5.5 Solving Exponential and Logarithmic Equations

Essential Question

How can you solve exponential and logarithmic equations?

EXPLORATION 1 **Solving Exponential and Logarithmic Equations**

Work with a partner. Match each equation with the graph of its related system of equations. Explain your reasoning. Then use the graph to solve the equation.

a. $e^x = 2$

b. $\ln x = -1$

c. $2^x = 3^{-x}$

d. $\log_4 x = 1$

e. $\log_5 x = \frac{1}{2}$

f. $4^x = 2$

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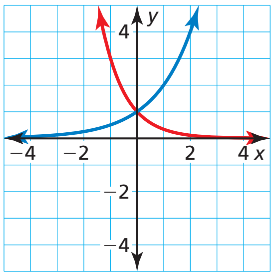

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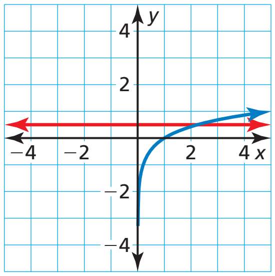

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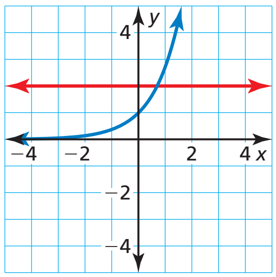

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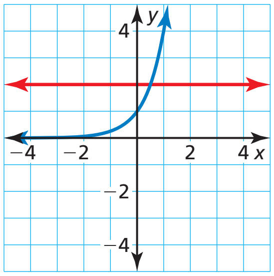

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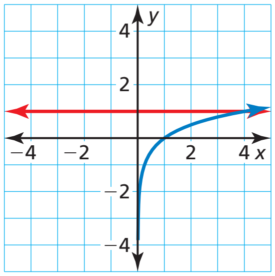

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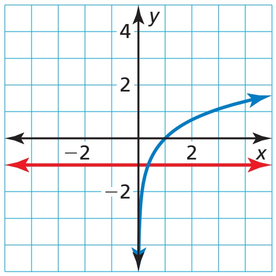


MAKING SENSE OF PROBLEMS

To be proficient in math, you need to plan a solution pathway rather than simply jumping into a solution attempt.

EXPLORATION 2 **Solving Exponential and Logarithmic Equations**

Work with a partner. Look back at the equations in Explorations 1(a) and 1(b). Suppose you want a more accurate way to solve the equations than using a graphical approach.

a. Show how you could use a *numerical approach* by creating a table. For instance, you might use a spreadsheet to solve the equations.

b. Show how you could use an *analytical approach*. For instance, you might try solving the equations by using the inverse properties of exponents and logarithms.

Communicate Your Answer

3. How can you solve exponential and logarithmic equations?

4. Solve each equation using any method. Explain your choice of method.

a. $16^x = 2$

b. $2^x = 4^{2x+1}$

c. $2^x = 3^{x+1}$

d. $\log x = \frac{1}{2}$

e. $\ln x = 2$

f. $\log_3 x = \frac{3}{2}$

5.5 Lesson

Core Vocabulary

exponential equations, *p. 282*
logarithmic equations, *p. 283*

Previous
extraneous solution
inequality

What You Will Learn

- Solve exponential equations.
- Solve logarithmic equations.
- Solve exponential and logarithmic inequalities.

Solving Exponential Equations

Exponential equations are equations in which variable expressions occur as exponents. The result below is useful for solving certain exponential equations.

Core Concept

Property of Equality for Exponential Equations

Algebra If b is a positive real number other than 1, then $b^x = b^y$ if and only if $x = y$.

Example If $3^x = 3^5$, then $x = 5$. If $x = 5$, then $3^x = 3^5$.

The preceding property is useful for solving an exponential equation when each side of the equation uses the same base (or can be rewritten to use the same base). When it is not convenient to write each side of an exponential equation using the same base, you can try to solve the equation by taking a logarithm of each side.

