# 8.2 Angles and Radian Measure

## Essential Question

How can you find the measure of an angle in radians?

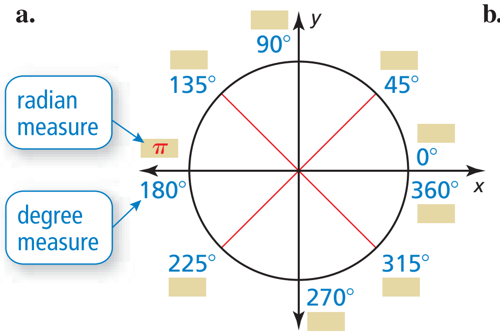
Let the vertex of an angle be at the origin, with one side of the angle on the positive $x$-axis. The *radian measure* of the angle is a measure of the intercepted arc length on a circle of radius 1. To convert between degree and radian measure, use the fact that

$$\frac{\pi \text{ radians}}{180^\circ} = 1.$$

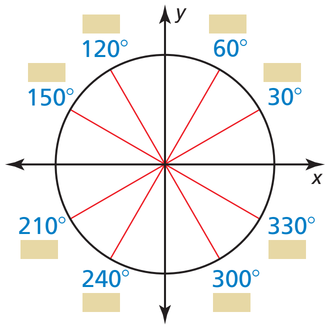
### EXPLORATION 1 Writing Radian Measures of Angles

**Work with a partner.** Write the radian measure of each angle with the given degree measure. Explain your reasoning.

**a.**

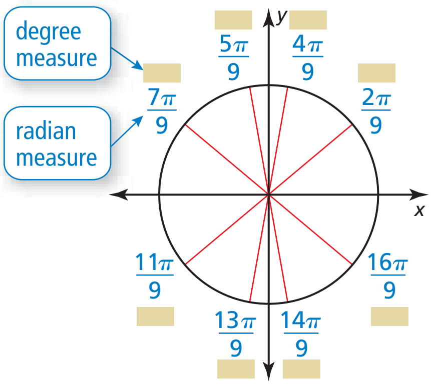



**b.**




### EXPLORATION 2 Writing Degree Measures of Angles

**Work with a partner.** Write the degree measure of each angle with the given radian measure. Explain your reasoning.



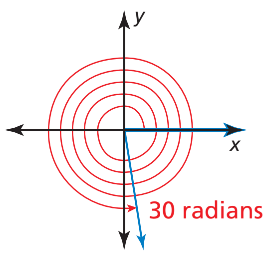

**REASONING ABSTRACTLY**

To be proficient in math, you need to make sense of quantities and their relationships in problem situations.

## Communicate Your Answer

**3.** How can you find the measure of an angle in radians?

**4.** The figure shows an angle whose measure is 30 radians. What is the measure of the angle in degrees? How many times greater is 30 radians than 30 degrees? Justify your answers.



# 8.2 Lesson

## Core Vocabulary

initial side, *p. 418*
terminal side, *p. 418*
standard position, *p. 418*
coterminal, *p. 419*
radian, *p. 419*
sector, *p. 420*
central angle, *p. 420*

***Previous***
radius of a circle
circumference of a circle

## What You Will Learn

- Draw angles in standard position.
- Find coterminal angles.
- Use radian measure.

## Drawing Angles in Standard Position

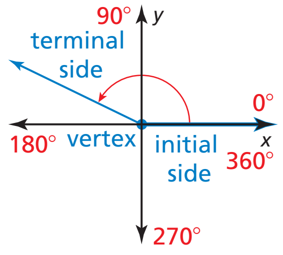
In this lesson, you will expand your study of angles to include angles with measures that can be any real numbers.

## Core Concept

### Angles in Standard Position

In a coordinate plane, an angle can be formed by fixing one ray, called the **initial side**, and rotating the other ray, called the **terminal side**, about the vertex.

An angle is in **standard position** when its vertex is at the origin and its initial side lies on the positive $x$-axis.




The measure of an angle is positive when the rotation of its terminal side is counterclockwise and negative when the rotation is clockwise. The terminal side of an angle can rotate more than 360°.

