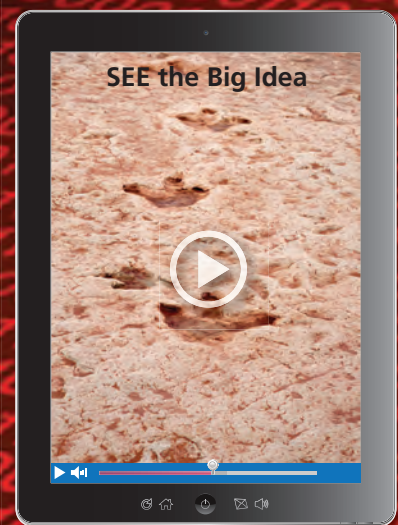


# 9 Trigonometric Identities and Formulas

- 9.1 Using Trigonometric Identities
- 9.2 Using Sum and Difference Formulas
- 9.3 Law of Sines
- 9.4 Law of Cosines



Zip Line (p. 497)



Step Angle (p. 495)



Observation Deck (p. 491)



Sundial (p. 476)



Leaning Tower of Pisa (p. 491)

# Maintaining Mathematical Proficiency

## Solving Proportions

**Example 1** Solve  $\frac{6}{2x+7} = \frac{15}{26}$ .

$$\frac{6}{2x+7} = \frac{15}{26}$$

Write the proportion.

$$6 \cdot 26 = (2x + 7) \cdot 15$$

Cross Products Property

$$156 = 30x + 105$$

Multiply.

$$1.7 = x$$

Solve for  $x$ .

Solve the proportion.

1.  $\frac{3x-5}{10} = \frac{13}{25}$

2.  $\frac{9}{32} = \frac{1+5x}{8}$

3.  $\frac{4}{21} = \frac{16}{8x+7}$

4.  $\frac{5}{18-6x} = \frac{2}{11}$

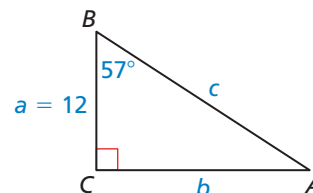
5.  $\frac{18}{x} = \frac{x}{2}$

6.  $\frac{x}{10} = \frac{40}{x}$

## Solving Right Triangles

**Example 2** Solve  $\triangle ABC$ .

Because the triangle is a right triangle,  $A$  and  $B$  are complementary angles. So,  $A = 90^\circ - 57^\circ = 33^\circ$ .



Write two equations, one that involves the ratio of  $b$  and 12, and one that involves the ratio of  $c$  and 12. Solve the first equation for  $b$  and the second equation for  $c$ .

$$\tan 57^\circ = \frac{\text{opp.}}{\text{adj.}}$$

Write trigonometric equation.

$$\sec 57^\circ = \frac{\text{hyp.}}{\text{adj.}}$$

$$\tan 57^\circ = \frac{b}{12}$$

Substitute.

$$\sec 57^\circ = \frac{c}{12}$$

$$12(\tan 57^\circ) = b$$

Solve for the variable.

$$12\left(\frac{1}{\cos 57^\circ}\right) = c$$

$$18.48 \approx b$$

Use a calculator.

$$22.03 \approx c$$

► So,  $A = 33^\circ$ ,  $b \approx 18.48$ , and  $c \approx 22.03$ .

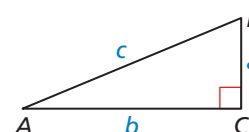
Solve  $\triangle ABC$  using the diagram and the given measurements.

7.  $B = 73^\circ$ ,  $c = 9$

8.  $B = 35^\circ$ ,  $a = 12$

9.  $A = 42^\circ$ ,  $c = 16$

10.  $A = 29^\circ$ ,  $b = 40.5$



11. **ABSTRACT REASONING** You know that  $\tan A = \frac{x}{y}$  in a right triangle. Can you determine the value of  $\sec A$  without finding the angle measure of  $A$ ? Explain.

## Using a Graphing Calculator

### Core Concept

#### Using Graphs to Find Trigonometric Identities

A trigonometric equation that is true for all values of the variable for which both sides of the equation are defined is called a **trigonometric identity**. You can use graphs of trigonometric functions to help you write and verify trigonometric identities. For example, knowing whether a trigonometric function is *even* or *odd* can help you write a trigonometric identity.

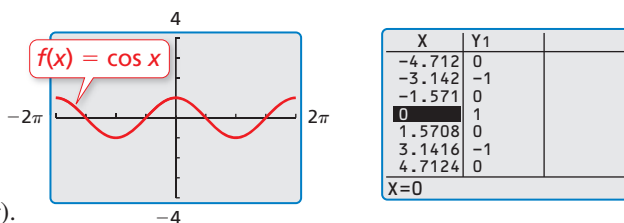
Recall that a function  $f$  is an *even function* when  $f(-x) = f(x)$  for all  $x$  in its domain. The graph of an even function is *symmetric about the y-axis*. A function  $f$  is an *odd function* when  $f(-x) = -f(x)$  for all  $x$  in its domain. The graph of an odd function is *symmetric about the origin*.

#### EXAMPLE 1 Writing a Trigonometric Identity

Use a graphing calculator to graph  $f(x) = \cos x$ . Determine whether  $f$  is *even* or *odd* and then use the corresponding definition to write a trigonometric identity.

#### SOLUTION

Graph the function. Notice that the graph appears to be symmetric about the  $y$ -axis. Use a table of values to confirm this symmetry.



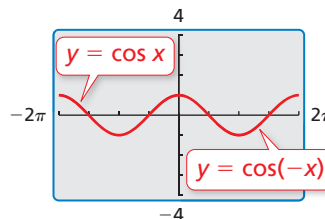
Because the graph is symmetric about the  $y$ -axis,  $f$  is even, meaning that  $f(-x) = f(x)$ . Use this equation to write an identity.

$$f(-x) = f(x) \quad \text{Definition of an even function}$$

$$\cos(-x) = \cos x \quad \text{Substitute using } f(x) = \cos x.$$

Notice that you can confirm your identity by observing that the graphs of  $y = \cos(-x)$  and  $y = \cos x$  coincide.

► So,  $f(x) = \cos x$  is an even function and  $\cos(-x) = \cos x$ .



### Monitoring Progress

Use a graphing calculator to graph the function. Determine whether the function is *even* or *odd* and then use the corresponding definition to write a trigonometric identity.

1.  $f(x) = \sin x$

2.  $f(x) = \tan x$

3.  $f(x) = \sec x$

4. Use a graphing calculator to graph  $f(x) = \sin^2 x + \cos^2 x$ . What do you notice? Use your observation to write a trigonometric identity.

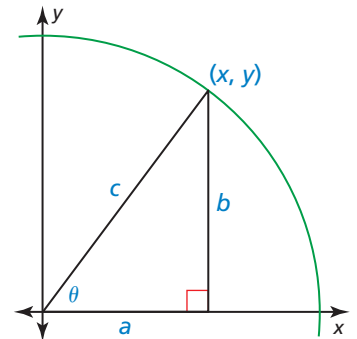
# 9.1 Using Trigonometric Identities

**Essential Question** How can you verify a trigonometric identity?

## EXPLORATION 1 Writing a Trigonometric Identity

**Work with a partner.** In the figure,  $(x, y)$  is on a circle of radius  $c$  with center at the origin.

- Write an equation that relates  $a$ ,  $b$ , and  $c$ .
- Write expressions for the sine and cosine ratios of angle  $\theta$ .
- Use the results from parts (a) and (b) to find the sum of  $\sin^2 \theta$  and  $\cos^2 \theta$ .
- Use a table to verify that the identity you wrote in part (c) is valid for angles in each of the four quadrants.



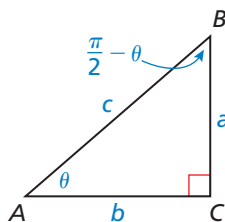
### REASONING ABSTRACTLY

To be proficient in math, you need to know and flexibly use different properties of operations and objects.

- The trigonometric identity you derived in part (c) is called a *Pythagorean identity*. There are two other Pythagorean identities. Divide each side of the identity in part (c) by  $\cos^2 \theta$  and simplify. Divide each side of the identity in part (c) by  $\sin^2 \theta$  and simplify.

## EXPLORATION 2 Writing Cofunction Identities

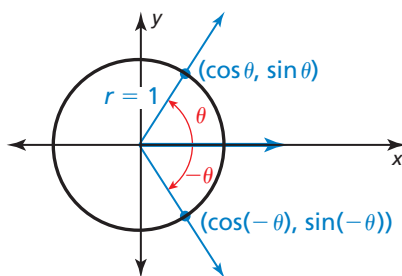
**Work with a partner.** Use the figure at the left. Recall that the two acute angles of a right triangle are complementary. Given the radian measure of one of the acute angles of a right triangle is  $\theta$ , the measure of the other must be  $\frac{\pi}{2} - \theta$ .



- Write expressions for the six trigonometric functions of the angle  $\theta$  and of the angle  $\frac{\pi}{2} - \theta$ .
- Which expressions in part (a) are equivalent? Use the results to write trigonometric identities for the six trigonometric functions of the angle  $\frac{\pi}{2} - \theta$ .

## EXPLORATION 3 Writing Negative Angle Identities

**Work with a partner.** Use the figure at the left.



- Describe the transformation of the point  $(\cos \theta, \sin \theta)$  that results from replacing  $\theta$  with  $-\theta$ . Explain your reasoning.
- Use the coordinate rule for the transformation you described in part (a) to write trigonometric identities for  $\cos(-\theta)$  and  $\sin(-\theta)$ .
- Use the results of part (b) to write trigonometric identities for the remaining four trigonometric functions of the angle  $-\theta$ .

## Communicate Your Answer

- How can you verify a trigonometric identity?
- Is  $\sin \theta = \cos \theta$  a trigonometric identity? Explain your reasoning.

## 9.1 Lesson

### Core Vocabulary

trigonometric identity, p. 472

Previous  
unit circle

### STUDY TIP

Note that  $\sin^2 \theta$  represents  $(\sin \theta)^2$  and  $\cos^2 \theta$  represents  $(\cos \theta)^2$ .

## What You Will Learn

- ▶ Use trigonometric identities to evaluate trigonometric functions and simplify trigonometric expressions.
- ▶ Verify trigonometric identities.

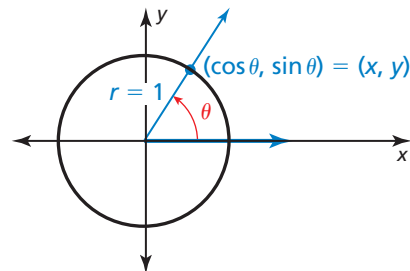
## Using Trigonometric Identities

Recall that when an angle  $\theta$  is in standard position with its terminal side intersecting the unit circle at  $(x, y)$ , then  $x = \cos \theta$  and  $y = \sin \theta$ . Because  $(x, y)$  is on a circle centered at the origin with radius 1, it follows that

$$x^2 + y^2 = 1$$

and

$$\cos^2 \theta + \sin^2 \theta = 1.$$



The equation  $\cos^2 \theta + \sin^2 \theta = 1$  is true for any value of  $\theta$ . A trigonometric equation that is true for all values of the variable for which both sides of the equation are defined is called a **trigonometric identity**. In Section 8.1, you used reciprocal identities to find the values of the cosecant, secant, and cotangent functions. These and other fundamental trigonometric identities are listed below.

## Core Concept

### Fundamental Trigonometric Identities

#### Reciprocal Identities

$$\sin \theta = \frac{1}{\csc \theta}$$

$$\cos \theta = \frac{1}{\sec \theta}$$

$$\tan \theta = \frac{1}{\cot \theta}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

#### Tangent and Cotangent Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

#### Pythagorean Identities

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

#### Cofunction Identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec \theta$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \csc \theta$$

$$\cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$$

#### Negative Angle Identities

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = -\tan \theta$$

$$\csc(-\theta) = -\csc \theta$$

$$\sec(-\theta) = \sec \theta$$

$$\cot(-\theta) = -\cot \theta$$

### EXAMPLE 1 Finding Trigonometric Values

Given that  $\sin \theta = \frac{4}{5}$  and  $\frac{\pi}{2} < \theta < \pi$ , find the values of the other five trigonometric functions of  $\theta$ .

#### SOLUTION

**Step 1** Find  $\cos \theta$ .

$$\sin^2 \theta + \cos^2 \theta = 1$$

Write Pythagorean identity.

$$\left(\frac{4}{5}\right)^2 + \cos^2 \theta = 1$$

Substitute  $\frac{4}{5}$  for  $\sin \theta$ .

$$\cos^2 \theta = 1 - \left(\frac{4}{5}\right)^2$$

Subtract  $\left(\frac{4}{5}\right)^2$  from each side.

$$\cos^2 \theta = \frac{9}{25}$$

Simplify.

$$\cos \theta = \pm \frac{3}{5}$$

Take square root of each side.

$$\cos \theta = -\frac{3}{5}$$

Because  $\theta$  is in Quadrant II,  $\cos \theta$  is negative.

**Step 2** Find the values of the other four trigonometric functions of  $\theta$  using the values of  $\sin \theta$  and  $\cos \theta$ .

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{-\frac{3}{5}}{\frac{4}{5}} = -\frac{3}{4}$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{1}{\frac{4}{5}} = \frac{5}{4} \qquad \sec \theta = \frac{1}{\cos \theta} = \frac{1}{-\frac{3}{5}} = -\frac{5}{3}$$

### EXAMPLE 2 Simplifying Trigonometric Expressions

Simplify (a)  $\tan\left(\frac{\pi}{2} - \theta\right)\sin \theta$  and (b)  $\sec \theta \tan^2 \theta + \sec \theta$ .