EXAMPLE 1 Solving Exponential Equations

Solve each equation.

a. $100^x = \left(\frac{1}{10}\right)^{x-3}$ **b.** $2^x = 7$

SOLUTION

a.

$100^x = \left(\frac{1}{10}\right)^{x-3}$	Write original equation.
$(10^2)^x = (10^{-1})^{x-3}$	Rewrite 100 and $\frac{1}{10}$ as powers with base 10.
$10^{2x} = 10^{-x+3}$	Power of a Power Property
$2x = -x + 3$	Property of Equality for Exponential Equations
$x = 1$	Solve for x.

Check

$$100^1 \stackrel{?}{=} \left(\frac{1}{10}\right)^{1-3}$$

$$100 \stackrel{?}{=} \left(\frac{1}{10}\right)^{-2}$$

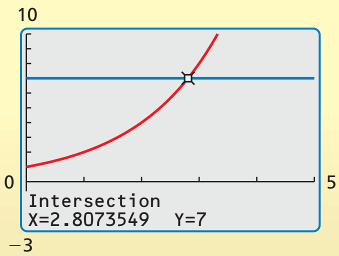
$$100 = 100 \checkmark$$

b.

$2^x = 7$	Write original equation.
$\log_2 2^x = \log_2 7$	Take $\log_2$ of each side.
$x = \log_2 7$	$\log_b b^x = x$
$x \approx 2.807$	Use a calculator.

Check

Enter $y = 2^x$ and $y = 7$ in a graphing calculator. Use the *intersect* feature to find the intersection point of the graphs. The graphs intersect at about (2.807, 7). So, the solution of $2^x = 7$ is about 2.807. ✓



An important application of exponential equations is *Newton's Law of Cooling*. This law states that for a cooling substance with initial temperature T_0, the temperature T after t minutes can be modeled by

$$T = (T_0 - T_R)e^{-rt} + T_R$$

where T_R is the surrounding temperature and r is the cooling rate of the substance.

LOOKING FOR STRUCTURE

Notice that Newton's Law of Cooling models the temperature of a cooling body by adding a constant function, T_R, to a decaying exponential function, $(T_0 - T_R)e^{-rt}$.

EXAMPLE 2 **Solving a Real-Life Problem**

You are cooking *aleecha*, an Ethiopian stew. When you take it off the stove, its temperature is 212°F. The room temperature is 70°F, and the cooling rate of the stew is $r = 0.046$. How long will it take to cool the stew to a serving temperature of 100°F?

SOLUTION

Use Newton's Law of Cooling with $T = 100$, $T_0 = 212$, $T_R = 70$, and $r = 0.046$.

$T = (T_0 - T_R)e^{-rt} + T_R$	Newton's Law of Cooling
$100 = (212 - 70)e^{-0.046t} + 70$	Substitute for T, T_0, T_R, and r.
$30 = 142e^{-0.046t}$	Subtract 70 from each side.
$0.211 \approx e^{-0.046t}$	Divide each side by 142.
$\ln 0.211 \approx \ln e^{-0.046t}$	Take natural log of each side.
$-1.556 \approx -0.046t$	$\ln e^x = \log_e e^x = x$
$33.8 \approx t$	Divide each side by -0.046.

▶ You should wait about 34 minutes before serving the stew.

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Solve the equation.

1. $2^x = 5$
2. $7^{9x} = 15$
3. $4e^{-0.3x} - 7 = 13$
4. **WHAT IF?** In Example 2, how long will it take to cool the stew to 100°F when the room temperature is 75°F?

Solving Logarithmic Equations

Logarithmic equations are equations that involve logarithms of variable expressions. You can use the next property to solve some types of logarithmic equations.

Core Concept

Property of Equality for Logarithmic Equations

Algebra If b, x, and y are positive real numbers with $b \neq 1$, then $\log_b x = \log_b y$ if and only if $x = y$.