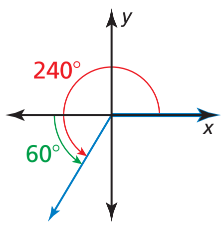
### EXAMPLE 1 Drawing Angles in Standard Position

Draw an angle with the given measure in standard position.

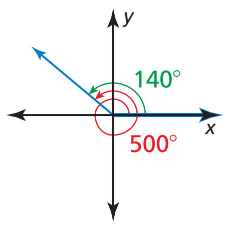
**a.** 240° **b.** 500° **c.** −50°

#### SOLUTION

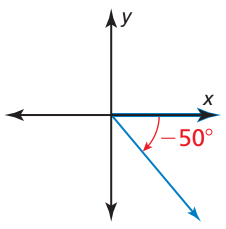
**a.** Because 240° is 60° more than 180°, the terminal side is 60° counterclockwise past the negative $x$-axis.




**b.** Because 500° is 140° more than 360°, the terminal side makes one complete rotation 360° counterclockwise plus 140° more.




**c.** Because −50° is negative, the terminal side is 50° clockwise from the positive $x$-axis.




## Monitoring Progress

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**Draw an angle with the given measure in standard position.**

**1.** 65° **2.** 300° **3.** −120° **4.** −450°

## Finding Coterminal Angles

In Example 1(b), the angles 500° and 140° are **coterminal** because their terminal sides coincide. An angle coterminal with a given angle can be found by adding or subtracting multiples of 360°.

**STUDY TIP**

If two angles differ by a multiple of 360°, then the angles are coterminal.

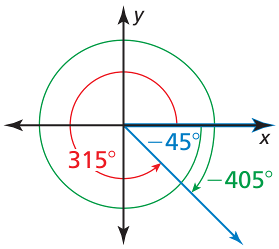
**EXAMPLE 2** Finding Coterminal Angles

Find one positive angle and one negative angle that are coterminal with (a) −45° and (b) 395°.

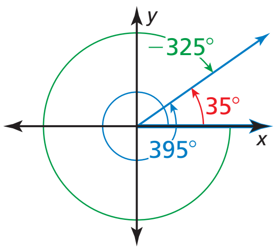
**SOLUTION**

There are many such angles, depending on what multiple of 360° is added or subtracted.

**a.** $-45° + 360° = 315°$
$-45° - 360° = -405°$




**b.** $395° - 360° = 35°$
$395° - 2(360°) = -325°$




### Monitoring Progress

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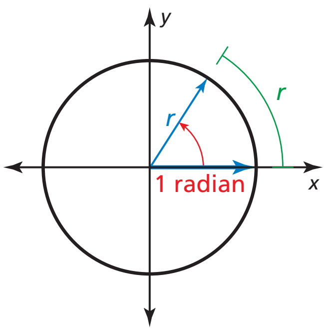
**Find one positive angle and one negative angle that are coterminal with the given angle.**

**5.** 80° **6.** 230° **7.** 740° **8.** −135°

## Using Radian Measure

Angles can also be measured in *radians*. To define a radian, consider a circle with radius $r$ centered at the origin, as shown. One **radian** is the measure of an angle in standard position whose terminal side intercepts an arc of length $r$.




Because the circumference of a circle is $2\pi r$, there are $2\pi$ radians in a full circle. So, degree measure and radian measure are related by the equation $360° = 2\pi$ radians, or $180° = \pi$ radians.

**STUDY TIP**

Notice that 1 radian is approximately equal to 57.3°.

$$180° = \pi \text{ radians}$$

$$\frac{180°}{\pi} = 1 \text{ radian}$$

$$57.3° \approx 1 \text{ radian}$$

### Core Concept

**Converting Between Degrees and Radians**

**Degrees to radians**

Multiply degree measure by

$$\frac{\pi \text{ radians}}{180°}.$$

**Radians to degrees**

Multiply radian measure by

$$\frac{180°}{\pi \text{ radians}}.$$

## EXAMPLE 3 Convert Between Degrees and Radians

Convert the degree measure to radians or the radian measure to degrees.

