6.4 Adding and Subtracting Rational Expressions

Essential Question

How can you determine the domain of the sum or difference of two rational expressions?

You can add and subtract rational expressions in much the same way that you add and subtract fractions.

$$\frac{x}{x+1} + \frac{2}{x+1} = \frac{x+2}{x+1}$$ Sum of rational expressions

$$\frac{1}{x} - \frac{1}{2x} = \frac{2}{2x} - \frac{1}{2x} = \frac{1}{2x}$$ Difference of rational expressions

EXPLORATION 1 Adding and Subtracting Rational Expressions

Work with a partner. Find the sum or difference of the two rational expressions. Then match the sum or difference with its domain. Explain your reasoning.

Sum or Difference	Domain
a. $\frac{1}{x-1} + \frac{3}{x-1} =$	**A.** all real numbers except -2
b. $\frac{1}{x-1} + \frac{1}{x} =$	**B.** all real numbers except -1 and 1
c. $\frac{1}{x-2} + \frac{1}{2-x} =$	**C.** all real numbers except 1
d. $\frac{1}{x-1} + \frac{-1}{x+1} =$	**D.** all real numbers except 0
e. $\frac{x}{x+2} - \frac{x+1}{2+x} =$	**E.** all real numbers except -2 and 1
f. $\frac{x}{x-2} - \frac{x+1}{x} =$	**F.** all real numbers except 0 and 1
g. $\frac{x}{x+2} - \frac{x}{x-1} =$	**G.** all real numbers except 2
h. $\frac{x+2}{x} - \frac{x+1}{x} =$	**H.** all real numbers except 0 and 2

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to justify your conclusions and communicate them to others.

EXPLORATION 2 Writing a Sum or Difference

Work with a partner. Write a sum or difference of rational expressions that has the given domain. Justify your answer.

a. all real numbers except -1

b. all real numbers except -1 and 3

c. all real numbers except -1, 0, and 3

Communicate Your Answer

3. How can you determine the domain of the sum or difference of two rational expressions?

4. Your friend found a sum as follows. Describe and correct the error(s).

$$\frac{x}{x+4} + \frac{3}{x-4} = \frac{x+3}{2x}$$

6.4 Lesson

Core Vocabulary

complex fraction, *p. 335*

Previous
rational numbers
reciprocal

What You Will Learn

- Add or subtract rational expressions.
- Rewrite rational functions.
- Simplify complex fractions.

Adding or Subtracting Rational Expressions

As with numerical fractions, the procedure used to add (or subtract) two rational expressions depends upon whether the expressions have like or unlike denominators. To add (or subtract) rational expressions with like denominators, simply add (or subtract) their numerators. Then place the result over the common denominator.

Core Concept

Adding or Subtracting with Like Denominators

Let a, b, and c be expressions with $c \neq 0$.

Addition

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$$

Subtraction

$$\frac{a}{c} - \frac{b}{c} = \frac{a - b}{c}$$

EXAMPLE 1 Adding or Subtracting with Like Denominators

a. $\frac{7}{4x} + \frac{3}{4x} = \frac{7 + 3}{4x} = \frac{10}{4x} = \frac{5}{2x}$ Add numerators and simplify.

b. $\frac{2x}{x + 6} - \frac{5}{x + 6} = \frac{2x - 5}{x + 6}$ Subtract numerators.

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Find the sum or difference.

1. $\frac{8}{12x} - \frac{5}{12x}$ **2.** $\frac{2}{3x^2} + \frac{1}{3x^2}$ **3.** $\frac{4x}{x - 2} - \frac{x}{x - 2}$ **4.** $\frac{2x^2}{x^2 + 1} + \frac{2}{x^2 + 1}$

To add (or subtract) two rational expressions with *unlike* denominators, find a common denominator. Rewrite each rational expression using the common denominator. Then add (or subtract).

Core Concept

Adding or Subtracting with Unlike Denominators

Let a, b, c, and d be expressions with $c \neq 0$ and $d \neq 0$.

Addition

$$\frac{a}{c} + \frac{b}{d} = \frac{ad}{cd} + \frac{bc}{cd} = \frac{ad + bc}{cd}$$

Subtraction

$$\frac{a}{c} - \frac{b}{d} = \frac{ad}{cd} - \frac{bc}{cd} = \frac{ad - bc}{cd}$$

You can always find a common denominator of two rational expressions by multiplying the denominators, as shown above. However, when you use the least common denominator (LCD), which is the least common multiple (LCM) of the denominators, simplifying your answer may take fewer steps.