Example If $\log_2 x = \log_2 7$, then $x = 7$. If $x = 7$, then $\log_2 x = \log_2 7$.

The preceding property implies that if you are given an equation $x = y$, then you can exponentiate each side to obtain an equation of the form $b^x = b^y$. This technique is useful for solving some logarithmic equations.

EXAMPLE 3 Solving Logarithmic Equations

Solve (a) $\ln(4x - 7) = \ln(x + 5)$ and (b) $\log_2(5x - 17) = 3$.

SOLUTION

a. $\ln(4x - 7) = \ln(x + 5)$ Write original equation.

$4x - 7 = x + 5$ Property of Equality for Logarithmic Equations

$3x - 7 = 5$ Subtract x from each side.

$3x = 12$ Add 7 to each side.

$x = 4$ Divide each side by 3.

Check

$\ln(4 \cdot 4 - 7) \stackrel{?}{=} \ln(4 + 5)$

$\ln(16 - 7) \stackrel{?}{=} \ln 9$

$\ln 9 = \ln 9$ ✓

b. $\log_2(5x - 17) = 3$ Write original equation.

$2^{\log_2(5x - 17)} = 2^3$ Exponentiate each side using base 2.

$5x - 17 = 8$ $b^{\log_b x} = x$

$5x = 25$ Add 17 to each side.

$x = 5$ Divide each side by 5.

Check

$\log_2(5 \cdot 5 - 17) \stackrel{?}{=} 3$

$\log_2(25 - 17) \stackrel{?}{=} 3$

$\log_2 8 \stackrel{?}{=} 3$

Because $2^3 = 8$, $\log_2 8 = 3$. ✓

Because the domain of a logarithmic function generally does not include all real numbers, be sure to check for extraneous solutions of logarithmic equations. You can do this algebraically or graphically.

EXAMPLE 4 Solving a Logarithmic Equation

Solve $\log 2x + \log(x - 5) = 2$.

SOLUTION

$\log 2x + \log(x - 5) = 2$ Write original equation.

$\log[2x(x - 5)] = 2$ Product Property of Logarithms

$10^{\log[2x(x - 5)]} = 10^2$ Exponentiate each side using base 10.

$2x(x - 5) = 100$ $b^{\log_b x} = x$

$2x^2 - 10x = 100$ Distributive Property

$2x^2 - 10x - 100 = 0$ Write in standard form.

$x^2 - 5x - 50 = 0$ Divide each side by 2.

$(x - 10)(x + 5) = 0$ Factor.

$x = 10$ or $x = -5$ Zero-Product Property

▶ The apparent solution $x = -5$ is extraneous. So, the only solution is $x = 10$.

Check

$\log(2 \cdot 10) + \log(10 - 5) \stackrel{?}{=} 2$

$\log 20 + \log 5 \stackrel{?}{=} 2$

$\log 100 \stackrel{?}{=} 2$

$2 = 2$ ✓

$\log[2 \cdot (-5)] + \log(-5 - 5) \stackrel{?}{=} 2$

$\log(-10) + \log(-10) \stackrel{?}{=} 2$

Because $\log(-10)$ is not defined, -5 is not a solution. ✗

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Solve the equation. Check for extraneous solutions.

5. $\ln(7x - 4) = \ln(2x + 11)$

6. $\log_2(x - 6) = 5$

7. $\log 5x + \log(x - 1) = 2$

8. $\log_4(x + 12) + \log_4 x = 3$

Solving Exponential and Logarithmic Inequalities

Exponential inequalities are inequalities in which variable expressions occur as exponents, and *logarithmic inequalities* are inequalities that involve logarithms of variable expressions. To solve exponential and logarithmic inequalities algebraically, use these properties. Note that the properties are true for $\leq$ and $\geq$.

STUDY TIP
Be sure you understand that these properties of inequality are only true for values of $b > 1$.