**a.** $120°$ **b.** $-\frac{\pi}{12}$

> **READING**
> The unit "radians" is often omitted. For instance, the measure $-\frac{\pi}{12}$ radians may be written simply as $-\frac{\pi}{12}$.

### SOLUTION

**a.** $120° = 120\ \cancel{\text{degrees}}\left(\frac{\pi \text{ radians}}{180\ \cancel{\text{degrees}}}\right)$

$= \frac{2\pi}{3}$

**b.** $-\frac{\pi}{12} = \left(-\frac{\pi}{12}\ \cancel{\text{radians}}\right)\left(\frac{180°}{\pi\ \cancel{\text{radians}}}\right)$

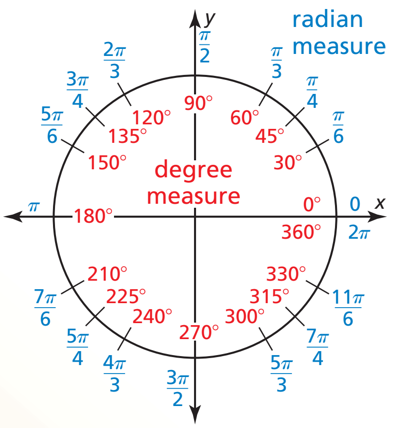
$= -15°$

## Concept Summary

### Degree and Radian Measures of Special Angles

The diagram shows equivalent degree and radian measures for special angles from $0°$ to $360°$ (0 radians to $2\pi$ radians).

You may find it helpful to memorize the equivalent degree and radian measures of special angles in the first quadrant and for $90° = \frac{\pi}{2}$ radians. All other special angles shown are multiples of these angles.




## Monitoring Progress

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**Convert the degree measure to radians or the radian measure to degrees.**

**9.** $135°$ **10.** $-40°$ **11.** $\frac{5\pi}{4}$ **12.** $-6.28$

A **sector** is a region of a circle that is bounded by two radii and an arc of the circle. The **central angle** $\theta$ of a sector is the angle formed by the two radii. There are simple formulas for the arc length and area of a sector when the central angle is measured in radians.

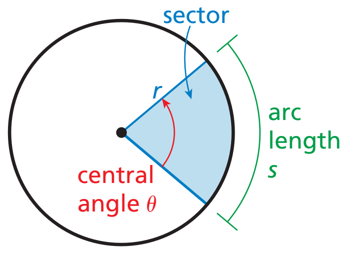
## Core Concept

### Arc Length and Area of a Sector

The arc length $s$ and area $A$ of a sector with radius $r$ and central angle $\theta$ (measured in radians) are as follows.

**Arc length:** $s = r\theta$

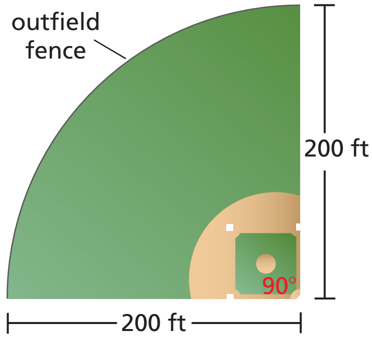
**Area:** $A = \frac{1}{2}r^2\theta$



## EXAMPLE 4 Modeling with Mathematics

A softball field forms a sector with the dimensions shown. Find the length of the outfield fence and the area of the field.

### SOLUTION




1. **Understand the Problem** You are given the dimensions of a softball field. You are asked to find the length of the outfield fence and the area of the field.

2. **Make a Plan** Find the measure of the central angle in radians. Then use the arc length and area of a sector formulas.