To find the LCM of two (or more) expressions, factor the expressions completely. The LCM is the product of the highest power of each factor that appears in any of the expressions.

EXAMPLE 2 Finding a Least Common Multiple (LCM)

Find the least common multiple of $4x^2 - 16$ and $6x^2 - 24x + 24$.

SOLUTION

Step 1 Factor each polynomial. Write numerical factors as products of primes.

$$4x^2 - 16 = 4(x^2 - 4) = (2^2)(x + 2)(x - 2)$$

$$6x^2 - 24x + 24 = 6(x^2 - 4x + 4) = (2)(3)(x - 2)^2$$

Step 2 The LCM is the product of the highest power of each factor that appears in either polynomial.

$$\text{LCM} = (2^2)(3)(x + 2)(x - 2)^2 = 12(x + 2)(x - 2)^2$$

EXAMPLE 3 Adding with Unlike Denominators

Find the sum $\dfrac{7}{9x^2} + \dfrac{x}{3x^2 + 3x}$.

SOLUTION

Method 1 Use the definition for adding rational expressions with unlike denominators.

$$\frac{7}{9x^2} + \frac{x}{3x^2 + 3x} = \frac{7(3x^2 + 3x) + x(9x^2)}{9x^2(3x^2 + 3x)} \qquad \frac{a}{c} + \frac{b}{d} = \frac{ad + bc}{cd}$$

$$= \frac{21x^2 + 21x + 9x^3}{9x^2(3x^2 + 3x)} \qquad \text{Distributive Property}$$

$$= \frac{\cancel{3x}(3x^2 + 7x + 7)}{9x^2(x + 1)(\cancel{3x})} \qquad \text{Factor. Divide out common factors.}$$

$$= \frac{3x^2 + 7x + 7}{9x^2(x + 1)} \qquad \text{Simplify.}$$

Method 2 Find the LCD and then add. To find the LCD, factor each denominator and write each factor to the highest power that appears in either denominator. Note that $9x^2 = 3^2x^2$ and $3x^2 + 3x = 3x(x + 1)$, so the LCD is $9x^2(x + 1)$.

$$\frac{7}{9x^2} + \frac{x}{3x^2 + 3x} = \frac{7}{9x^2} + \frac{x}{3x(x + 1)} \qquad \text{Factor second denominator.}$$

$$= \frac{7}{9x^2} \cdot \frac{x + 1}{x + 1} + \frac{x}{3x(x + 1)} \cdot \frac{3x}{3x} \qquad \text{LCD is } 9x^2(x + 1).$$

$$= \frac{7x + 7}{9x^2(x + 1)} + \frac{3x^2}{9x^2(x + 1)} \qquad \text{Multiply.}$$

$$= \frac{3x^2 + 7x + 7}{9x^2(x + 1)} \qquad \text{Add numerators.}$$

Note in Examples 1 and 3 that when adding or subtracting rational expressions, the result is a rational expression. In general, similar to rational numbers, rational expressions are closed under addition and subtraction.

EXAMPLE 4 Subtracting with Unlike Denominators

Find the difference $\dfrac{x+2}{2x-2} - \dfrac{-2x-1}{x^2-4x+3}$.

SOLUTION

COMMON ERROR

When subtracting rational expressions, remember to distribute the negative sign to all the terms in the quantity that is being subtracted.

$$\frac{x+2}{2x-2} - \frac{-2x-1}{x^2-4x+3} = \frac{x+2}{2(x-1)} - \frac{-2x-1}{(x-1)(x-3)}$$ Factor each denominator.

$$= \frac{x+2}{2(x-1)} \cdot \frac{x-3}{x-3} - \frac{-2x-1}{(x-1)(x-3)} \cdot \frac{2}{2}$$ LCD is $2(x-1)(x-3)$.

$$= \frac{x^2-x-6}{2(x-1)(x-3)} - \frac{-4x-2}{2(x-1)(x-3)}$$ Multiply.

$$= \frac{x^2-x-6-(-4x-2)}{2(x-1)(x-3)}$$ Subtract numerators.

$$= \frac{x^2+3x-4}{2(x-1)(x-3)}$$ Simplify numerator.

$$= \frac{\cancel{(x-1)}(x+4)}{2\cancel{(x-1)}(x-3)}$$ Factor numerator. Divide out common factor.

$$= \frac{x+4}{2(x-3)},\ x \neq -1$$ Simplify.