Exponential Property of Inequality: If b is a positive real number greater than 1, then $b^x > b^y$ if and only if $x > y$, and $b^x < b^y$ if and only if $x < y$.

Logarithmic Property of Inequality: If b, x, and y are positive real numbers with $b > 1$, then $\log_b x > \log_b y$ if and only if $x > y$, and $\log_b x < \log_b y$ if and only if $x < y$.

You can also solve an inequality by taking a logarithm of each side or by exponentiating.

EXAMPLE 5 Solving an Exponential Inequality

Solve $3^x < 20$.

SOLUTION

$3^x < 20$ Write original inequality.

$\log_3 3^x < \log_3 20$ Take $\log_3$ of each side.

$x < \log_3 20$ $\log_b b^x = x$

▶ The solution is $x < \log_3 20$. Because $\log_3 20 \approx 2.727$, the approximate solution is $x < 2.727$.

EXAMPLE 6 Solving a Logarithmic Inequality

Solve $\log x \leq 2$.

SOLUTION

Method 1 Use an algebraic approach.

$\log x \leq 2$ Write original inequality.

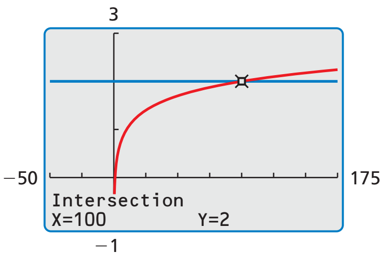
$10^{\log_{10} x} \leq 10^2$ Exponentiate each side using base 10.

$x \leq 100$ $b^{\log_b x} = x$

▶ Because $\log x$ is only defined when $x > 0$, the solution is $0 < x \leq 100$.

Method 2 Use a graphical approach.

Graph $y = \log x$ and $y = 2$ in the same viewing window. Use the *intersect* feature to determine that the graphs intersect when $x = 100$. The graph of $y = \log x$ is on or below the graph of $y = 2$ when $0 < x \leq 100$.



▶ The solution is $0 < x \leq 100$.

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Solve the inequality.

9. $e^x < 2$ **10.** $10^{2x-6} > 3$ **11.** $\log x + 9 < 45$ **12.** $2 \ln x - 1 > 4$

5.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The equation $3^{x-1} = 34$ is an example of a(n) ___________ equation.

2. **WRITING** Compare the methods for solving exponential and logarithmic equations.

3. **WRITING** When do logarithmic equations have extraneous solutions?

4. **COMPLETE THE SENTENCE** If b is a positive real number other than 1, then $b^x = b^y$ if and only if _________.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–16, solve the equation. *(See Example 1.)*

5. $2^{3x+5} = 2^{1-x}$

6. $e^{2x} = e^{3x-1}$

7. $5^{x-3} = 25^{x-5}$

8. $6^{2x-6} = 36^{3x-5}$

9. $3^x = 7$

10. $10^x = 33$

11. $100^{5x+2} = \left(\frac{1}{10}\right)^{11-x}$

12. $512^{5x-1} = \left(\frac{1}{8}\right)^{-4-x}$

13. $5(7)^{5x} = 60$

14. $3(2)^{6x} = 99$

15. $3e^{4x} + 9 = 15$

16. $2e^{2x} - 7 = 5$

17. **MODELING WITH MATHEMATICS** The length ℓ (in centimeters) of a scalloped hammerhead shark can be modeled by the function

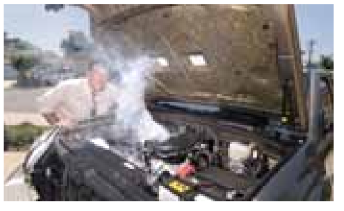
$$\ell = 266 - 219e^{-0.05t}$$

where t is the age (in years) of the shark. How old is a shark that is 175 centimeters long?