#### SOLUTION

a.  $\tan\left(\frac{\pi}{2} - \theta\right)\sin \theta = \cot \theta \sin \theta$

Cofunction identity

$$= \left(\frac{\cos \theta}{\sin \theta}\right)(\sin \theta)$$

Cotangent identity

$$= \cos \theta$$

Simplify.

b.  $\sec \theta \tan^2 \theta + \sec \theta = \sec \theta (\sec^2 \theta - 1) + \sec \theta$

Pythagorean identity

$$= \sec^3 \theta - \sec \theta + \sec \theta$$

Distributive Property

$$= \sec^3 \theta$$

Simplify.

### Monitoring Progress



Help in English and Spanish at [BigIdeasMath.com](http://BigIdeasMath.com)

1. Given that  $\cos \theta = \frac{1}{6}$  and  $0 < \theta < \frac{\pi}{2}$ , find the values of the other five trigonometric functions of  $\theta$ .

Simplify the expression.

2.  $\sin x \cot x \sec x$       3.  $\cos \theta - \cos \theta \sin^2 \theta$       4.  $\frac{\sec^2 x - \tan^2 x}{\cos(-x) \tan x}$

## Verifying Trigonometric Identities

You can use the fundamental identities you have already learned to verify new trigonometric identities. When verifying an identity, begin with the expression on one side. Use algebra and trigonometric properties to manipulate the expression until it is identical to the other side.

### EXAMPLE 3 Verifying a Trigonometric Identity

Verify the identity  $\frac{\sec^2 \theta - 1}{\sec^2 \theta} = \sin^2 \theta$ .

#### SOLUTION

$$\begin{aligned}\frac{\sec^2 \theta - 1}{\sec^2 \theta} &= \frac{\sec^2 \theta}{\sec^2 \theta} - \frac{1}{\sec^2 \theta} && \text{Write as separate fractions.} \\ &= 1 - \left(\frac{1}{\sec \theta}\right)^2 && \text{Simplify.} \\ &= 1 - \cos^2 \theta && \text{Reciprocal identity} \\ &= \sin^2 \theta && \text{Pythagorean identity}\end{aligned}$$

Notice that verifying an identity is not the same as solving an equation. When verifying an identity, you cannot assume that the two sides of the equation are equal because you are trying to verify that they are equal. So, you cannot use any properties of equality, such as adding the same quantity to each side of the equation.

### EXAMPLE 4 Verifying a Trigonometric Identity

Verify the identity  $\sec x + \tan x = \frac{\cos x}{1 - \sin x}$ .

#### SOLUTION

$$\begin{aligned}\sec x + \tan x &= \frac{1}{\cos x} + \tan x && \text{Reciprocal identity} \\ &= \frac{1}{\cos x} + \frac{\sin x}{\cos x} && \text{Tangent identity} \\ &= \frac{1 + \sin x}{\cos x} && \text{Add fractions.} \\ &= \frac{1 + \sin x}{\cos x} \cdot \frac{1 - \sin x}{1 - \sin x} && \text{Multiply by } \frac{1 - \sin x}{1 - \sin x} \\ &= \frac{1 - \sin^2 x}{\cos x(1 - \sin x)} && \text{Simplify numerator.} \\ &= \frac{\cos^2 x}{\cos x(1 - \sin x)} && \text{Pythagorean identity} \\ &= \frac{\cos x}{1 - \sin x} && \text{Simplify.}\end{aligned}$$

### LOOKING FOR STRUCTURE

To verify the identity, you must introduce  $1 - \sin x$  into the denominator. Multiply the numerator and the denominator by  $1 - \sin x$  so you get an equivalent expression.



### Monitoring Progress Help in English and Spanish at [BigIdeasMath.com](http://BigIdeasMath.com)

Verify the identity.

- $\cos \theta \cot\left(\frac{\pi}{2} - \theta\right) = \sin \theta$
- $\frac{\csc^2 x - \cot^2 x}{\cos x \tan(-x)} = -\csc x$
- $\csc^2 x(1 - \sin^2 x) = \cot^2 x$
- $(\tan^2 x + 1)(\cos^2 x - 1) = -\tan^2 x$

# 9.1 Exercises

## Vocabulary and Core Concept Check

- WRITING** Describe the difference between a trigonometric identity and a trigonometric equation.
- WHICH ONE DOESN'T BELONG?** Which trigonometric expression does *not* belong with the other three? Explain your reasoning.

$$\cos\left(\frac{\pi}{2} - \theta\right)$$

$$\sin(-\theta)$$

$$\cos(-\theta) \tan \theta$$

$$\frac{1}{\csc \theta}$$

## Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, find the values of the other five trigonometric functions of  $\theta$ . (See Example 1.)

- $\sin \theta = \frac{1}{3}, 0 < \theta < \frac{\pi}{2}$
- $\sin \theta = -\frac{7}{10}, \pi < \theta < \frac{3\pi}{2}$
- $\tan \theta = -\frac{3}{7}, \frac{\pi}{2} < \theta < \pi$
- $\cot \theta = -\frac{2}{5}, \frac{\pi}{2} < \theta < \pi$
- $\cos \theta = -\frac{5}{6}, \pi < \theta < \frac{3\pi}{2}$
- $\sec \theta = \frac{9}{4}, \frac{3\pi}{2} < \theta < 2\pi$
- $\cot \theta = -3, \frac{3\pi}{2} < \theta < 2\pi$
- $\csc \theta = -\frac{5}{3}, \pi < \theta < \frac{3\pi}{2}$

In Exercises 11–22, simplify the expression. (See Example 2.)

- $\sin x \cot x$
- $\cos \theta(1 + \tan^2 \theta)$
- $\frac{\sin(-\theta)}{\cos(-\theta)}$
- $\frac{\cos^2 x}{\cot^2 x}$
- $\frac{\sin x + \cos x}{1 - \tan(-x)}$
- $\sin\left(\frac{\pi}{2} - \theta\right) \sec \theta$
- $\cot(-x) \csc\left(\frac{\pi}{2} - x\right)$
- $\cos \theta \sec(-\theta)$
- $\frac{\csc^2 x - \cot^2 x}{\sin(-x) \cot x}$
- $\frac{\cos^2 x \tan^2(-x) - 1}{\cos^2 x}$

$$21. \frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\csc \theta} + \cos^2 \theta \quad 22. \frac{\sec x \sin x + \cos\left(\frac{\pi}{2} - x\right)}{1 + \sec x}$$

- ERROR ANALYSIS** Describe and correct the error in simplifying the expression.



$$\begin{aligned} 1 - \sin^2 \theta &= 1 - (1 + \cos^2 \theta) \\ &= 1 - 1 - \cos^2 \theta \\ &= -\cos^2 \theta \end{aligned}$$

- REASONING** Explain how you can use a graphing calculator to determine which of the six trigonometric functions is equal to  $\cot x \cos x + \sin x$ .

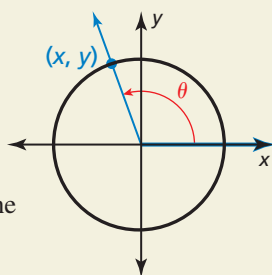
In Exercises 25–34, verify the identity. (See Examples 3 and 4.)

- $\sin x \csc x = 1$
- $\tan \theta \csc \theta \cos \theta = 1$
- $\cos\left(\frac{\pi}{2} - x\right) \cot x = \cos x$
- $\tan x + \tan\left(\frac{\pi}{2} - x\right) = \csc x \sec x$
- $\frac{\cos\left(\frac{\pi}{2} - \theta\right) + 1}{1 - \sin(-\theta)} = 1$
- $\frac{\sin^2(-x)}{\tan^2 x} = \cos^2 x$
- $\frac{1 + \cos x}{\sin x} + \frac{\sin x}{1 + \cos x} = 2 \csc x$
- $\frac{\sin x}{1 - \cos(-x)} = \csc x + \cot x$
- $\frac{2 \sin \theta + \csc(-\theta)}{1 - \cot^2 \theta} = \sin \theta$
- $\frac{2 \cos \theta - \sec(-\theta)}{1 - \tan^2 \theta} = \cos \theta$

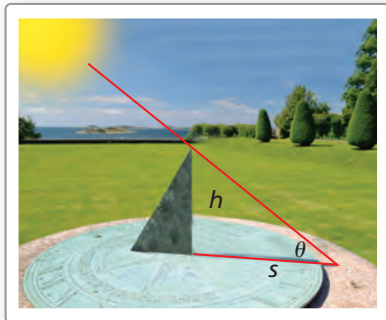


- 35. USING STRUCTURE** A function  $f$  is *odd* when  $f(-x) = -f(x)$ . A function  $f$  is *even* when  $f(-x) = f(x)$ . Which of the six trigonometric functions are odd? Which are even? Justify your answers using identities and graphs.
- 36. ANALYZING RELATIONSHIPS** As the value of  $\cos \theta$  increases, what happens to the value of  $\sec \theta$ ? Explain your reasoning.
- 37. MAKING AN ARGUMENT** Your friend simplifies an expression and obtains  $\sec x \tan x - \sin x$ . You simplify the same expression and obtain  $\sin x \tan^2 x$ . Are your answers equivalent? Justify your answer.

**38. HOW DO YOU SEE IT?** The figure shows the unit circle and the angle  $\theta$ .



- Is  $\sin \theta$  positive or negative?  $\cos \theta$ ?  $\tan \theta$ ?
  - In what quadrant does the terminal side of  $-\theta$  lie?
  - Is  $\sin(-\theta)$  positive or negative?  $\cos(-\theta)$ ?  $\tan(-\theta)$ ?
- 39. MODELING WITH MATHEMATICS** A vertical *gnomon* (the part of a sundial that projects a shadow) has height  $h$ . The length  $s$  of the shadow cast by the gnomon when the angle of the Sun above the horizon is  $\theta$  can be modeled by the equation below. Show that the equation below is equivalent to  $s = h \cot \theta$ .



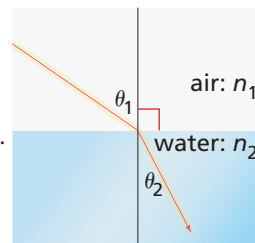
$$s = \frac{h \sin(90^\circ - \theta)}{\sin \theta}$$

**40. THOUGHT PROVOKING** Explain how you can use a trigonometric identity to find all the values of  $x$  for which  $\sin x = \cos x$ .

- 41. DRAWING CONCLUSIONS** *Static friction* is the amount of force necessary to keep a stationary object on a flat surface from moving. Suppose a book weighing  $W$  pounds is lying on a ramp inclined at an angle  $\theta$ . The coefficient of static friction  $u$  for the book can be found using the equation  $uW \cos \theta = W \sin \theta$ .
- Solve the equation for  $u$  and simplify the result.
  - Use the equation from part (a) to determine what happens to the value of  $u$  as the angle  $\theta$  increases from  $0^\circ$  to  $90^\circ$ .

**42. PROBLEM SOLVING** When light traveling in a medium (such as air) strikes the surface of a second medium (such as water) at an angle  $\theta_1$ , the light begins to travel at a different angle  $\theta_2$ . This change of direction is defined by Snell's law,  $n_1 \sin \theta_1 = n_2 \sin \theta_2$ , where  $n_1$  and  $n_2$  are the *indices of refraction* for the two mediums. Snell's law can be derived from the equation

$$\frac{n_1}{\sqrt{\cot^2 \theta_1 + 1}} = \frac{n_2}{\sqrt{\cot^2 \theta_2 + 1}}$$



- Simplify the equation to derive Snell's law.
  - What is the value of  $n_1$  when  $\theta_1 = 55^\circ$ ,  $\theta_2 = 35^\circ$ , and  $n_2 = 2$ ?
  - If  $\theta_1 = \theta_2$ , then what must be true about the values of  $n_1$  and  $n_2$ ? Explain when this situation would occur.
- 43. WRITING** Explain how transformations of the graph of the parent function  $f(x) = \sin x$  support the cofunction identity  $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$ .

- 44. USING STRUCTURE** Verify each identity.
- $\ln|\sec \theta| = -\ln|\cos \theta|$
  - $\ln|\tan \theta| = \ln|\sin \theta| - \ln|\cos \theta|$

## Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the function without using a calculator. (Section 8.3)

- $\sin(-210^\circ)$
- $\tan \frac{4\pi}{3}$
- $\csc 135^\circ$
- $\cos\left(-\frac{13\pi}{6}\right)$
- Describe the transformation of the graph of  $f(x) = \sin x$  represented by the function  $g(x) = 3 \sin 4x - 1$ . (Section 8.4)

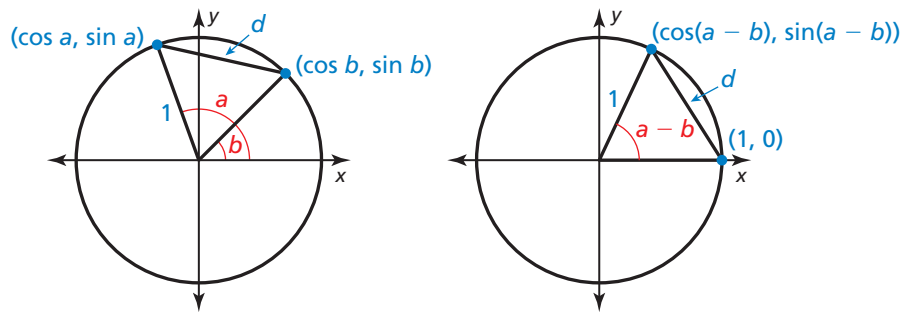
## 9.2 Using Sum and Difference Formulas

**Essential Question** How can you evaluate trigonometric functions of the sum or difference of two angles?

### EXPLORATION 1 Deriving a Difference Formula

Work with a partner.

- a. Explain why the two triangles shown are congruent.



- b. Use the Distance Formula to write an expression for  $d$  in the first unit circle.  
 c. Use the Distance Formula to write an expression for  $d$  in the second unit circle.  
 d. Write an equation that relates the expressions in parts (b) and (c). Then simplify this equation to obtain a formula for  $\cos(a - b)$ .

### CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results.

### EXPLORATION 2 Deriving a Sum Formula

Work with a partner. Use the difference formula you derived in Exploration 1 to write a formula for  $\cos(a + b)$  in terms of sine and cosine of  $a$  and  $b$ . *Hint:* Use the fact that

$$\cos(a + b) = \cos[a - (-b)].$$

### EXPLORATION 3 Deriving Difference and Sum Formulas

Work with a partner. Use the formulas you derived in Explorations 1 and 2 to write formulas for  $\sin(a - b)$  and  $\sin(a + b)$  in terms of sine and cosine of  $a$  and  $b$ . *Hint:* Use the cofunction identities

$$\sin\left(\frac{\pi}{2} - a\right) = \cos a \text{ and } \cos\left(\frac{\pi}{2} - a\right) = \sin a$$

and the fact that

$$\cos\left[\left(\frac{\pi}{2} - a\right) + b\right] = \sin(a - b) \text{ and } \sin(a + b) = \sin[a - (-b)].$$

### Communicate Your Answer

4. How can you evaluate trigonometric functions of the sum or difference of two angles?
5. a. Find the exact values of  $\sin 75^\circ$  and  $\cos 75^\circ$  using sum formulas. Explain your reasoning.  
 b. Find the exact values of  $\sin 75^\circ$  and  $\cos 75^\circ$  using difference formulas. Compare your answers to those in part (a).

## 9.2 Lesson

### Core Vocabulary

Previous  
ratio

## What You Will Learn

- ▶ Use sum and difference formulas to evaluate and simplify trigonometric expressions.
- ▶ Use sum and difference formulas to solve trigonometric equations and rewrite real-life formulas.

## Using Sum and Difference Formulas

In this lesson, you will study formulas that allow you to evaluate trigonometric functions of the sum or difference of two angles.

## Core Concept

### Sum and Difference Formulas

#### Sum Formulas

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

#### Difference Formulas

$$\sin(a - b) = \sin a \cos b - \cos a \sin b$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

In general,  $\sin(a + b) \neq \sin a + \sin b$ . Similar statements can be made for the other trigonometric functions of sums and differences.

### EXAMPLE 1 Evaluating Trigonometric Expressions

Find the exact value of (a)  $\sin 15^\circ$  and (b)  $\tan \frac{7\pi}{12}$ .

#### SOLUTION

#### Check

$$\begin{array}{l} \sin(15^\circ) \\ (\sqrt{(6)} - \sqrt{(2)}) / 4 \end{array} \begin{array}{l} .2588190451 \\ .2588190451 \end{array}$$

a.  $\sin 15^\circ = \sin(60^\circ - 45^\circ)$

$$= \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ$$

$$= \frac{\sqrt{3}}{2} \left( \frac{\sqrt{2}}{2} \right) - \frac{1}{2} \left( \frac{\sqrt{2}}{2} \right)$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

Substitute  $60^\circ - 45^\circ$  for  $15^\circ$ .

Difference formula for sine

Evaluate.

Simplify.

- ▶ The exact value of  $\sin 15^\circ$  is  $\frac{\sqrt{6} - \sqrt{2}}{4}$ . Check this with a calculator.

#### Check

$$\begin{array}{l} \tan(7\pi/12) \\ -2 - \sqrt{(3)} \end{array} \begin{array}{l} -3.732050808 \\ -3.732050808 \end{array}$$

b.  $\tan \frac{7\pi}{12} = \tan\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

$$= \frac{\tan \frac{\pi}{3} + \tan \frac{\pi}{4}}{1 - \tan \frac{\pi}{3} \tan \frac{\pi}{4}}$$

$$= \frac{\sqrt{3} + 1}{1 - \sqrt{3} \cdot 1}$$

$$= -2 - \sqrt{3}$$

Substitute  $\frac{\pi}{3} + \frac{\pi}{4}$  for  $\frac{7\pi}{12}$ .

Sum formula for tangent

Evaluate.

Simplify.

- ▶ The exact value of  $\tan \frac{7\pi}{12}$  is  $-2 - \sqrt{3}$ . Check this with a calculator.

## ANOTHER WAY

You can also use a Pythagorean identity and quadrant signs to find  $\sin a$  and  $\cos b$ .

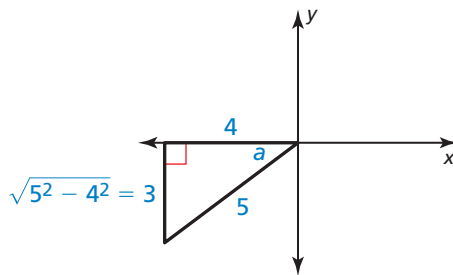
### EXAMPLE 2 Using a Difference Formula

Find  $\cos(a - b)$  given that  $\cos a = -\frac{4}{5}$  with  $\pi < a < \frac{3\pi}{2}$  and  $\sin b = \frac{5}{13}$  with  $0 < b < \frac{\pi}{2}$ .

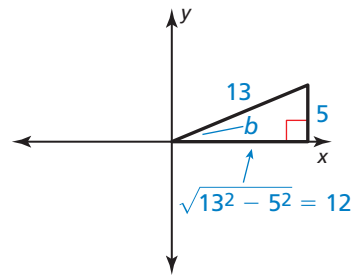
#### SOLUTION

**Step 1** Find  $\sin a$  and  $\cos b$ .

Because  $\cos a = -\frac{4}{5}$  and  $a$  is in Quadrant III,  $\sin a = -\frac{3}{5}$ , as shown in the figure.



Because  $\sin b = \frac{5}{13}$  and  $b$  is in Quadrant I,  $\cos b = \frac{12}{13}$ , as shown in the figure.



**Step 2** Use the difference formula for cosine to find  $\cos(a - b)$ .

$$\begin{aligned}\cos(a - b) &= \cos a \cos b + \sin a \sin b && \text{Difference formula for cosine} \\ &= -\frac{4}{5}\left(\frac{12}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) && \text{Evaluate.} \\ &= -\frac{63}{65} && \text{Simplify.}\end{aligned}$$

► The value of  $\cos(a - b)$  is  $-\frac{63}{65}$ .

### EXAMPLE 3 Simplifying an Expression

Simplify the expression  $\cos(x + \pi)$ .

#### SOLUTION

$$\begin{aligned}\cos(x + \pi) &= \cos x \cos \pi - \sin x \sin \pi && \text{Sum formula for cosine} \\ &= (\cos x)(-1) - (\sin x)(0) && \text{Evaluate.} \\ &= -\cos x && \text{Simplify.}\end{aligned}$$

## Monitoring Progress



Help in English and Spanish at [BigIdeasMath.com](http://BigIdeasMath.com)

Find the exact value of the expression.

- $\sin 105^\circ$
- $\cos 15^\circ$
- $\tan \frac{5\pi}{12}$
- $\cos \frac{\pi}{12}$
- Find  $\sin(a - b)$  given that  $\sin a = \frac{8}{17}$  with  $0 < a < \frac{\pi}{2}$  and  $\cos b = -\frac{24}{25}$  with  $\pi < b < \frac{3\pi}{2}$ .

Simplify the expression.

- $\sin(x + \pi)$
- $\cos(x - 2\pi)$
- $\tan(x - \pi)$

## Solving Equations and Rewriting Formulas

### EXAMPLE 4 Solving a Trigonometric Equation

Solve  $\sin\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{3}\right) = 1$  for  $0 \leq x < 2\pi$ .

#### SOLUTION

$$\sin\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{3}\right) = 1 \quad \text{Write equation.}$$

$$\sin x \cos \frac{\pi}{3} + \cos x \sin \frac{\pi}{3} + \sin x \cos \frac{\pi}{3} - \cos x \sin \frac{\pi}{3} = 1 \quad \text{Use formulas.}$$

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x = 1 \quad \text{Evaluate.}$$

$$\sin x = 1 \quad \text{Simplify.}$$

▶ In the interval  $0 \leq x < 2\pi$ , the solution is  $x = \frac{\pi}{2}$ .

### EXAMPLE 5 Rewriting a Real-Life Formula

The *index of refraction* of a transparent material is the ratio of the speed of light in a vacuum to the speed of light in the material. A triangular prism, like the one shown, can be used to measure the index of refraction using the formula

$$n = \frac{\sin\left(\frac{\theta}{2} + \frac{\alpha}{2}\right)}{\sin \frac{\theta}{2}}$$

For  $\alpha = 60^\circ$ , show that the formula can be rewritten as  $n = \frac{\sqrt{3}}{2} + \frac{1}{2} \cot \frac{\theta}{2}$ .

#### SOLUTION

$$n = \frac{\sin\left(\frac{\theta}{2} + 30^\circ\right)}{\sin \frac{\theta}{2}} \quad \text{Write formula with } \frac{\alpha}{2} = \frac{60^\circ}{2} = 30^\circ.$$

$$= \frac{\sin \frac{\theta}{2} \cos 30^\circ + \cos \frac{\theta}{2} \sin 30^\circ}{\sin \frac{\theta}{2}} \quad \text{Sum formula for sine}$$

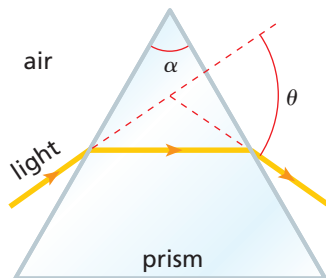
$$= \frac{\left(\sin \frac{\theta}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\cos \frac{\theta}{2}\right)\left(\frac{1}{2}\right)}{\sin \frac{\theta}{2}} \quad \text{Evaluate.}$$

$$= \frac{\frac{\sqrt{3}}{2} \sin \frac{\theta}{2}}{\sin \frac{\theta}{2}} + \frac{\frac{1}{2} \cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \quad \text{Write as separate fractions.}$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2} \cot \frac{\theta}{2} \quad \text{Simplify.}$$

#### ANOTHER WAY

You can also solve the equation by using a graphing calculator. First, graph each side of the original equation. Then use the *intersect* feature to find the  $x$ -value(s) where the expressions are equal.



**Monitoring Progress**  Help in English and Spanish at [BigIdeasMath.com](http://BigIdeasMath.com)

9. Solve  $\sin\left(\frac{\pi}{4} - x\right) - \sin\left(x + \frac{\pi}{4}\right) = 1$  for  $0 \leq x < 2\pi$ .

# 9.2 Exercises

## Vocabulary and Core Concept Check

- COMPLETE THE SENTENCE** Write the expression  $\cos 130^\circ \cos 40^\circ - \sin 130^\circ \sin 40^\circ$  as the cosine of an angle.
- WRITING** Explain how to evaluate  $\tan 75^\circ$  using either the sum or difference formula for tangent.

## Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, find the exact value of the expression.

(See Example 1.)

- |                            |   |
|----------------------------|---|
| 3. $\tan(-15^\circ)$       | 4. $\tan 195^\circ$                     |
| 5. $\sin \frac{23\pi}{12}$ | 6. $\sin(-165^\circ)$                   |
| 7. $\cos 105^\circ$        | 8. $\cos \frac{11\pi}{12}$              |
| 9. $\tan \frac{17\pi}{12}$ | 10. $\sin\left(-\frac{7\pi}{12}\right)$ |

In Exercises 11–16, evaluate the expression given

that  $\cos a = \frac{4}{5}$  with  $0 < a < \frac{\pi}{2}$  and  $\sin b = -\frac{15}{17}$  with  $\frac{3\pi}{2} < b < 2\pi$ . (See Example 2.)

- |                   |                   |
|-------------------|-------------------|
| 11. $\sin(a + b)$ | 12. $\sin(a - b)$ |
| 13. $\cos(a - b)$ | 14. $\cos(a + b)$ |
| 15. $\tan(a + b)$ | 16. $\tan(a - b)$ |

In Exercises 17–22, simplify the expression.

(See Example 3.)

- |   |  |
|---|--|
| 17. $\tan(x + \pi)$                       | 18. $\cos\left(x - \frac{\pi}{2}\right)$ |
| 19. $\cos(x + 2\pi)$                      | 20. $\tan(x - 2\pi)$                     |
| 21. $\sin\left(x - \frac{3\pi}{2}\right)$ | 22. $\tan\left(x + \frac{\pi}{2}\right)$ |

**ERROR ANALYSIS** In Exercises 23 and 24, describe and correct the error in simplifying the expression.

23.



$$\begin{aligned} \tan\left(x + \frac{\pi}{4}\right) &= \frac{\tan x + \tan \frac{\pi}{4}}{1 + \tan x \tan \frac{\pi}{4}} \\ &= \frac{\tan x + 1}{1 + \tan x} \\ &= 1 \end{aligned}$$

24.



$$\begin{aligned} \sin\left(x - \frac{\pi}{4}\right) &= \sin \frac{\pi}{4} \cos x - \cos \frac{\pi}{4} \sin x \\ &= \frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \sin x \\ &= \frac{\sqrt{2}}{2} (\cos x - \sin x) \end{aligned}$$

25. What are the solutions of the equation  $2 \sin x - 1 = 0$  for  $0 \leq x < 2\pi$ ?

- |                      |                      |
|----------------------|----------------------|
| (A) $\frac{\pi}{3}$  | (B) $\frac{\pi}{6}$  |
| (C) $\frac{2\pi}{3}$ | (D) $\frac{5\pi}{6}$ |

26. What are the solutions of the equation  $\tan x + 1 = 0$  for  $0 \leq x < 2\pi$ ?

- |                      |                      |
|----------------------|----------------------|
| (A) $\frac{\pi}{4}$  | (B) $\frac{3\pi}{4}$ |
| (C) $\frac{5\pi}{4}$ | (D) $\frac{7\pi}{4}$ |

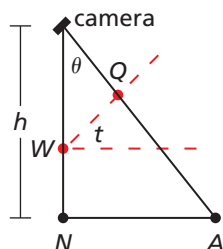
In Exercises 27–32, solve the equation for  $0 \leq x < 2\pi$ .

(See Example 4.)