Monitoring Progress

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5. Find the least common multiple of $5x^3$ and $10x^2 - 15x$.

Find the sum or difference.

6. $\dfrac{3}{4x} - \dfrac{1}{7}$ **7.** $\dfrac{1}{3x^2} + \dfrac{x}{9x^2-12}$ **8.** $\dfrac{x}{x^2-x-12} + \dfrac{5}{12x-48}$

Rewriting Rational Functions

Rewriting a rational function may reveal properties of the function and its graph. In Example 4 of Section 6.2, you used long division to rewrite a rational function. In the next example, you will use inspection.

EXAMPLE 5 Rewriting and Graphing a Rational Function

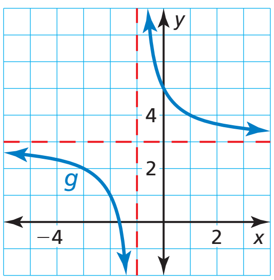
Rewrite $g(x) = \dfrac{3x+5}{x+1}$ in the form $g(x) = \dfrac{a}{x-h} + k$. Graph the function. Describe the graph of g as a transformation of the graph of $f(x) = \dfrac{a}{x}$.

SOLUTION

Rewrite by inspection:

$$\frac{3x+5}{x+1} = \frac{3x+3+2}{x+1} = \frac{3(x+1)+2}{x+1} = \frac{3\cancel{(x+1)}}{\cancel{x+1}} + \frac{2}{x+1} = 3 + \frac{2}{x+1}$$

▶ The rewritten function is $g(x) = \dfrac{2}{x+1} + 3$. The graph of g is a translation 1 unit left and 3 units up of the graph of $f(x) = \dfrac{2}{x}$.



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9. Rewrite $g(x) = \dfrac{2x-4}{x-3}$ in the form $g(x) = \dfrac{a}{x-h} + k$. Graph the function. Describe the graph of g as a transformation of the graph of $f(x) = \dfrac{a}{x}$.

Complex Fractions

A **complex fraction** is a fraction that contains a fraction in its numerator or denominator. A complex fraction can be simplified using either of the methods below.

Core Concept

Simplifying Complex Fractions

Method 1 If necessary, simplify the numerator and denominator by writing each as a single fraction. Then divide by multiplying the numerator by the reciprocal of the denominator.

Method 2 Multiply the numerator and the denominator by the LCD of *every* fraction in the numerator and denominator. Then simplify.

EXAMPLE 6 Simplifying a Complex Fraction

Simplify $\dfrac{\frac{5}{x+4}}{\frac{1}{x+4}+\frac{2}{x}}$.

SOLUTION

Method 1

$$\frac{\frac{5}{x+4}}{\frac{1}{x+4}+\frac{2}{x}} = \frac{\frac{5}{x+4}}{\frac{3x+8}{x(x+4)}} \qquad \text{Add fractions in denominator.}$$

$$= \frac{5}{x+4} \cdot \frac{x(x+4)}{3x+8} \qquad \text{Multiply by reciprocal.}$$

$$= \frac{5x\cancel{(x+4)}}{\cancel{(x+4)}(3x+8)} \qquad \text{Divide out common factors.}$$

$$= \frac{5x}{3x+8},\ x \neq -4,\ x \neq 0 \qquad \text{Simplify.}$$

Method 2 The LCD of all the fractions in the numerator and denominator is $x(x+4)$.

$$\frac{\frac{5}{x+4}}{\frac{1}{x+4}+\frac{2}{x}} = \frac{\frac{5}{x+4}}{\frac{1}{x+4}+\frac{2}{x}} \cdot \frac{x(x+4)}{x(x+4)} \qquad \text{Multiply numerator and denominator by the LCD.}$$

$$= \frac{\frac{5}{\cancel{x+4}} \cdot x\cancel{(x+4)}}{\frac{1}{\cancel{x+4}} \cdot x\cancel{(x+4)} + \frac{2}{\cancel{x}} \cdot \cancel{x}(x+4)} \qquad \text{Divide out common factors.}$$

$$= \frac{5x}{x+2(x+4)} \qquad \text{Simplify.}$$

$$= \frac{5x}{3x+8},\ x \neq -4,\ x \neq 0 \qquad \text{Simplify.}$$

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Simplify the complex fraction.

