t days. Determine whether the function represents *exponential growth* or *exponential decay*. Identify the initial amount and interpret the growth factor or decay factor.

Rewrite the function in the form $f(x) = ab^x$ to determine whether it represents *exponential*
***growth* or *exponential decay*.**

16. $f(x) = 4(1.25)^{x+3}$

17. $f(x) = (1.06)^{8x}$

18. $f(x) = 6(0.84)^{x-4}$

19. You deposit $750 in a savings account that earns 5% annual interest compounded quarterly. (a) Write a function that represents the balance after t years. (b) What is the balance of the account after 4 years?

20. The value of a TV is $1500. Its value decreases by 14% each year. (a) Write a function that represents the value y (in dollars) of the TV after t years. (b) Find the approximate monthly percent decrease in value. (c) Graph the function from part (a). Identify and interpret any asymptotes of the graph. (d) Estimate the value of the TV after 3 years.

6.5 Geometric Sequences *(pp. 311–318)*

Write the next three terms of the geometric sequence 2, 6, 18, 54, ...

Use a table to organize the terms and extend the sequence.

Position	1	2	3	4	5	6	7
Term	2	6	18	54	162	486	1458

Each term is 3 times the previous term. So, the common ratio is 3.

$\times 3$ $\times 3$ $\times 3$ $\times 3$ $\times 3$ $\times 3$

Multiply a term by 3 to find the next term.

▶ The next three terms are 162, 486, and 1458.

Decide whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning. If the sequence is geometric, write the next three terms and graph the sequence.

21. 3, 12, 48, 192, . . .   **22.** 9, −18, 27, −36, . . .   **23.** 375, −75, 15, −3, . . .

Write an equation for the nth term of the geometric sequence. Then find a_9.

24. 1, 4, 16, 64, . . .   **25.** 5, −10, 20, −40, . . .   **26.** 486, 162, 54, 18, . . .

6.6 Recursively Defined Sequences *(pp. 319–326)*

Write a recursive rule for the sequence 5, 12, 19, 26, 33, ...

Use a table to organize the terms and find the pattern.

Position, n	1	2	3	4	5
Term, a_n	5	12	19	26	33

$+ 7$ $+ 7$ $+ 7$ $+ 7$

The sequence is arithmetic, with first term $a_1 = 5$ and common difference $d = 7$.

$a_n = a_{n-1} + d$ Recursive equation for an arithmetic sequence

$a_n = a_{n-1} + 7$ Substitute 7 for d.

▶ So, a recursive rule for the sequence is $a_1 = 5$, $a_n = a_{n-1} + 7$.

Write the first six terms of the sequence. Then graph the sequence.

27. $a_1 = 4$, $a_n = a_{n-1} + 5$ **28.** $a_1 = -4$, $a_n = -3a_{n-1}$ **29.** $a_1 = 32$, $a_n = \frac{1}{4}a_{n-1}$

Write a recursive rule for the sequence.

30. 3, 8, 13, 18, 23, . . .   **31.** 3, 6, 12, 24, 48, . . .   **32.** 7, 6, 13, 19, 32, . . .

33. The first term of a sequence is 8. Each term of the sequence is 5 times the preceding term. Graph the first four terms of the sequence. Write a recursive rule and an explicit rule for the sequence.

Evaluate the expression.

1. $-\sqrt[4]{16}$

2. $729^{1/6}$

3. $(-32)^{7/5}$

Simplify the expression. Write your answer using only positive exponents.

4. $z^{-1/4} \cdot z^{3/5}$

5. $\dfrac{b^{-5}}{a^0 b^{-8}}$

6. $\left(\dfrac{2c^4}{5}\right)^{-3}$

Write and graph a function that represents the situation.

7. Your starting annual salary of \$42,500 increases by 3% each year.

8. You deposit \$500 in an account that earns 6.5% annual interest compounded yearly.

Write an explicit rule and a recursive rule for the sequence.

9.

n	1	2	3	4
a_n	-6	8	22	36

10.

n	1	2	3	4
a_n	400	100	25	6.25

11. Graph $f(x) = 2(6)^x$. Compare the graph to the graph of $g(x) = 6^x$. Identify the y-intercepts and asymptotes of the graphs. Describe the domain and range of f.

12. The function $y = ab^t$ represents the population y (in millions of people) of a country after t years. Choose values of a and b so that the population is initially 13 million people, but is below 10 million people after 3 years.

Use the equation to complete the statement "a ▢ b" with the symbol $<$, $>$, or $=$. Do not attempt to solve the equation.