3. **Solve the Problem**

**Step 1** Convert the measure of the central angle to radians.

$$90^\circ = 90\ \cancel{\text{degrees}}\left(\frac{\pi \text{ radians}}{180\ \cancel{\text{degrees}}}\right)$$

$$= \frac{\pi}{2} \text{ radians}$$

**Step 2** Find the arc length and the area of the sector.

**Arc length:** $s = r\theta$

$$= 200\left(\frac{\pi}{2}\right)$$
$$= 100\pi$$
$$\approx 314$$

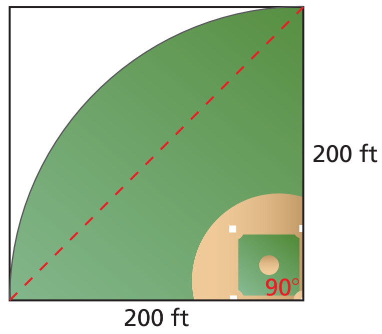
**Area:** $A = \frac{1}{2}r^2\theta$

$$= \frac{1}{2}(200)^2\left(\frac{\pi}{2}\right)$$
$$= 10{,}000\pi$$
$$\approx 31{,}416$$

▶ The length of the outfield fence is about 314 feet. The area of the field is about 31,416 square feet.

4. **Look Back** To check the area of the field, consider the square formed using the two 200-foot sides.

By drawing the diagonal, you can see that the area of the field is less than the area of the square but greater than one-half of the area of the square.




$\frac{1}{2}$ • (area of square)

area of square

$$\frac{1}{2}(200)^2 \overset{?}{<} 31{,}416 \overset{?}{<} 200^2$$

$$20{,}000 < 31{,}416 < 40{,}000 \ ✓$$

**COMMON ERROR**

You must write the measure of an angle in radians when using these formulas for the arc length and area of a sector.

**ANOTHER WAY**

Because the central angle is 90°, the sector represents $\frac{1}{4}$ of a circle with a radius of 200 feet. So,

$$s = \frac{1}{4} \cdot 2\pi r$$
$$= \frac{1}{4} \cdot 2\pi(200)$$
$$= 100\pi$$

and

$$A = \frac{1}{4} \cdot \pi r^2$$
$$= \frac{1}{4} \cdot \pi(200)^2$$
$$= 10{,}000\pi.$$

### Monitoring Progress

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**13.** **WHAT IF?** In Example 4, the outfield fence is 220 feet from home plate. Estimate the length of the outfield fence and the area of the field.

# 8.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

## Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** An angle is in standard position when its vertex is at the __________ and its __________ lies on the positive $x$-axis.

2. **WRITING** Explain how the sign of an angle measure determines its direction of rotation.

3. **VOCABULARY** In your own words, define a radian.

4. **WHICH ONE DOESN'T BELONG?** Which angle does *not* belong with the other three? Explain your reasoning.

| | | | |
|---|---|---|---|
| $-90°$ | $450°$ | $90°$ | $-270°$ |

## Monitoring Progress and Modeling with Mathematics

**In Exercises 5–8, draw an angle with the given measure in standard position.** *(See Example 1.)*

5. $110°$

6. $450°$

7. $-900°$

8. $-10°$

**In Exercises 9–12, find one positive angle and one negative angle that are coterminal with the given angle.** *(See Example 2.)*

9. $70°$

10. $255°$

11. $-125°$

12. $-800°$

**In Exercises 13–20, convert the degree measure to radians or the radian measure to degrees.** *(See Example 3.)*

13. $40°$

14. $315°$

15. $-260°$

16. $-500°$

17. $\frac{\pi}{9}$

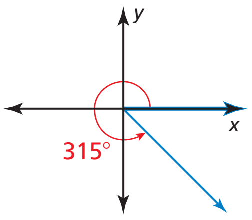
18. $\frac{3\pi}{4}$

19. $-5$

20. $12$

21. **WRITING** The terminal side of an angle in standard position rotates one-sixth of a revolution counterclockwise from the positive $x$-axis. Describe how to find the measure of the angle in both degree and radian measures.

22. **OPEN-ENDED** Using radian measure, give one positive angle and one negative angle that are coterminal with the angle shown. Justify your answers.