10. $\dfrac{\frac{x}{6}-\frac{x}{3}}{\frac{x}{5}-\frac{7}{10}}$

11. $\dfrac{\frac{2}{x}-4}{\frac{2}{x}+3}$

12. $\dfrac{\frac{3}{x+5}}{\frac{2}{x-3}+\frac{1}{x+5}}$

6.4 Exercises

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Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A fraction that contains a fraction in its numerator or denominator is called a(n) __________.

2. **WRITING** Explain how adding and subtracting rational expressions is similar to adding and subtracting numerical fractions.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, find the sum or difference. *(See Example 1.)*

3. $\dfrac{15}{4x} + \dfrac{5}{4x}$

4. $\dfrac{x}{16x^2} - \dfrac{4}{16x^2}$

5. $\dfrac{9}{x+1} - \dfrac{2x}{x+1}$

6. $\dfrac{3x^2}{x-8} + \dfrac{6x}{x-8}$

7. $\dfrac{5x}{x+3} + \dfrac{15}{x+3}$

8. $\dfrac{4x^2}{2x-1} - \dfrac{1}{2x-1}$

In Exercises 9–16, find the least common multiple of the expressions. *(See Example 2.)*

9. $3x, 3(x-2)$

10. $2x^2, 4x+12$

11. $2x, 2x(x-5)$

12. $24x^2, 8x^2-16x$

13. $x^2-25, x-5$

14. $9x^2-16, 3x^2+x-4$

15. $x^2+3x-40, x-8$

16. $x^2-2x-63, x+7$

ERROR ANALYSIS In Exercises 17 and 18, describe and correct the error in finding the sum.

17. ✗ $\dfrac{2}{5x} + \dfrac{4}{x^2} = \dfrac{2+4}{5x+x^2} = \dfrac{6}{x(5+x)}$

18. ✗ $\dfrac{x}{x+2} + \dfrac{4}{x-5} = \dfrac{x+4}{(x+2)(x-5)}$

In Exercises 19–26, find the sum or difference. *(See Examples 3 and 4.)*

19. $\dfrac{12}{5x} - \dfrac{7}{6x}$

20. $\dfrac{8}{3x^2} + \dfrac{5}{4x}$

21. $\dfrac{3}{x+4} - \dfrac{1}{x+6}$

22. $\dfrac{9}{x-3} + \dfrac{2x}{x+1}$

23. $\dfrac{12}{x^2+5x-24} + \dfrac{3}{x-3}$

24. $\dfrac{x^2-5}{x^2+5x-14} - \dfrac{x+3}{x+7}$

25. $\dfrac{x+2}{x-4} + \dfrac{2}{x} + \dfrac{5x}{3x-1}$

26. $\dfrac{x+3}{x^2-25} - \dfrac{x-1}{x-5} + \dfrac{3}{x+3}$

REASONING In Exercises 27 and 28, tell whether the statement is *always*, *sometimes*, or *never* true. Explain.

27. The LCD of two rational expressions is the product of the denominators.

28. The LCD of two rational expressions will have a degree greater than or equal to that of the denominator with the higher degree.

29. **ANALYZING EQUATIONS** How would you begin to rewrite the function $g(x) = \dfrac{4x+1}{x+2}$ to obtain the form $g(x) = \dfrac{a}{x-h} + k$?

Ⓐ $g(x) = \dfrac{4(x+2)-7}{x+2}$

Ⓑ $g(x) = \dfrac{4(x+2)+1}{x+2}$

Ⓒ $g(x) = \dfrac{(x+2)+(3x-1)}{x+2}$

Ⓓ $g(x) = \dfrac{4x+2-1}{x+2}$

30. **ANALYZING EQUATIONS** How would you begin to rewrite the function $g(x) = \dfrac{x}{x-5}$ to obtain the form $g(x) = \dfrac{a}{x-h} + k$?