- |  |  |
|--|--|
| 27. $\sin\left(x + \frac{\pi}{2}\right) = \frac{1}{2}$   | 28. $\tan\left(x - \frac{\pi}{4}\right) = 0$ |
| 29. $\cos\left(x + \frac{\pi}{6}\right) - \cos\left(x - \frac{\pi}{6}\right) = 1$  |  |
| 30. $\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) = 0$  |  |
| 31. $\tan(x + \pi) - \tan(\pi - x) = 0$  |  |
| 32. $\sin(x + \pi) + \cos(x + \pi) = 0$  |  |
| 33. <b>USING EQUATIONS</b> Derive the cofunction identity $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$ using the difference formula for sine. |  |

**34. MAKING AN ARGUMENT** Your friend claims it is possible to use the difference formula for tangent to derive the cofunction identity  $\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$ . Is your friend correct? Explain your reasoning.

**35. MODELING WITH MATHEMATICS** A photographer is at a height  $h$  taking aerial photographs with a 35-millimeter camera. The ratio of the image length  $WQ$  to the length  $NA$  of the actual object is given by the formula



$$\frac{WQ}{NA} = \frac{35 \tan(\theta - t) + 35 \tan t}{h \tan \theta}$$

where  $\theta$  is the angle between the vertical line perpendicular to the ground and the line from the camera to point  $A$  and  $t$  is the tilt angle of the film. When  $t = 45^\circ$ , show that the formula can be rewritten as  $\frac{WQ}{NA} = \frac{70}{h(1 + \tan \theta)}$ . (See Example 5.)

**36. MODELING WITH MATHEMATICS** When a wave travels through a taut string, the displacement  $y$  of each point on the string depends on the time  $t$  and the point's position  $x$ . The equation of a *standing wave* can be obtained by adding the displacements of two waves traveling in opposite directions. Suppose a standing wave can be modeled by the formula

$$y = A \cos\left(\frac{2\pi t}{3} - \frac{2\pi x}{5}\right) + A \cos\left(\frac{2\pi t}{3} + \frac{2\pi x}{5}\right).$$

When  $t = 1$ , show that the formula can be rewritten as  $y = -A \cos \frac{2\pi x}{5}$ .

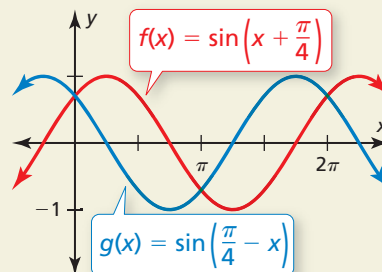
**37. MODELING WITH MATHEMATICS** The busy signal on a touch-tone phone is a combination of two tones with frequencies of 480 hertz and 620 hertz. The individual tones can be modeled by the equations:

**480 hertz:**  $y_1 = \cos 960\pi t$

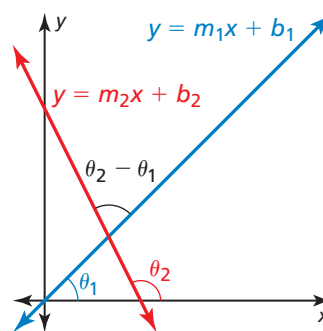
**620 hertz:**  $y_2 = \cos 1240\pi t$

The sound of the busy signal can be modeled by  $y_1 + y_2$ . Show that  $y_1 + y_2 = 2 \cos 1100\pi t \cos 140\pi t$ .

**38. HOW DO YOU SEE IT?** Explain how to use the figure to solve the equation  $\sin\left(x + \frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4} - x\right) = 0$  for  $0 \leq x < 2\pi$ .



**39. MATHEMATICAL CONNECTIONS** The figure shows the acute angle of intersection,  $\theta_2 - \theta_1$ , of two lines with slopes  $m_1$  and  $m_2$ .



- Use the difference formula for tangent to write an equation for  $\tan(\theta_2 - \theta_1)$  in terms of  $m_1$  and  $m_2$ .
- Use the equation from part (a) to find the acute angle of intersection of the lines  $y = x - 1$  and  $y = \left(\frac{1}{\sqrt{3} - 2}\right)x + \frac{4 - \sqrt{3}}{2 - \sqrt{3}}$ .

**40. THOUGHT PROVOKING** Rewrite each function. Justify your answers.

- Write  $\sin 3x$  as a function of  $\sin x$ .
- Write  $\cos 3x$  as a function of  $\cos x$ .
- Write  $\tan 3x$  as a function of  $\tan x$ .

## Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution(s). (Section 6.5)

41.  $1 - \frac{9}{x-2} = -\frac{7}{2}$

42.  $\frac{12}{x} + \frac{3}{4} = \frac{8}{x}$

43.  $\frac{2x-3}{x+1} = \frac{10}{x^2-1} + 5$

# 9.1–9.2 What Did You Learn?

## Core Vocabulary

trigonometric identity, *p.* 472

## Core Concepts

### Section 9.1

Fundamental Trigonometric Identities, *p.* 472

Verifying Trigonometric Identities, *p.* 474

### Section 9.2

Sum and Difference Formulas, *p.* 478

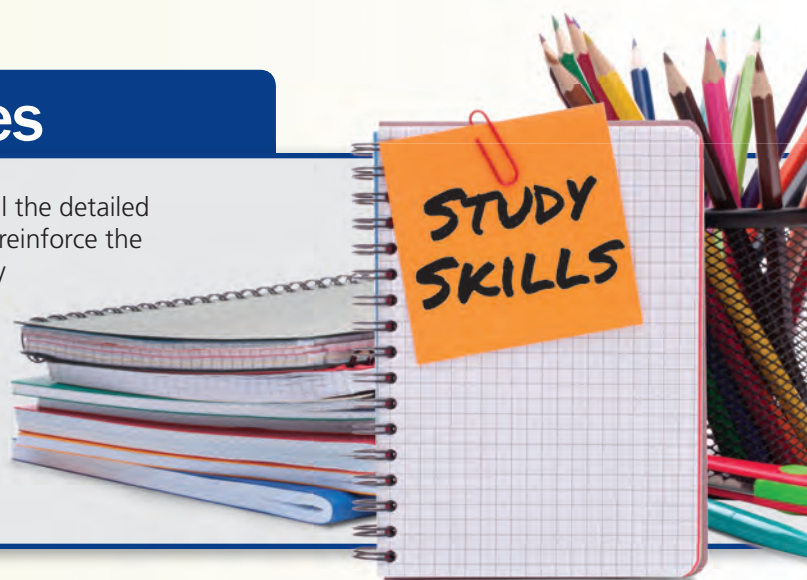
## Mathematical Practices

1. How can you use tools to verify the identity in Exercise 32 on page 475.
2. How can you use definitions to relate the slope of a line with the tangent of an angle in Exercise 39 on page 482?

## Reworking Your Notes

It's almost impossible to write down in your notes all the detailed information you are taught in class. A good way to reinforce the concepts and put them into your long-term memory is to rework your notes. When you take notes, leave extra space on the pages. You can go back after class and fill in:

- important definitions and rules
- additional examples
- questions you have about the material





## 9.1–9.2 Quiz

Find the values of the other five trigonometric functions of  $\theta$ . (Section 9.1)

1.  $\sin \theta = -\frac{7}{25}, \pi < \theta < \frac{3\pi}{2}$       2.  $\sec \theta = \frac{13}{5}, \frac{3\pi}{2} < \theta < 2\pi$       3.  $\cot \theta = -\frac{5}{6}, \frac{\pi}{2} < \theta < \pi$

Simplify the expression. (Section 9.1)

4.  $\cos\left(\frac{\pi}{2} - \theta\right) \csc \theta$       5.  $\frac{\tan x \csc x}{\sec x}$   
6.  $\tan(-x) \sin\left(\frac{\pi}{2} - x\right)$       7.  $\frac{(\csc x + 1)(\csc x - 1)}{\cot\left(\frac{\pi}{2} - x\right)}$

Verify the identity. (Section 9.1)

8.  $\cos \theta \sec(-\theta) = 1$       9.  $\sin x \csc(-x) = -1$   
10.  $\tan x \cot x - \sin^2 x = \cos^2 x$       11.  $\frac{\sin x \csc x}{\sec^2 x - 1} = \cot^2 x$   
12.  $\sec\left(\frac{\pi}{2} - x\right) \tan x = \sec x$       13.  $\frac{\tan \theta - 2 \tan\left(\frac{\pi}{2} - \theta\right)}{\tan(-\theta)} = 2 \cot^2 \theta - 1$   
14. Describe two different ways you can verify the identity  $\csc x + \csc x \cot^2 x = \csc^3 x$  algebraically. (Section 9.1)  
15. Explain how to use trigonometric identities to verify that  $\sec(-\theta) = \sec \theta$ . (Section 9.1)

Find the exact value of the expression. (Section 9.2)

16.  $\sin(-105^\circ)$       17.  $\cos\left(-\frac{13\pi}{12}\right)$   
18.  $\tan\left(\frac{13\pi}{12}\right)$       19.  $\tan 255^\circ$   
20. Find  $\sin(a - b)$  given that  $\sin a = -\frac{3}{5}$  with  $\pi < a < \frac{3\pi}{2}$  and  $\cos b = -\frac{12}{13}$  with  $\pi < b < \frac{3\pi}{2}$ . (Section 9.2)

Simplify the expression. (Section 9.2)

21.  $\cos(x - \pi)$       22.  $\tan(x + 2\pi)$       23.  $\sin\left(x + \frac{\pi}{2}\right)$

Solve the equation for  $0 \leq x < 2\pi$ . (Section 9.2)

24.  $\cos\left(x + \frac{\pi}{4}\right) - \cos\left(x - \frac{\pi}{4}\right) = 1$       25.  $\sin\left(x + \frac{\pi}{2}\right) - \sin\left(x - \frac{\pi}{2}\right) = \sqrt{2}$   
26. Derive the cofunction identity  $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$  using the difference formula for cosine. (Section 9.2)

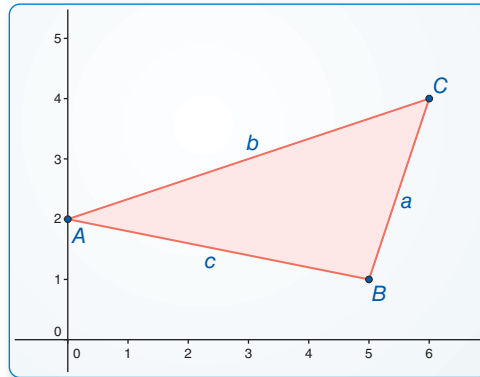
# 9.3 Law of Sines

**Essential Question** What is the Law of Sines?

## EXPLORATION 1 Discovering the Law of Sines

Work with a partner.

- a. Copy and complete the table for the triangle shown. What can you conclude?



**Sample Segments**  
 $a = 3.16$   
 $b = 6.32$   
 $c = 5.10$   
**Angles**  
 $A = 29.74^\circ$   
 $B = 97.13^\circ$   
 $C = 53.13^\circ$

### USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technology to compare predictions with data.

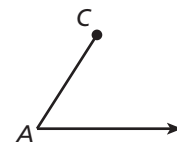
$A$	$a$	$\frac{\sin A}{a}$	$B$	$b$	$\frac{\sin B}{b}$	$C$	$c$	$\frac{\sin C}{c}$

- b. Use dynamic geometry software to draw two other triangles. Copy and complete the table in part (a) for each triangle. Use your results to write a conjecture about the relationship between the sines of the angles and the lengths of the sides of a triangle.

## EXPLORATION 2 Discovering Cases of the Law of Sines

Work with a partner.

- a. Begin constructing  $\triangle ABC$  by drawing side  $AC$  and acute angle  $A$ , as shown.



- b. Set a compass to the distance from point  $C$  to the other side of the angle. How many triangles can you construct using this setting as the length of side  $BC$ ? Explain your reasoning.
- c. Repeat part (b) using a compass setting shorter than the setting in part (b).
- d. Repeat part (b) using a compass setting longer than the setting in part (b).

### Communicate Your Answer

- What is the Law of Sines?
- When would you use the Law of Sines to solve a triangle?

## 9.3 Lesson

### Core Vocabulary

Law of Sines, p. 487

## What You Will Learn

- ▶ Find areas of triangles.
- ▶ Use the Law of Sines to solve triangles.

## Finding Areas of Triangles

Previously, you used trigonometric ratios to solve right triangles. In this lesson, you will learn how to solve any triangle. When the triangle is obtuse, you may need to find a trigonometric ratio for an obtuse angle.

### EXAMPLE 1 Finding Trigonometric Ratios for Obtuse Angles

Evaluate each trigonometric function using a calculator. Round your answer to four decimal places.

- a.  $\tan 150^\circ$                       b.  $\sin 120^\circ$                       c.  $\cos 95^\circ$

#### SOLUTION

- a.  $\tan 150^\circ \approx -0.5774$               b.  $\sin 120^\circ \approx 0.8660$               c.  $\cos 95^\circ \approx -0.0872$

### Monitoring Progress



Help in English and Spanish at [BigIdeasMath.com](http://BigIdeasMath.com)

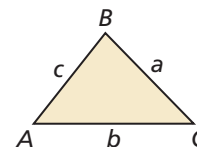
Evaluate the trigonometric function using a calculator. Round your answer to four decimal places.

1.  $\tan 110^\circ$                       2.  $\sin 97^\circ$                       3.  $\cos 165^\circ$

## Core Concept

### Area of a Triangle

The area of any triangle is given by one-half the product of the lengths of two sides times the sine of their included angle. For  $\triangle ABC$  shown, there are three ways to calculate the area.



$$\text{Area} = \frac{1}{2}bc \sin A \qquad \text{Area} = \frac{1}{2}ac \sin B \qquad \text{Area} = \frac{1}{2}ab \sin C$$

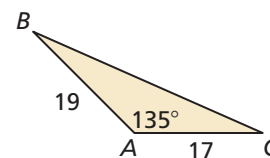
### EXAMPLE 2 Finding the Area of a Triangle

Find the area of the triangle. Round your answer to the nearest tenth.