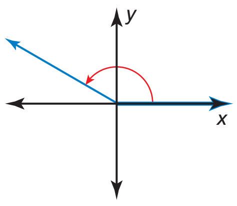

**ANALYZING RELATIONSHIPS** **In Exercises 23–26, match the angle measure with the angle.**

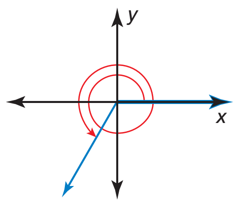
23. $600°$

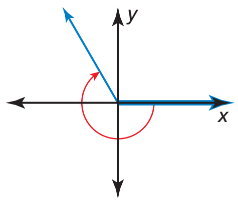
24. $-\frac{9\pi}{4}$

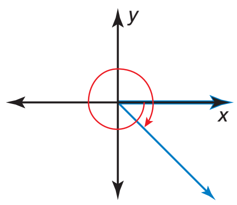
25. $\frac{5\pi}{6}$

26. $-240°$

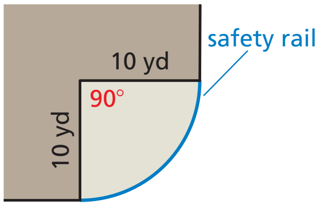
A.



B.


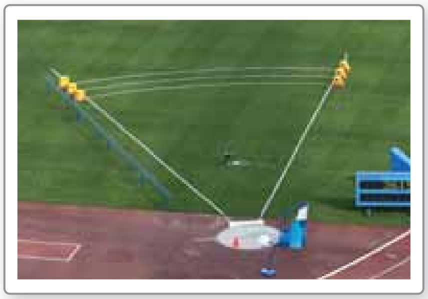

C.



D.


**27.** **MODELING WITH MATHEMATICS** The observation deck of a building forms a sector with the dimensions shown. Find the length of the safety rail and the area of the deck. *(See Example 4.)*




**28.** **MODELING WITH MATHEMATICS** In the men's shot put event at the 2012 Summer Olympic Games, the length of the winning shot was 21.89 meters. A shot put must land within a sector having a central angle of 34.92° to be considered fair.



**a.** The officials draw an arc across the fair landing area, marking the farthest throw. Find the length of the arc.

**b.** All fair throws in the 2012 Olympics landed within a sector bounded by the arc in part (a). What is the area of this sector?

**29.** **ERROR ANALYSIS** Describe and correct the error in converting the degree measure to radians.

$$24° = 24\ \cancel{\text{degrees}}\left(\frac{180\ \cancel{\text{degrees}}}{\pi\ \text{radians}}\right)$$

$$= \frac{4320}{\pi}\ \text{radians}$$

$$\approx 1375.1\ \text{radians}$$

**30.** **ERROR ANALYSIS** Describe and correct the error in finding the area of a sector with a radius of 6 centimeters and a central angle of 40°.

$$A = \frac{1}{2}(6)^2(40) = 720\ \text{cm}^2$$

**31.** **PROBLEM SOLVING** When a CD player reads information from the outer edge of a CD, the CD spins about 200 revolutions per minute. At that speed, through what angle does a point on the CD spin in one minute? Give your answer in both degree and radian measures.

**32.** **PROBLEM SOLVING** You work every Saturday from 9:00 A.M. to 5:00 P.M. Draw a diagram that shows the rotation completed by the hour hand of a clock during this time. Find the measure of the angle generated by the hour hand in both degrees and radians. Compare this angle with the angle generated by the minute hand from 9:00 A.M. to 5:00 P.M.

**USING TOOLS** **In Exercises 33–38, use a calculator to evaluate the trigonometric function.**

**33.** $\cos \frac{4\pi}{3}$

**34.** $\sin \frac{7\pi}{8}$

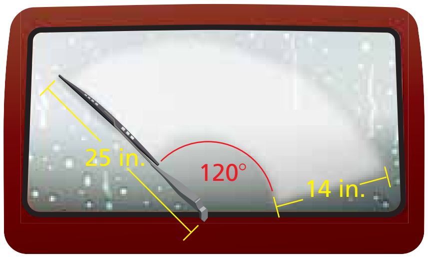
**35.** $\csc \frac{10\pi}{11}$

**36.** $\cot\left(-\frac{6\pi}{5}\right)$

**37.** $\cot(-14)$

**38.** $\cos 6$

**39.** **MODELING WITH MATHEMATICS** The rear windshield wiper of a car rotates 120°, as shown. Find the area cleared by the wiper.