Ⓐ $g(x) = \dfrac{x(x+5)(x-5)}{x-5}$

Ⓑ $g(x) = \dfrac{x-5+5}{x-5}$

Ⓒ $g(x) = \dfrac{x}{x-5+5}$

Ⓓ $g(x) = \dfrac{x}{x} - \dfrac{x}{5}$

In Exercises 31–38, rewrite the function in the form $g(x) = \dfrac{a}{x-h} + k$. Graph the function. Describe the graph of g as a transformation of the graph of $f(x) = \dfrac{a}{x}$. *(See Example 5.)*

31. $g(x) = \dfrac{5x-7}{x-1}$

32. $g(x) = \dfrac{6x+4}{x+5}$

33. $g(x) = \dfrac{12x}{x-5}$

34. $g(x) = \dfrac{8x}{x+13}$

35. $g(x) = \dfrac{2x+3}{x}$

36. $g(x) = \dfrac{4x-6}{x}$

37. $g(x) = \dfrac{3x+11}{x-3}$

38. $g(x) = \dfrac{7x-9}{x+10}$

In Exercises 39–44, simplify the complex fraction. *(See Example 6.)*

39. $\dfrac{\frac{x}{3}-6}{10+\frac{4}{x}}$

40. $\dfrac{15-\frac{2}{x}}{\frac{x}{5}+4}$

41. $\dfrac{\frac{1}{2x-5}-\frac{7}{8x-20}}{\frac{x}{2x-5}}$

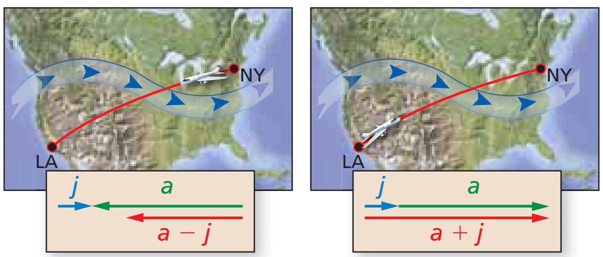
42. $\dfrac{\frac{16}{x-2}}{\frac{4}{x+1}+\frac{6}{x}}$

43. $\dfrac{\frac{1}{3x^2-3}}{\frac{5}{x+1}-\frac{x+4}{x^2-3x-4}}$

44. $\dfrac{\frac{3}{x-2}-\frac{6}{x^2-4}}{\frac{3}{x+2}+\frac{1}{x-2}}$

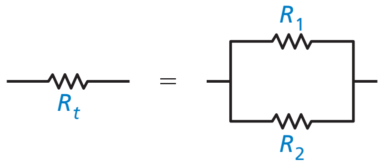
45. PROBLEM SOLVING The total time T (in hours) needed to fly from New York to Los Angeles and back can be modeled by the equation below, where d is the distance (in miles) each way, a is the average airplane speed (in miles per hour), and j is the average speed (in miles per hour) of the jet stream. Simplify the equation. Then find the total time it takes to fly 2468 miles when $a = 510$ miles per hour and $j = 115$ miles per hour.

$$T = \frac{d}{a-j} + \frac{d}{a+j}$$



46. REWRITING A FORMULA The total resistance R_t of two resistors in a parallel circuit with resistances R_1 and R_2 (in ohms) is given by the equation shown. Simplify the complex fraction. Then find the total resistance when $R_1 = 2000$ ohms and $R_2 = 5600$ ohms.

$$R_t = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$



47. PROBLEM SOLVING You plan a trip that involves a 40-mile bus ride and a train ride. The entire trip is 140 miles. The time (in hours) the bus travels is $y_1 = \dfrac{40}{x}$, where x is the average speed (in miles per hour) of the bus. The time (in hours) the train travels is $y_2 = \dfrac{100}{x+30}$. Write and simplify a model that shows the total time y of the trip.

48. PROBLEM SOLVING You participate in a sprint triathlon that involves swimming, bicycling, and running. The table shows the distances (in miles) and your average speed for each portion of the race.

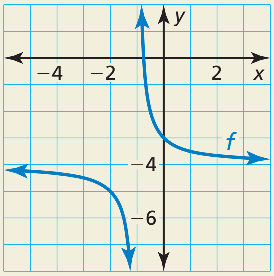
	Distance (miles)	Speed (miles per hour)
Swimming	0.5	r
Bicycling	22	$15r$
Running	6	$r+5$

a. Write a model in simplified form for the total time (in hours) it takes to complete the race.

b. How long does it take to complete the race if you can swim at an average speed of 2 miles per hour? Justify your answer.