13. $\dfrac{5^a}{5^b} = 5^{-3}$

14. $9^a \cdot 9^{-b} = 1$

15. The first two terms of a sequence are $a_1 = 3$ and $a_2 = -12$. Let a_3 be the third term when the sequence is arithmetic and let b_3 be the third term when the sequence is geometric. Find $a_3 - b_3$.

16. At sea level, Earth's atmosphere exerts a pressure of 1 atmosphere. Atmospheric pressure P (in atmospheres) decreases with altitude. It can be modeled by $P = (0.99988)^a$, where a is the altitude (in meters).

   a. Identify the initial amount, decay factor, and decay rate.

   b. Use a graphing calculator to graph the function. Use the graph to estimate the atmospheric pressure at an altitude of 5000 feet.

17. You follow the training schedule from your coach.

   a. Write an explicit rule and a recursive rule for the geometric sequence.

   b. On what day do you run approximately 3 kilometers?

> **Training On Your Own**
> **Day 1:** Run 1 km.
> **Each day after Day 1:** Run 20% farther than the previous day.

1. Which graph represents the function $f(x) = -3^x$? *(TEKS A.9.D)*

- Ⓐ *A*
- Ⓑ *B*
- Ⓒ *C*
- Ⓓ *D*

2. **GRIDDED ANSWER** What number should you use to complete the expression so that the statement is true? *(TEKS A.11.B)*

$$\frac{x^{5/3} \cdot x^{-1} \cdot \sqrt[3]{x}}{x^{-2} \cdot x^0} = x^{\boxed{}}$$

3. Which symbol should you use to complete Inequality 2 so that the system of linear inequalities has no solution? *(TEKS A.3.H)*

Inequality 1 $y - 2x \leq 4$

Inequality 2 $6x - 3y \boxed{} -12$

- Ⓕ $<$
- Ⓖ $>$
- Ⓗ $\leq$
- Ⓙ $\geq$

4. The second term of a sequence is 7. Each term of the sequence is 10 more than the preceding term. Which equation gives the nth term of the sequence? *(TEKS A.12.D)*

- Ⓐ $a_n = -3n + 13$
- Ⓑ $a_n = 10n - 13$
- Ⓒ $a_n = 10n - 3$
- Ⓓ $a_n = 10n + 7$

5. Which statement is true about the function $f(x) = -2(9)^x$? *(TEKS A.9.A)*

- Ⓕ The domain is $x < 0$. The range is $y < 0$.
- Ⓖ The domain is all real numbers. The range is $y < 0$.
- Ⓗ The domain is all real numbers. The range is $y > 0$.
- Ⓙ The domain is all real numbers. The range is all real numbers.

6. For 10 years, the population of a city increases in a pattern that is approximately exponential. Using exponential regression, you fit the function $p(t) = 15(1.064)^t$ to the population data, where p is the population (in thousands) of the city in year t. What is a reasonable prediction for the future population of the city? *(TEKS A.9.E)*

    (A) The population will be under 7000 in Year 12.

    (B) The population will be over 40,000 in Year 15.

    (C) The population will be under 50,000 in Year 18.

    (D) The population will be over 300,000 by Year 20.

7. Which equation represents the line that passes through the two points shown? *(TEKS A.2.B)*

    (F) $y = \frac{5}{2}x - 13$

    (G) $y = \frac{2}{5}x - \frac{23}{5}$

    (H) $-5x + 2y = -14$

    (J) $-2x + 5y = 7$
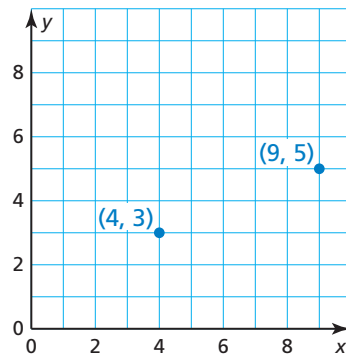
8. Which of the relations is *not* a function? *(TEKS A.12.A)*

(A)

x	−2	0	0	2
y	1	−3	3	5

(B)

x	0	2	4	6
y	−5	−5	−5	−5

(C)

x	1	2	3	4
y	−1	6	13	21

(D)

x	−1	0	1	2
y	7	6	7	10

9. The graph shows the value of a business over time. Which equation models the value v (in dollars) of the business over time t (in years)? *(TEKS A.9.C)*

    (F) $v = 15,000(1.30)^t$

    (G) $v = 15,000(0.70)^t$

    (H) $v = 15,000(0.50)^t$

    (J) $v = 15,000(0.30)^t$