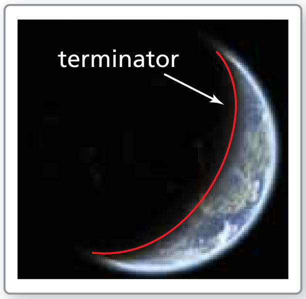



**40.** **MODELING WITH MATHEMATICS** A scientist performed an experiment to study the effects of gravitational force on humans. In order for humans to experience twice Earth's gravity, they were placed in a centrifuge 58 feet long and spun at a rate of about 15 revolutions per minute.

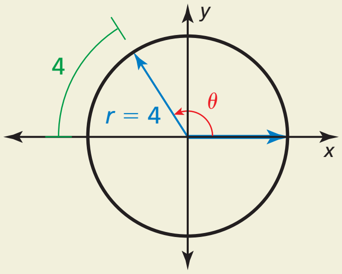
**a.** Through how many radians did the people rotate each second?

**b.** Find the length of the arc through which the people rotated each second.

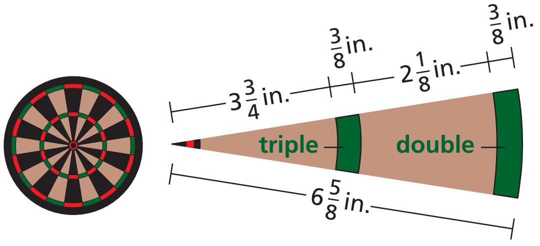
**41. REASONING** In astronomy, the *terminator* is the day-night line on a planet that divides the planet into daytime and nighttime regions. The terminator moves across the surface of a planet as the planet rotates. It takes about 4 hours for Earth's terminator to move across the continental United States. Through what angle has Earth rotated during this time? Give your answer in both degree and radian measures.




**42. HOW DO YOU SEE IT?** Use the graph to find the measure of $\theta$. Explain your reasoning.




**43. MODELING WITH MATHEMATICS** A dartboard is divided into 20 sectors. Each sector is worth a point value from 1 to 20 and has shaded regions that double or triple this value. A sector is shown below. Find the areas of the entire sector, the double region, and the triple region.




**44. THOUGHT PROVOKING** $\pi$ is an irrational number, which means that it cannot be written as the ratio of two whole numbers. $\pi$ can, however, be written exactly as a *continued fraction*, as follows.

$$3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}}}$$

Show how to use this continued fraction to obtain a decimal approximation for $\pi$.

**45. MAKING AN ARGUMENT** Your friend claims that when the arc length of a sector equals the radius, the area can be given by $A = \frac{s^2}{2}$. Is your friend correct? Explain.

**46. PROBLEM SOLVING** A spiral staircase has 15 steps. Each step is a sector with a radius of 42 inches and a central angle of $\frac{\pi}{8}$.

**a.** What is the length of the arc formed by the outer edge of a step?

**b.** Through what angle would you rotate by climbing the stairs?

**c.** How many square inches of carpeting would you need to cover the 15 steps?

**47. MULTIPLE REPRESENTATIONS** There are 60 *minutes* in 1 degree of arc, and 60 *seconds* in 1 minute of arc. The notation $50^\circ\ 30'\ 10''$ represents an angle with a measure of 50 degrees, 30 minutes, and 10 seconds.

**a.** Write the angle measure $70.55^\circ$ using the notation above.

**b.** Write the angle measure $110^\circ\ 45'\ 30''$ to the nearest hundredth of a degree. Justify your answer.

## Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

**Find the distance between the two points.** *(Skills Review Handbook)*

**48.** (1, 4), (3, 6)

**49.** (−7, −13), (10, 8)

**50.** (2, 12), (8, −5)

**51.** (4, 16), (−1, 34)

**52.** What is the volume of the solid that is produced when the region enclosed by $y = 0$, $x = 0$, and $y = -\frac{1}{3}x + 3$ is rotated about (a) the $x$-axis and (b) the $y$-axis? *(Section 1.4)*