49. MAKING AN ARGUMENT Your friend claims that the least common multiple of two numbers is always greater than each of the numbers. Is your friend correct? Justify your answer.

50. HOW DO YOU SEE IT? Use the graph of the function $f(x) = \frac{a}{x - h} + k$ to determine the values of h and k.



51. REWRITING A FORMULA You borrow P dollars to buy a car and agree to repay the loan over t years at a monthly interest rate of i (expressed as a decimal). Your monthly payment M is given by either formula below.

$$M = \frac{Pi}{1 - \left(\frac{1}{1+i}\right)^{12t}} \quad \text{or} \quad M = \frac{Pi(1+i)^{12t}}{(1+i)^{12t} - 1}$$

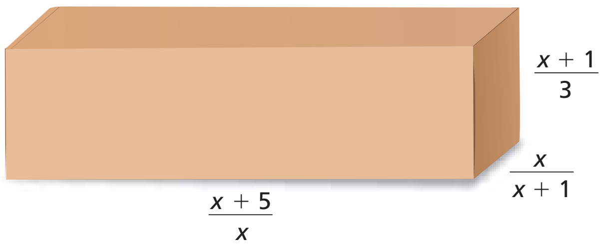
a. Show that the formulas are equivalent by simplifying the first formula.

b. Find your monthly payment when you borrow \$15,500 at a monthly interest rate of 0.5% and repay the loan over 4 years.

52. THOUGHT PROVOKING Is it possible to write two rational functions whose sum is a quadratic function? Justify your answer.

53. USING TOOLS Use technology to rewrite the function $g(x) = \frac{(97.6)(0.024) + x(0.003)}{12.2 + x}$ in the form $g(x) = \frac{a}{x - h} + k$. Describe the graph of g as a transformation of the graph of $f(x) = \frac{a}{x}$.

54. MATHEMATICAL CONNECTIONS Find an expression for the surface area of the box.



55. PROBLEM SOLVING You are hired to wash the new cars at a car dealership with two other employees. You take an average of 40 minutes to wash a car (R_1 = 1/40 car per minute). The second employee washes a car in x minutes. The third employee washes a car in $x + 10$ minutes.

a. Write a single expression R for the combined rate of cars washed per minute by the group.

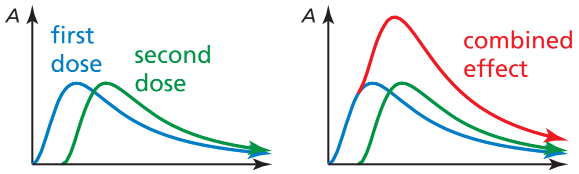
b. Evaluate your expression in part (a) when the second employee washes a car in 35 minutes. How many cars per hour does this represent? Explain your reasoning.

56. USING TOOLS The expression $2w + \frac{450}{w}$ models the perimeter of the corral in Section 1.1 Example 3. Find the sum of the terms. Then use a graph to justify the value of w found in the example. How is the graph different from previous graphs of rational functions?

57. MODELING WITH MATHEMATICS The amount A (in milligrams) of aspirin in a person's bloodstream can be modeled by

$$A = \frac{391t^2 + 0.112}{0.218t^4 + 0.991t^2 + 1}$$

where t is the time (in hours) after one dose is taken.



a. A second dose is taken 1 hour after the first dose. Write an equation to model the amount of the second dose in the bloodstream.

b. Write a model for the *total* amount of aspirin in the bloodstream after the second dose is taken.

58. FINDING A PATTERN Find the next two expressions in the pattern shown. Then simplify all five expressions. What value do the expressions approach?

$$1 + \frac{1}{2 + \frac{1}{2}}, \; 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, \; 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}}, \ldots$$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve $f(x) = g(x)$ by graphing and algebraic methods. *(Section 3.5)*

59. $f(x) = 2x^3 + 5$
$g(x) = x^3 - 3$

60. $f(x) = x^3 + x^2$
$g(x) = 9x + 9$

61. $f(x) = x^4 - 3x^2 + 2$
$g(x) = x^2 + 2$