#### SOLUTION

$$\text{Area} = \frac{1}{2}bc \sin A = \frac{1}{2}(17)(19) \sin 135^\circ \approx 114.2$$

- ▶ The area of the triangle is about 114.2 square units.



### Monitoring Progress



Help in English and Spanish at [BigIdeasMath.com](http://BigIdeasMath.com)

Find the area of  $\triangle ABC$ . Round your answer to the nearest tenth.

4.  $B = 60^\circ$ ,  $a = 19$ ,  $c = 14$                       5.  $C = 29^\circ$ ,  $a = 38$ ,  $b = 31$

## Using the Law of Sines

You can use the **Law of Sines** to solve triangles when two angles and the length of any side are known (AAS or ASA cases), or when the lengths of two sides and an angle opposite one of the two sides are known (SSA case).

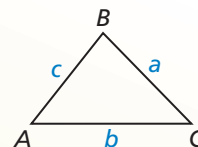
### Theorem

#### Law of Sines

The Law of Sines can be written in either of the following forms for  $\triangle ABC$  with sides of length  $a$ ,  $b$ , and  $c$ .

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

*Proof* Ex. 55, p. 492



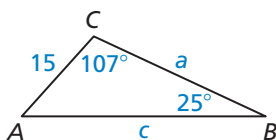
### EXAMPLE 3 Solving a Triangle for the AAS or ASA Case

Solve  $\triangle ABC$ . Round decimal answers to the nearest tenth.

a.  $B = 25^\circ$ ,  $C = 107^\circ$ ,  $b = 15$

b.  $A = 71^\circ$ ,  $C = 60^\circ$ ,  $b = 21$

#### SOLUTION



a. By the Triangle Sum Theorem,  $A = 180^\circ - 107^\circ - 25^\circ = 48^\circ$ .

By the Law of Sines, you can write  $\frac{a}{\sin 48^\circ} = \frac{15}{\sin 25^\circ} = \frac{c}{\sin 107^\circ}$ .

$$\frac{a}{\sin 48^\circ} = \frac{15}{\sin 25^\circ}$$

Write two equations,  
each with one variable.

$$\frac{c}{\sin 107^\circ} = \frac{15}{\sin 25^\circ}$$

$$a = \frac{15 \sin 48^\circ}{\sin 25^\circ}$$

Solve for each variable.

$$c = \frac{15 \sin 107^\circ}{\sin 25^\circ}$$

$$a \approx 26.4$$

Use a calculator.

$$c \approx 33.9$$

► In  $\triangle ABC$ ,  $A = 48^\circ$ ,  $a \approx 26.4$ , and  $c \approx 33.9$ .

b. By the Triangle Sum Theorem,  $B = 180^\circ - 71^\circ - 60^\circ = 49^\circ$ .

By the Law of Sines, you can write  $\frac{a}{\sin 71^\circ} = \frac{21}{\sin 49^\circ} = \frac{c}{\sin 60^\circ}$ .

$$\frac{a}{\sin 71^\circ} = \frac{21}{\sin 49^\circ}$$

Write two equations,  
each with one variable.

$$\frac{c}{\sin 60^\circ} = \frac{21}{\sin 49^\circ}$$

$$a = \frac{21 \sin 71^\circ}{\sin 49^\circ}$$

Solve for each variable.

$$c = \frac{21 \sin 60^\circ}{\sin 49^\circ}$$

$$a \approx 26.3$$

Use a calculator.

$$c \approx 24.1$$

► In  $\triangle ABC$ ,  $B = 49^\circ$ ,  $a \approx 26.3$ , and  $c \approx 24.1$ .

### Monitoring Progress



Help in English and Spanish at [BigIdeasMath.com](http://BigIdeasMath.com)

Solve  $\triangle ABC$ . Round decimal answers to the nearest tenth.

6.  $A = 29^\circ$ ,  $B = 85^\circ$ ,  $b = 9$

7.  $B = 70^\circ$ ,  $C = 81^\circ$ ,  $a = 10$

Two angles and one side (AAS or ASA) determine exactly one triangle. Two sides and an angle opposite one of the sides (SSA) may determine no triangle, one triangle, or two triangles.

### STUDY TIP

Because the SSA case can result in 0, 1, or 2 triangles, it is called the *ambiguous case*.



## Core Concept

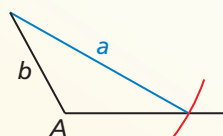
### Possible Triangles in the SSA Case

Consider  $\triangle ABC$ , where you are given  $a$ ,  $b$ , and  $A$ . By fixing side  $b$  and angle  $A$ , you can sketch the possible positions of side  $a$  to figure out how many triangles can be formed. In the diagrams below, note that  $h = b \sin A$ .

**A is obtuse.**

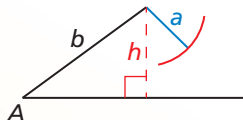


$a \leq b$   
No triangle

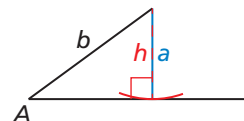


$a > b$   
One triangle

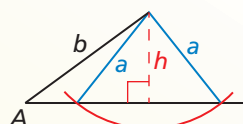
**A is acute.**



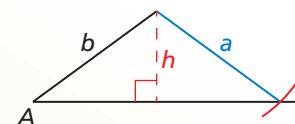
$h > a$   
No triangle



$h = a$   
One triangle



$h < a < b$   
Two triangles



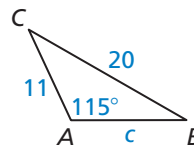
$a \geq b$   
One triangle

### EXAMPLE 4 Solving the SSA Case with One Solution

Solve  $\triangle ABC$  with  $A = 115^\circ$ ,  $a = 20$ , and  $b = 11$ . Round decimal answers to the nearest tenth.

#### SOLUTION

First make a sketch. Because  $A$  is obtuse and the side opposite  $A$  is longer than the given adjacent side, you know that only one triangle can be formed. Use the Law of Sines to find  $B$ .



$$\frac{\sin B}{11} = \frac{\sin 115^\circ}{20}$$

Law of Sines

$$\sin B = \frac{11 \sin 115^\circ}{20}$$

Multiply each side by 11.

$$B \approx 29.9^\circ$$

Use a calculator.

By the Triangle Sum Theorem,  $C \approx 180^\circ - 115^\circ - 29.9^\circ = 35.1^\circ$ .

Use the Law of Sines again to find the remaining side length  $c$  of the triangle.

$$\frac{c}{\sin 35.1^\circ} = \frac{20}{\sin 115^\circ}$$

Law of Sines

$$c = \frac{20 \sin 35.1^\circ}{\sin 115^\circ}$$

Multiply each side by  $\sin 35.1^\circ$ .

$$c \approx 12.7$$

Use a calculator.

► In  $\triangle ABC$ ,  $B \approx 29.9^\circ$ ,  $C \approx 35.1^\circ$ , and  $c \approx 12.7$ .

### Monitoring Progress



Help in English and Spanish at [BigIdeasMath.com](http://BigIdeasMath.com)

Solve  $\triangle ABC$ . Round decimal answers to the nearest tenth.

8.  $A = 110^\circ$ ,  $a = 15$ ,  $b = 10$

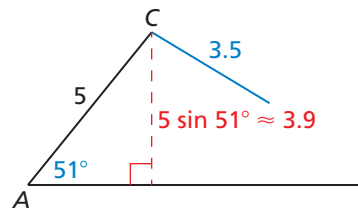
9.  $B = 63^\circ$ ,  $a = 7$ ,  $b = 12$

### EXAMPLE 5 Examining the SSA Case with No Solution

Solve  $\triangle ABC$  with  $A = 51^\circ$ ,  $a = 3.5$ , and  $b = 5$ . Round decimal answers to the nearest tenth.

#### SOLUTION

First make a sketch. Begin by drawing a horizontal line. On one end form a  $51^\circ$  angle ( $A$ ) and draw a segment 5 units long ( $\overline{AC}$ , or  $b$ ). At vertex  $C$ , draw a segment 3.5 units long ( $a$ ). You can see that  $a$  needs to be at least  $5 \sin 51^\circ \approx 3.9$  units long to reach the horizontal side and form a triangle. So, it is not possible to draw the indicated triangle.



### EXAMPLE 6 Solving the SSA Case with Two Solutions

Solve  $\triangle ABC$  with  $A = 40^\circ$ ,  $a = 13$ , and  $b = 16$ . Round decimal answers to the nearest tenth.

#### SOLUTION

First make a sketch. Because  $b \sin A = 16 \sin 40^\circ \approx 10.3$ , and  $10.3 < 13 < 16$  ( $h < a < b$ ), two triangles can be formed.

Use the Law of Sines to find the possible measures of  $B$ .

$$\frac{\sin B}{16} = \frac{\sin 40^\circ}{13} \quad \text{Law of Sines}$$

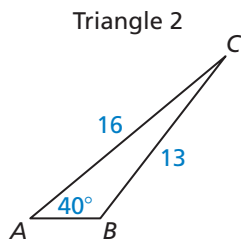
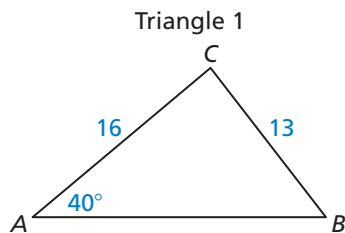
$$\sin B = \frac{16 \sin 40^\circ}{13} \quad \text{Multiply each side by 16.}$$

$$\sin B \approx 0.7911 \quad \text{Use a calculator.}$$

There are two angles  $B$  between  $0^\circ$  and  $180^\circ$  for which  $\sin B \approx 0.7911$ . One is acute and the other is obtuse. Use your calculator to find the acute angle:  $\sin^{-1} 0.7911 \approx 52.3^\circ$ .

The obtuse angle has  $52.3^\circ$  as a reference angle, so its measure is  $180^\circ - 52.3^\circ = 127.7^\circ$ . So,  $B \approx 52.3^\circ$  or  $B \approx 127.7^\circ$ .

Now find the remaining angle  $C$  and side length  $c$  for each triangle.



#### Triangle 1

$$C \approx 180^\circ - 40^\circ - 52.3^\circ = 87.7^\circ$$

$$\frac{c}{\sin 87.7^\circ} = \frac{13}{\sin 40^\circ}$$

$$c = \frac{13 \sin 87.7^\circ}{\sin 40^\circ}$$

$$c \approx 20.2$$

#### Triangle 2

$$C \approx 180^\circ - 40^\circ - 127.7^\circ = 12.3^\circ$$

$$\frac{c}{\sin 12.3^\circ} = \frac{13}{\sin 40^\circ}$$

$$c = \frac{13 \sin 12.3^\circ}{\sin 40^\circ}$$

$$c \approx 4.3$$

► In Triangle 1,  $B \approx 52.3^\circ$ ,  $C \approx 87.7^\circ$ , and  $c \approx 20.2$ .

► In Triangle 2,  $B \approx 127.7^\circ$ ,  $C \approx 12.3^\circ$ , and  $c \approx 4.3$ .

### Monitoring Progress



Help in English and Spanish at [BigIdeasMath.com](http://BigIdeasMath.com)

Solve  $\triangle ABC$ , if possible. Round decimal answers to the nearest tenth.

10.  $A = 44^\circ$ ,  $a = 11$ ,  $b = 14$

11.  $A = 65^\circ$ ,  $a = 6$ ,  $b = 8$

# 9.3 Exercises

## Vocabulary and Core Concept Check

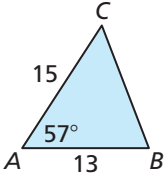
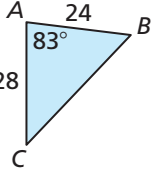
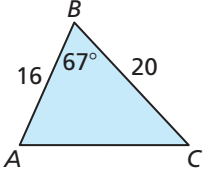
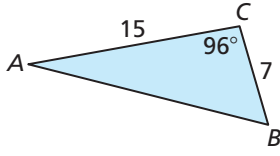
- VOCABULARY** What information do you need to use the Law of Sines?
- WRITING** Suppose  $a$ ,  $b$ , and  $A$  are given for  $\triangle ABC$  and  $A < 90^\circ$ . Under what conditions would you have no triangle? one triangle? two triangles?

## Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, evaluate the trigonometric function using a calculator. Round your answer to four decimal places. (See Example 1.)

- $\sin 127^\circ$
- $\sin 98^\circ$
- $\cos 139^\circ$
- $\cos 108^\circ$
- $\tan 165^\circ$
- $\tan 116^\circ$

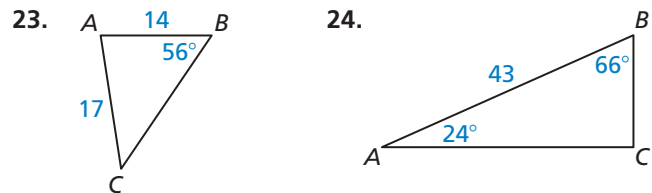
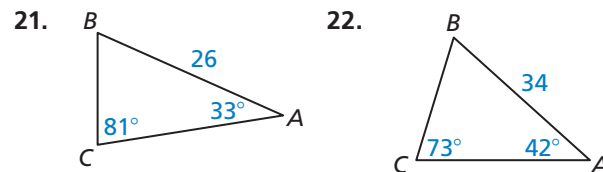
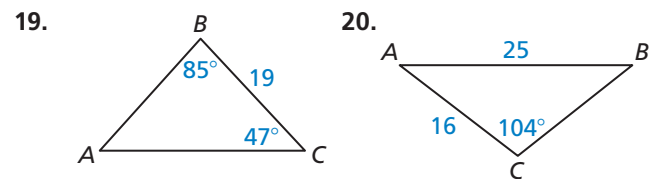
In Exercises 9–12, find the area of  $\triangle ABC$ . Round your answer to the nearest tenth. (See Example 2.)

- 
- 
- 
- 

In Exercises 13–18, state the case (AAS, ASA, or SSA) applicable to the given measurements. Then decide whether the measurements determine *one triangle*, *two triangles*, or *no triangle*.

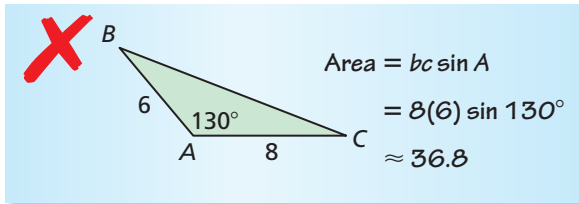
- $A = 49^\circ$ ,  $a = 24$ ,  $b = 22$
- $A = 50^\circ$ ,  $B = 30^\circ$ ,  $b = 8$
- $B = 65^\circ$ ,  $a = 21$ ,  $b = 20$
- $B = 95^\circ$ ,  $C = 46^\circ$ ,  $a = 14$
- $C = 120^\circ$ ,  $b = 12$ ,  $c = 11$
- $C = 135^\circ$ ,  $b = 25$ ,  $c = 30$

In Exercises 19–34, solve  $\triangle ABC$ , if possible. Round decimal answers to the nearest tenth. (See Examples 3, 4, 5, and 6.)

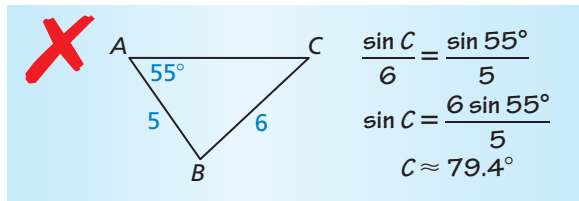


- $A = 24^\circ$ ,  $B = 68^\circ$ ,  $b = 14$
- $B = 35^\circ$ ,  $C = 92^\circ$ ,  $a = 15$
- $A = 55^\circ$ ,  $a = 12$ ,  $b = 8$
- $C = 26^\circ$ ,  $b = 5$ ,  $c = 5$
- $A = 150^\circ$ ,  $a = 9$ ,  $b = 16$
- $B = 80^\circ$ ,  $b = 10$ ,  $c = 20$
- $A = 75^\circ$ ,  $a = 15$ ,  $b = 15.5$
- $B = 10^\circ$ ,  $b = 6$ ,  $c = 6.5$
- $A = 108^\circ$ ,  $a = 10.2$ ,  $b = 7.8$
- $C = 115^\circ$ ,  $b = 21.9$ ,  $c = 17.8$

35. **ERROR ANALYSIS** Describe and correct the error in finding the area of  $\triangle ABC$ .

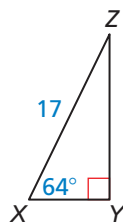


36. **ERROR ANALYSIS** Describe and correct the error in finding  $C$ .



**COMPARING METHODS** In Exercises 37–44, tell whether you would use the Law of Sines or the Pythagorean Theorem and trigonometric functions to solve the triangle with the given information. Explain your reasoning. Then solve the triangle.

37.  $A = 72^\circ, B = 44^\circ, b = 14$   
 38.  $B = 98^\circ, C = 37^\circ, a = 18$   
 39.  $B = 90^\circ, a = 15, c = 6$   
 40.  $C = 40^\circ, b = 27, c = 36$   
 41.  $B = 50^\circ, C = 80^\circ, c = 14$   
 42.  $A = 110^\circ, C = 48^\circ, a = 26$   
 43.  $A = 90^\circ, b = 19, c = 35$   
 44.  $C = 78^\circ, b = 50, c = 49$   
 45. **REASONING** Use  $\triangle XYZ$ .



- a. Can you use the Law of Sines to solve  $\triangle XYZ$ ? Explain your reasoning.  
 b. Can you use another method to solve  $\triangle XYZ$ ? Explain your reasoning.

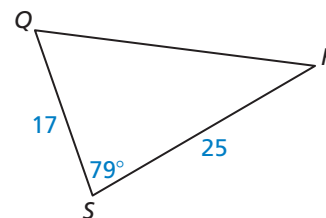
46. **MODELING WITH MATHEMATICS** The Leaning Tower of Pisa in Italy has a height of 183 feet and is  $4^\circ$  off vertical. Find the horizontal distance  $d$  that the top of the tower is off vertical.



47. **MODELING WITH MATHEMATICS** You are on the observation deck of the Empire State Building looking at the Chrysler Building. When you turn  $145^\circ$  clockwise, you see the Statue of Liberty. You know that the Chrysler Building and the Empire State Building are about 0.6 mile apart and that the Chrysler Building and the Statue of Liberty are about 5.6 miles apart. Estimate the distance between the Empire State Building and the Statue of Liberty.



48. **MAKING AN ARGUMENT** Your friend calculates the area of the triangle using the formula  $A = \frac{1}{2}qr \sin S$  and says that the area is approximately 208.6 square units. Is your friend correct? Explain your reasoning.

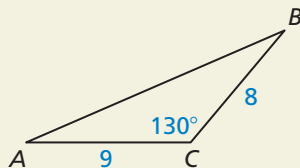




- 49. MODELING WITH MATHEMATICS** You are fertilizing a triangular garden. One side of the garden is 62 feet long, and another side is 54 feet long. The angle opposite the 62-foot side is  $58^\circ$ .
- Draw a diagram to represent this situation.
  - Use the Law of Sines to solve the triangle from part (a).
  - One bag of fertilizer covers an area of 200 square feet. How many bags of fertilizer will you need to cover the entire garden?

**50. HOW DO YOU SEE IT?**

Can you use the Law of Sines to solve the triangle? Explain.

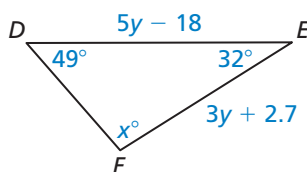


- 51. COMPARING METHODS** A building is constructed on top of a cliff that is 300 meters high. A person standing on level ground below the cliff observes that the angle of elevation to the top of the building is  $72^\circ$  and the angle of elevation to the top of the cliff is  $63^\circ$ .
- How far away is the person from the base of the cliff?
  - Describe two different methods you can use to find the height of the building. Use one of these methods to find the building's height.

- 52. THOUGHT PROVOKING** Explain why there is no case for SSA where the angle is obtuse and two triangles are possible.

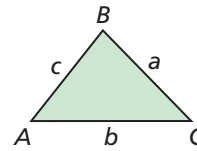
**53. MATHEMATICAL CONNECTIONS**

Find the values of  $x$  and  $y$ .



- 54. REWRITING A FORMULA** Follow the steps to derive the formula for the area of a triangle,  

$$\text{Area} = \frac{1}{2}ab \sin C.$$



- Draw the altitude from vertex  $B$  to  $\overline{AC}$ . Label the altitude as  $h$ . Write a formula for the area of the triangle using  $h$ .
- Write an equation for  $\sin C$ .
- Use the results of parts (a) and (b) to write a formula for the area of a triangle that does not include  $h$ .

- 55. PROVING A THEOREM** Follow the steps to use the formula for the area of a triangle to prove the Law of Sines.

- Use the derivation in Exercise 54 to explain how to derive the three related formulas for the area of a triangle.

$$\text{Area} = \frac{1}{2}bc \sin A,$$

$$\text{Area} = \frac{1}{2}ac \sin B,$$

$$\text{Area} = \frac{1}{2}ab \sin C$$

- Why can you use the formulas in part (a) to write the following statement?

$$\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$$

- Show how to rewrite the statement in part (b) to prove the Law of Sines. Justify each step.

## Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the trigonometric function using a calculator. (Section 8.1)

56.  $\cos 25^\circ$

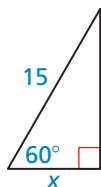
57.  $\sin 48^\circ$

58.  $\tan 20^\circ$

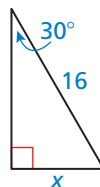
59.  $\sec 36^\circ$

Find the value of  $x$  for the right triangle. (Section 8.1)

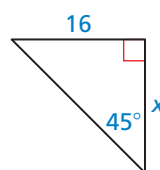
60.



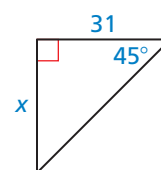
61.



62.



63.



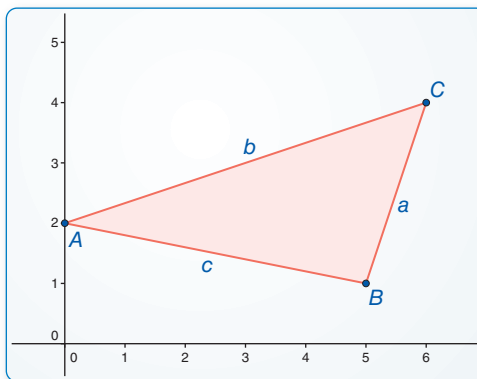
# 9.4 Law of Cosines

**Essential Question** What is the Law of Cosines?

## EXPLORATION 1 Discovering the Law of Cosines

Work with a partner.

- a. Copy and complete the table for the triangle shown. What can you conclude?



**Sample**  
Segments  
 $a = 3.16$   
 $b = 6.32$   
 $c = 5.10$   
Angles  
 $A = 29.74^\circ$   
 $B = 97.13^\circ$   
 $C = 53.13^\circ$

### USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technology to compare predictions with data.

$c$	$c^2$	$a$	$a^2$	$b$	$b^2$	$C$	$a^2 + b^2 - 2ab \cos C$

- b. Use dynamic geometry software to draw two other triangles. Copy and complete the table in part (a) for each triangle. Use your results to write a conjecture about what you observe in the completed tables.

## EXPLORATION 2 Discovering Heron's Area Formula

Work with a partner.

- a. Copy and complete the table for the triangle in Exploration 1(a). What can you conclude?

$a$	$A$	$b$	$c$	$s = \frac{1}{2}(a + b + c)$	$\sqrt{s(s - a)(s - b)(s - c)}$	Area = $\frac{1}{2}bc \sin A$

- b. Use dynamic geometry software to draw two other triangles. Copy and complete the table in part (a) for each triangle. Use your results to write a conjecture about what you observe in the completed tables.

### Communicate Your Answer

- What is the Law of Cosines?
- When would you use the Law of Cosines to solve a triangle?

# 9.4 Lesson

## Core Vocabulary

Law of Cosines, p. 494

## What You Will Learn

- ▶ Use the Law of Cosines to solve triangles.
- ▶ Find areas of triangles.

## Using the Law of Cosines

You can use the **Law of Cosines** to solve triangles when two sides and the included angle are known (SAS case), or when all three sides are known (SSS case).

## Theorem

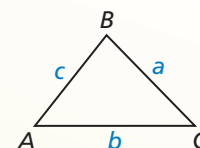
### Law of Cosines

If  $\triangle ABC$  has sides of length  $a$ ,  $b$ , and  $c$ , as shown, then the following are true.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

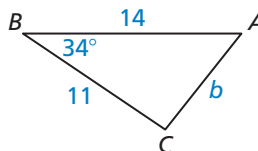
$$c^2 = a^2 + b^2 - 2ab \cos C$$



*Proof* Ex. 26, p. 498

### EXAMPLE 1 Using the Law of Cosines (SAS Case)

Solve  $\triangle ABC$ . Round decimal answers to the nearest tenth.



### SOLUTION

Use the Law of Cosines to find side length  $b$ .

$$b^2 = a^2 + c^2 - 2ac \cos B$$

Law of Cosines

$$b^2 = 11^2 + 14^2 - 2(11)(14) \cos 34^\circ$$

Substitute.

$$b^2 = 317 - 308 \cos 34^\circ$$

Simplify.

$$b = \sqrt{317 - 308 \cos 34^\circ}$$

Find the positive square root.

$$b \approx 7.9$$

Use a calculator.

### ANOTHER WAY

When you know all three sides and one angle, you can use the Law of Cosines or the Law of Sines to find the measure of a second angle.

Use the Law of Sines to find  $A$ . You want to find the smaller remaining angle first because the inverse sine feature of a calculator only gives angle measures from  $0^\circ$  to  $90^\circ$ .

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Law of Sines

$$\frac{\sin A}{11} = \frac{\sin 34^\circ}{\sqrt{317 - 308 \cos 34^\circ}}$$

Substitute.

$$\sin A = \frac{11 \sin 34^\circ}{\sqrt{317 - 308 \cos 34^\circ}}$$

Multiply each side by 11.

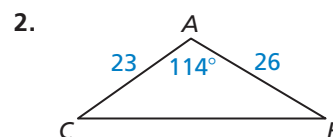
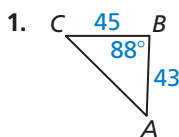
$$A \approx 51.6^\circ$$

Use a calculator.

By the Triangle Sum Theorem,  $C \approx 180^\circ - 34^\circ - 51.6^\circ = 94.4^\circ$ .

- ▶ In  $\triangle ABC$ ,  $b \approx 7.9$ ,  $A \approx 51.6^\circ$ , and  $C \approx 94.4^\circ$ .

Solve  $\triangle ABC$ . Round decimal answers to the nearest tenth.

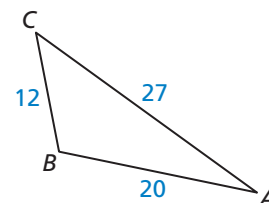


**COMMON ERROR**

In Example 2, the largest angle is found first to make sure that the other two angles are acute. This way, when you use the Law of Sines to find another angle measure, you will know that it is between  $0^\circ$  and  $90^\circ$ .

**EXAMPLE 2 Using the Law of Cosines (SSS Case)**

Solve  $\triangle ABC$ . Round decimal answers to the nearest tenth.



**SOLUTION**

First, find the angle opposite the longest side,  $\overline{AC}$ . Use the Law of Cosines to find  $B$ .

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$27^2 = 12^2 + 20^2 - 2(12)(20) \cos B$$

$$\frac{27^2 - 12^2 - 20^2}{-2(12)(20)} = \cos B$$

$$B \approx 112.7^\circ$$

Law of Cosines

Substitute.

Solve for  $\cos B$ .

Use a calculator.

Now, use the Law of Sines to find  $A$ .

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Law of Sines

$$\frac{\sin A}{12} = \frac{\sin 112.7^\circ}{27}$$

Substitute.

$$\sin A = \frac{12 \sin 112.7^\circ}{27}$$

Multiply each side by 12.

$$A \approx 24.2^\circ$$

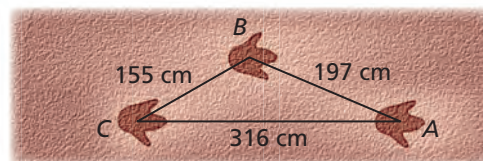
Use a calculator.

By the Triangle Sum Theorem,  $C \approx 180^\circ - 24.2^\circ - 112.7^\circ = 43.1^\circ$ .

► In  $\triangle ABC$ ,  $A \approx 24.2^\circ$ ,  $B \approx 112.7^\circ$ , and  $C \approx 43.1^\circ$ .

**EXAMPLE 3 Solving a Real-Life Problem**

An organism's step angle is a measure of walking efficiency. The closer the step angle is to  $180^\circ$ , the more efficiently the organism walked. The diagram shows a set of footprints for a dinosaur. Find the step angle  $B$ .



**SOLUTION**

$$b^2 = a^2 + c^2 - 2ac \cos B$$

Law of Cosines

$$316^2 = 155^2 + 197^2 - 2(155)(197) \cos B$$

Substitute.

$$\frac{316^2 - 155^2 - 197^2}{-2(155)(197)} = \cos B$$

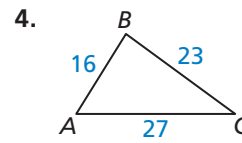
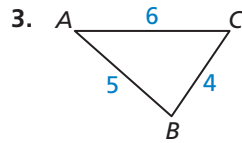
Solve for  $\cos B$ .

$$127.3^\circ \approx B$$

Use a calculator.

► The step angle  $B$  is about  $127.3^\circ$ .

Solve  $\triangle ABC$ . Round decimal answers to the nearest tenth.



## Finding Areas of Triangles

In the previous section, you used a formula involving sine to find the areas of triangles. Another formula for the area of a triangle is Heron's area formula, which can be derived using the Law of Cosines.

### Core Concept

#### Heron's Area Formula

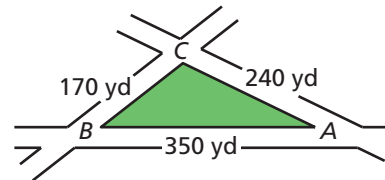
The area of any triangle with sides of length  $a$ ,  $b$ , and  $c$  is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where  $s = \frac{1}{2}(a + b + c)$ . The variable  $s$  is called the *semiperimeter*, or half-perimeter, of the triangle.

#### EXAMPLE 4 Finding the Area of a Triangle

The intersection of three streets forms a piece of land called a traffic triangle. Find the area of the traffic triangle shown. Round your answer to the nearest tenth.



#### SOLUTION

**Step 1** Find the semiperimeter  $s$ .

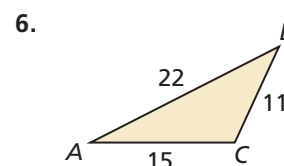
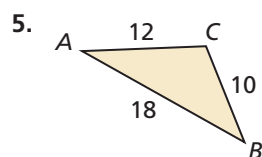
$$\begin{aligned} s &= \frac{1}{2}(a + b + c) \\ &= \frac{1}{2}(170 + 240 + 350) \\ &= 380 \end{aligned}$$

**Step 2** Use Heron's formula to find the area of  $\triangle ABC$ .

$$\begin{aligned} \text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{380(380-170)(380-240)(380-350)} \\ &\approx 18,307.4 \end{aligned}$$

▶ The area of the traffic triangle is about 18,307.4 square yards.

Find the area of  $\triangle ABC$ . Round your answer to the nearest tenth.



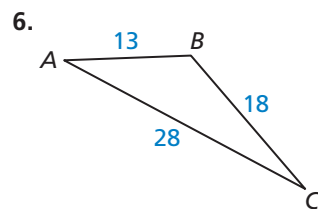
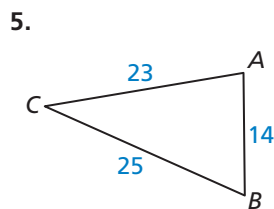
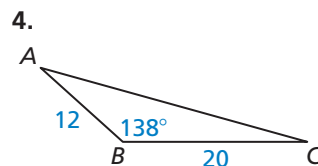
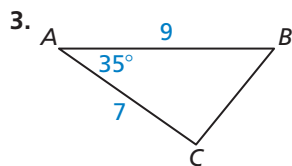
# 9.4 Exercises

## Vocabulary and Core Concept Check

- VOCABULARY** What information do you need to use the Law of Cosines?
- WRITING** After using the Law of Cosines for the SSS case to find the first angle measure, why can you use the Law of Sines to find the second angle measure?

## Monitoring Progress and Modeling with Mathematics

In Exercises 3–12, solve  $\triangle ABC$ . Round decimal answers to the nearest tenth. (See Examples 1 and 2.)



7.  $B = 63^\circ, a = 29, c = 38$

8.  $a = 12, b = 5, c = 13$

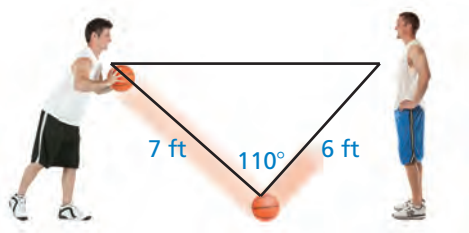
9.  $a = 8, b = 9, c = 12$

10.  $A = 46^\circ, b = 14, c = 16$

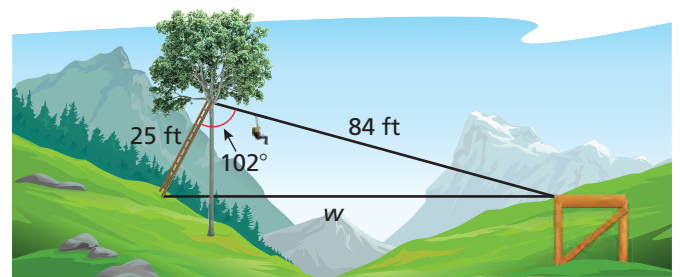
11.  $B = 154^\circ, a = 7, c = 10$

12.  $a = 5, b = 11, c = 15$

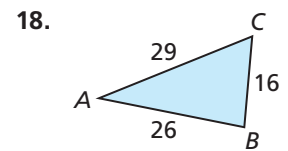
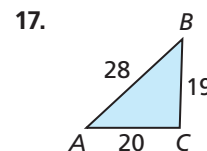
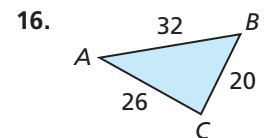
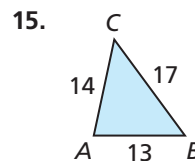
13. **MODELING WITH MATHEMATICS** You and your friend are standing on the baseline of a basketball court. You bounce a basketball to your friend, as shown in the diagram. What is the distance between you and your friend? (See Example 3.)



14. **MODELING WITH MATHEMATICS** A zip line is constructed across a valley, as shown in the diagram. What is the width  $w$  of the valley?



In Exercises 15–18, find the area of  $\triangle ABC$ . Round your answer to the nearest tenth. (See Example 4.)



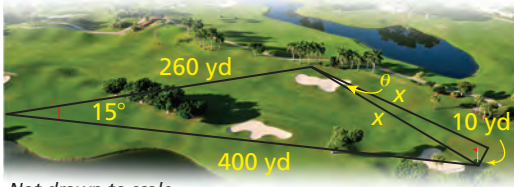
19. **ERROR ANALYSIS** Describe and correct the error in finding  $A$  in  $\triangle ABC$  when  $a = 19, b = 21$ , and  $c = 11$ .



$$\cos A = \frac{19^2 - 21^2 - 11^2}{-2(19)(21)}$$

$$A \approx 75.4^\circ$$

20. **MODELING WITH MATHEMATICS** A golfer hits a drive 260 yards on a hole that is 400 yards long. The shot is  $15^\circ$  off target.



- a. What is the distance  $x$  from the golfer's ball to the hole?
- b. Assume the golfer is able to hit the ball precisely the distance found in part (a). What is the maximum angle  $\theta$  by which the ball can be off target in order to land no more than 10 yards from the hole?
21. **REWRITING A FORMULA** Simplify the Law of Cosines for when the given angle is a right angle.

26. **PROVING A THEOREM** Use the given information to complete the two-column proof of the Law of Cosines.

**Given**  $\overline{BD}$  is an altitude of  $\triangle ABC$ .

**Prove**  $a^2 = b^2 + c^2 - 2bc \cos A$

**STATEMENTS**

1.  $\overline{BD}$  is an altitude of  $\triangle ABC$ .

2.  $\triangle ADB$  and  $\triangle CDB$  are right triangles.

3.  $a^2 = (b - x)^2 + h^2$

4. \_\_\_\_\_

5.  $x^2 + h^2 = c^2$

6. \_\_\_\_\_

7.  $\cos A = \frac{x}{c}$

8.  $x = c \cos A$

9.  $a^2 = b^2 + c^2 - 2bc \cos A$

**REASONS**

1. Given

2. \_\_\_\_\_

3. \_\_\_\_\_

4. Expand binomial.

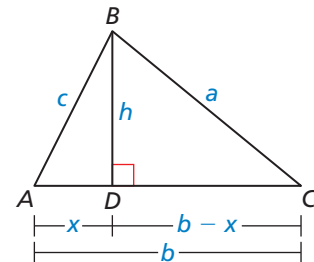
5. \_\_\_\_\_

6. Substitution Property of Equality

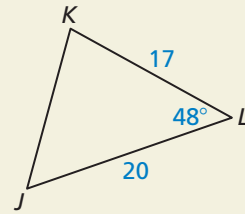
7. \_\_\_\_\_

8. \_\_\_\_\_

9. \_\_\_\_\_



22. **HOW DO YOU SEE IT?** Would you use the Law of Sines or the Law of Cosines to solve the triangle? Explain.



23. **ABSTRACT REASONING** Use the Law of Cosines to show that the measure of each angle of an equilateral triangle is  $60^\circ$ . Explain your reasoning.

24. **THOUGHT PROVOKING** How can you use the Law of Cosines to derive Heron's area formula?

25. **CRITICAL THINKING** An airplane flies  $55^\circ$  east of north from City A to City B, a distance of 470 miles. Another airplane flies  $7^\circ$  north of east from City A to City C, a distance of 890 miles. What is the distance between Cities B and C?

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Write the first six terms of the sequence. (Section 7.1)

27.  $a_n = n + 7$

28.  $a_n = n^2 - 3$

29.  $a_n = n^3 + 1$

30.  $a_n = 2^{n-1}$

## 9.3–9.4 What Did You Learn?

### Core Vocabulary

Law of Sines, *p.* 487

Law of Cosines, *p.* 494

### Core Concepts

#### Section 9.3

Area of a Triangle, *p.* 486

Law of Sines, *p.* 487

Possible Triangles in the SSA Case, *p.* 488

#### Section 9.4

Law of Cosines, *p.* 494

Heron's Area Formula, *p.* 496

### Mathematical Practices

1. Describe the overall step-by-step process you used to solve Exercise 49 on page 492.
2. In Exercise 53 on page 492, explain how you started solving the problem and why you started that way.
3. Create a diagram to model Exercise 25 on page 498.

### Performance Task:

## Step Angles

As we walk, our footsteps make patterns that reveal the efficiency of our stride. The Law of Cosines helps analyze those patterns. Using this law, can you compare the strides of other bipeds?

To explore the answer to this question and more, check out the Performance Task and Real-Life STEM video at [BigIdeasMath.com](http://BigIdeasMath.com).





**9.1** Using Trigonometric Identities (pp. 471–476)

Verify the identity  $\frac{\cot^2 \theta}{\csc \theta} = \csc \theta - \sin \theta$ .

$$\begin{aligned}\frac{\cot^2 \theta}{\csc \theta} &= \frac{\csc^2 \theta - 1}{\csc \theta} \\ &= \frac{\csc^2 \theta}{\csc \theta} - \frac{1}{\csc \theta} \\ &= \csc \theta - \frac{1}{\csc \theta} \\ &= \csc \theta - \sin \theta\end{aligned}$$

Pythagorean identity

Write as separate fractions.

Simplify.

Reciprocal identity

1. Given that  $\cos \theta = -\frac{8}{17}$  and  $\pi < \theta < \frac{3\pi}{2}$ , find the values of the other five trigonometric functions of  $\theta$ .

**Simplify the expression.**

2.  $\cot^2 x - \cot^2 x \cos^2 x$       3.  $\frac{(\sec x + 1)(\sec x - 1)}{\tan x}$       4.  $\sin\left(\frac{\pi}{2} - x\right) \tan x$

**Verify the identity.**

5.  $\frac{\cos x \sec x}{1 + \tan^2 x} = \cos^2 x$       6.  $\tan\left(\frac{\pi}{2} - x\right) \cot x = \csc^2 x - 1$

**9.2** Using Sum and Difference Formulas (pp. 477–482)

**Find the exact value of  $\sin 165^\circ$ .**

$$\begin{aligned}\sin 165^\circ &= \sin(135^\circ + 30^\circ) \\ &= \sin 135^\circ \cos 30^\circ + \cos 135^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \left(-\frac{\sqrt{2}}{2}\right) \cdot \frac{1}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

Substitute  $135^\circ + 30^\circ$  for  $165^\circ$ .

Sum formula for sine

Evaluate.

Simplify.

▶ The exact value of  $\sin 135^\circ$  is  $\frac{\sqrt{6} - \sqrt{2}}{4}$ .

**Find the exact value of the expression.**

7.  $\sin 195^\circ$       8.  $\tan(-105^\circ)$       9.  $\cos \frac{7\pi}{12}$
10. Find  $\tan(a + b)$ , given that  $\tan a = \frac{1}{4}$  with  $\pi < a < \frac{3\pi}{2}$  and  $\tan b = \frac{3}{7}$  with  $0 < b < \frac{\pi}{2}$ .

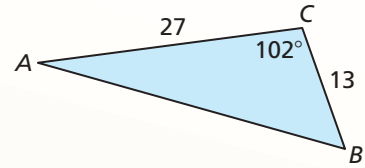
**Solve the equation for  $0 \leq x < 2\pi$ .**

11.  $\cos\left(x + \frac{3\pi}{4}\right) + \cos\left(x - \frac{3\pi}{4}\right) = 1$       12.  $\tan(x + \pi) + \cos\left(x + \frac{\pi}{2}\right) = 0$

### 9.3 Law of Sines (pp. 485–492)

- a. Find the area of the triangle. Round your answer to the nearest tenth.

$$\begin{aligned} \text{Area} &= \frac{1}{2}ab \sin C \\ &= \frac{1}{2}(13)(27) \sin 102^\circ \\ &\approx 171.7 \end{aligned}$$

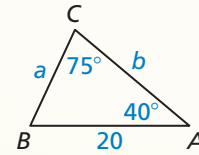


▶ The area of the triangle is about 171.7 square units.

- b. Solve  $\triangle ABC$ . Round decimal answers to the nearest tenth.

By the Triangle Sum Theorem,  
 $B = 180^\circ - 40^\circ - 75^\circ = 65^\circ$ .

By the Law of Sines, you can write  $\frac{a}{\sin 40^\circ} = \frac{b}{\sin 65^\circ} = \frac{20}{\sin 75^\circ}$ .



$$\frac{a}{\sin 40^\circ} = \frac{20}{\sin 75^\circ}$$

Write two equations, each with one variable.

$$\frac{b}{\sin 65^\circ} = \frac{20}{\sin 75^\circ}$$

$$a = \frac{20 \sin 40^\circ}{\sin 75^\circ}$$

Solve for each variable.

$$b = \frac{20 \sin 65^\circ}{\sin 75^\circ}$$

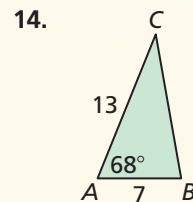
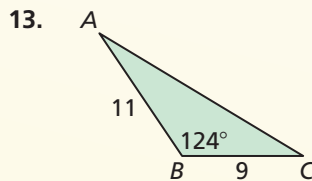
$$a \approx 13.3$$

Use a calculator.

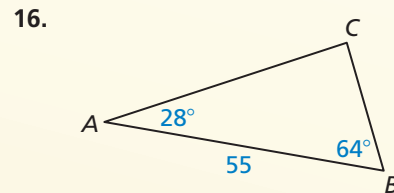
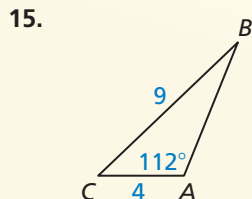
$$b \approx 18.8$$

▶ In  $\triangle ABC$ ,  $B = 65^\circ$ ,  $a \approx 13.3$ , and  $b \approx 18.8$ .

Find the area of  $\triangle ABC$ . Round your answer to the nearest tenth.



Solve  $\triangle ABC$ . Round decimal answers to the nearest tenth.



17.  $C = 48^\circ$ ,  $b = 20$ ,  $c = 28$

18.  $B = 102^\circ$ ,  $C = 43^\circ$ ,  $b = 21$

19.  $A = 50^\circ$ ,  $C = 56^\circ$ ,  $b = 5$

20.  $B = 31^\circ$ ,  $a = 12$ ,  $b = 10$

## 9.4 Law of Cosines (pp. 493–498)

a. Solve  $\triangle ABC$ . Round decimal answers to the nearest tenth.

First, find the angle opposite the longest side,  $\overline{AB}$ .  
Use the Law of Cosines to find  $C$ .

$$19^2 = 11^2 + 17^2 - 2(11)(17) \cos C$$

$$\frac{19^2 - 11^2 - 17^2}{-2(11)(17)} = \cos C$$

$$C \approx 82.5^\circ$$

Now, use the Law of Sines to find  $A$ .

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

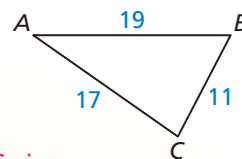
$$\frac{\sin A}{11} = \frac{\sin 82.5^\circ}{19}$$

$$\sin A = \frac{11 \sin 82.5^\circ}{19}$$

$$A \approx 35.0^\circ$$

By the Triangle Sum Theorem,  $B \approx 180^\circ - 35.0^\circ - 82.5^\circ = 62.5^\circ$ .

► In  $\triangle ABC$ ,  $A \approx 35.0^\circ$ ,  $B \approx 62.5^\circ$ , and  $C \approx 82.5^\circ$ .



Law of Cosines

Solve for  $\cos C$ .

Use a calculator.

Law of Sines

Substitute.

Multiply each side by 11.

Use a calculator.

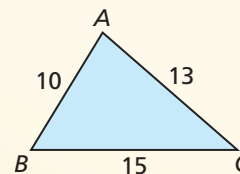
b. Find the area of  $\triangle ABC$ . Round your answer to the nearest tenth.

**Step 1** Find the semiperimeter  $s$ .

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(15 + 13 + 10) = 19$$

**Step 2** Use Heron's formula to find the area of  $\triangle ABC$ .

$$\begin{aligned} \text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{19(19-15)(19-13)(19-10)} \\ &\approx 64.1 \end{aligned}$$



► The area of the triangle is about 64.1 square units.

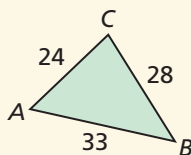
Solve  $\triangle ABC$ . Round decimal answers to the nearest tenth.

21.  $B = 25^\circ$ ,  $a = 8$ ,  $c = 3$

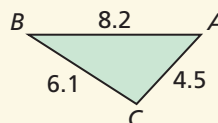
22.  $a = 10$ ,  $b = 3$ ,  $c = 12$

Find the area of  $\triangle ABC$ . Round your answer to the nearest tenth.

23.



24.



# 9 Chapter Test

Verify the identity.

1.  $\frac{\cos^2 x + \sin^2 x}{1 + \tan^2 x} = \cos^2 x$

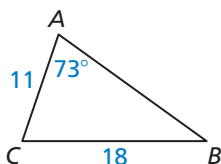
3.  $\cos\left(x + \frac{3\pi}{2}\right) = \sin x$

2.  $\frac{1 + \sin x}{\cos x} + \frac{\cos x}{1 + \sin x} = 2 \sec x$

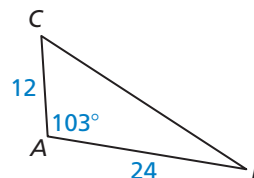
4.  $\frac{\sin \theta \cot \theta - \cot(-\theta)}{\sin \theta + 1} = \cot \theta$

Solve  $\triangle ABC$ . Round decimal answers to the nearest tenth.

5.



6.



7.  $A = 26^\circ, C = 35^\circ, b = 13$

8.  $a = 38, b = 31, c = 35$

9.  $C = 44^\circ, b = 15, c = 11$

10.  $A = 103^\circ, a = 24, b = 20$

11. Find the exact value of  $\tan\left(-\frac{\pi}{12}\right)$ .

12. Given that  $\tan \theta = -\frac{2}{5}$ , and  $\frac{\pi}{2} < \theta < \pi$ , find the values of the other five trigonometric functions of  $\theta$ .

Simplify the expression.

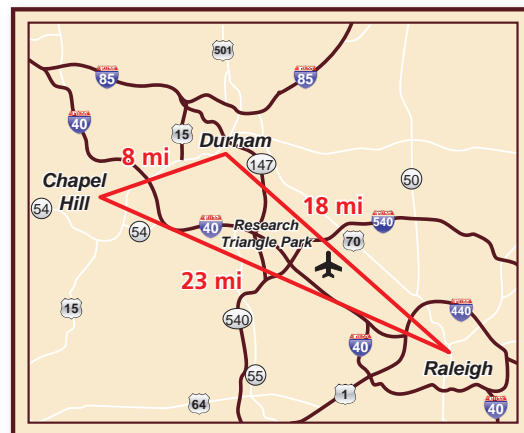
13.  $\sin x + \cot x \cos(-x)$       14.  $\frac{\cot^2 x - \csc^2 x}{\sin(-x)}$

15.  $\frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\sec \theta} + \sin^2 \theta$       16.  $\sin(x - \pi)$

17. Solve  $\sin\left(x + \frac{\pi}{2}\right) + \sin\left(x - \frac{3\pi}{2}\right) = 1$  for  $0 \leq x < 2\pi$ .

18. The Research Triangle in North Carolina is an area situated between the cities of Raleigh and Durham, and the town of Chapel Hill, as shown on the map. It is one of the most popular research and development centers in the United States.

- Find the area enclosed by Raleigh, Durham, and Chapel Hill.
- Solve the triangle formed by Raleigh, Durham, and Chapel Hill.



19. Use the diagram of the Bermuda Triangle shown.
- Find the area of the Bermuda Triangle.
  - Solve the triangle formed by the Bermuda Triangle.

# 9 Cumulative Assessment

1. Which expressions are equivalent to 1?

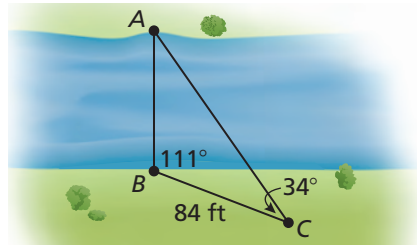
$$\tan x \sec x \cos x$$

$$\sin^2 x + \cos^2 x$$

$$\frac{\cos^2(-x) \tan^2 x}{\sin^2(-x)}$$

$$\cos\left(\frac{\pi}{2} - x\right) \csc x$$

2. A surveyor makes the measurements shown. What is the width of the river?



3. Order the acute angles from smallest to largest. Explain your reasoning.

$$\tan \theta_1 = 1$$

$$\tan \theta_2 = \frac{1}{2}$$

$$\tan \theta_3 = \frac{\sqrt{3}}{3}$$

$$\tan \theta_4 = \frac{23}{4}$$

$$\tan \theta_5 = \frac{38}{5}$$

$$\tan \theta_6 = \sqrt{3}$$

4. Which statements describe the transformation of the graph of  $f(x) = x^3 - x$  represented by  $g(x) = 4(x - 2)^3 - 4(x - 2)$ ?
- (A) a vertical stretch by a factor of 4
  - (B) a vertical shrink by a factor of  $\frac{1}{4}$
  - (C) a horizontal shrink by a factor of  $\frac{1}{4}$
  - (D) a horizontal stretch by a factor of 4
  - (E) a horizontal translation 2 units to the right
  - (F) a horizontal translation 2 units to the left
5. About 105,000 people live in a circular region with a 9-mile diameter. Find the population density in people per square mile.
6. Let the graph of  $g$  represent a vertical stretch and a reflection in the  $x$ -axis, followed by a translation left and down of the graph of  $f(x) = x^2$ . Use the tiles to write a rule for  $g$ .

-3

-1

$-\frac{1}{2}$

0

$\frac{1}{2}$

1

3

$x$

$g(x)$

+

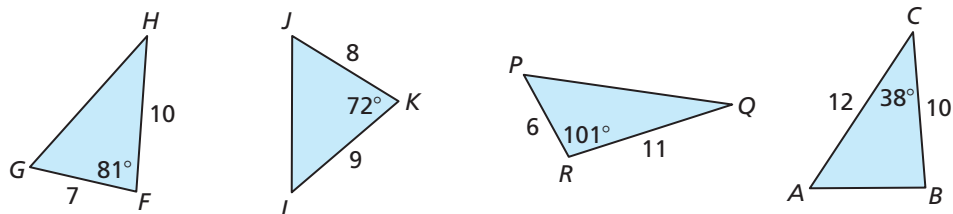
-

$\times$

$\div$

=

7. Order the triangles from smallest to largest according to their area.



8. Two points in front of a cliff are 75 meters apart. The angles of elevation to the top of the cliff from the two points are  $60^\circ$  and  $40^\circ$ . What is the height of the cliff?

- (A) about 35 meters                      (B) about 70 meters  
 (C) about 91 meters                      (D) about 122 meters

9. Match each trigonometric function with its correct value given that  $\cos \theta = -\frac{5}{8}$  and  $\frac{\pi}{2} < \theta < \pi$ .

sin $\theta$	tan $\theta$	csc $\theta$	sec $\theta$	cot $\theta$
$\frac{8\sqrt{39}}{39}$	$\frac{\sqrt{39}}{5}$	$-\frac{\sqrt{39}}{8}$	$-\frac{8}{5}$	$\frac{5\sqrt{39}}{39}$
$\frac{8}{5}$	$-\frac{\sqrt{39}}{5}$	$\frac{\sqrt{39}}{8}$	$-\frac{5\sqrt{39}}{39}$	$-\frac{8\sqrt{39}}{39}$

10. What is the exact value of  $\sin \frac{11\pi}{12}$ ?

- (A)  $-\frac{\sqrt{6} - \sqrt{2}}{4}$                       (B)  $\frac{\sqrt{6} - \sqrt{2}}{4}$   
 (C)  $\frac{\sqrt{6} + \sqrt{2}}{4}$                       (D)  $\frac{3 - \sqrt{2}}{4}$

11. A right triangle with legs of length 7 and 12 is rotated about its shorter leg. What is the volume of the solid of revolution that is formed?

- (A) about 615.75 cubic units                      (B) about 976.12 cubic units  
 (C) about 1055.58 cubic units                      (D) about 3166.73 cubic units