18. **MODELING WITH MATHEMATICS** One hundred grams of radium are stored in a container. The amount R (in grams) of radium present after t years can be modeled by $R = 100e^{-0.00043t}$. After how many years will only 5 grams of radium be present?

In Exercises 19 and 20, use Newton's Law of Cooling to solve the problem. *(See Example 2.)*

19. You are driving on a hot day when your car overheats and stops running. The car overheats at 280°F and can be driven again at 230°F. When it is 80°F outside, the cooling rate of the car is $r = 0.0058$. How long do you have to wait until you can continue driving?



20. You cook a turkey until the internal temperature reaches 180°F. The turkey is placed on the table until the internal temperature reaches 100°F and it can be carved. When the room temperature is 72°F, the cooling rate of the turkey is $r = 0.067$. How long do you have to wait until you can carve the turkey?

In Exercises 21–32, solve the equation. *(See Example 3.)*

21. $\ln(4x - 7) = \ln(x + 11)$

22. $\ln(2x - 4) = \ln(x + 6)$

23. $\log_2(3x - 4) = \log_2 5$

24. $\log(7x + 3) = \log 38$

25. $\log_2(4x + 8) = 5$

26. $\log_3(2x + 1) = 2$

27. $\log_7(4x + 9) = 2$

28. $\log_5(5x + 10) = 4$

29. $\log(12x - 9) = \log 3x$

30. $\log_6(5x + 9) = \log_6 6x$

31. $\log_2(x^2 - x - 6) = 2$

32. $\log_3(x^2 + 9x + 27) = 2$

In Exercises 33–40, solve the equation. Check for extraneous solutions. *(See Example 4.)*

33. $\log_2 x + \log_2(x - 2) = 3$

34. $\log_6 3x + \log_6(x - 1) = 3$

35. $\ln x + \ln(x + 3) = 4$

36. $\ln x + \ln(x - 2) = 5$

37. $\log_3 3x^2 + \log_3 3 = 2$

38. $\log_4(-x) + \log_4(x + 10) = 2$

39. $\log_3(x - 9) + \log_3(x - 3) = 2$

40. $\log_5(x + 4) + \log_5(x + 1) = 2$

ERROR ANALYSIS In Exercises 41 and 42, describe and correct the error in solving the equation.

41.

$$\log_3(5x - 1) = 4$$
$$3^{\log_3(5x-1)} = 4^3$$
$$5x - 1 = 64$$
$$5x = 65$$
$$x = 13$$

42.

$$\log_4(x + 12) + \log_4 x = 3$$
$$\log_4[(x + 12)(x)] = 3$$
$$4^{\log_4[(x+12)(x)]} = 4^3$$
$$(x + 12)(x) = 64$$
$$x^2 + 12x - 64 = 0$$
$$(x + 16)(x - 4) = 0$$
$$x = -16 \quad \text{or} \quad x = 4$$

43. **PROBLEM SOLVING** You deposit \$100 in an account that pays 6% annual interest. How long will it take for the balance to reach \$1000 for each frequency of compounding?

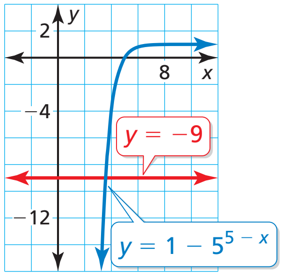
a. annual **b.** quarterly

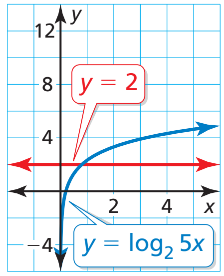
c. daily **d.** continuously

44. **MODELING WITH MATHEMATICS** The *apparent magnitude* of a star is a measure of the brightness of the star as it appears to observers on Earth. The apparent magnitude M of the dimmest star that can be seen with a telescope is $M = 5 \log D + 2$, where D is the diameter (in millimeters) of the telescope's objective lens. What is the diameter of the objective lens of a telescope that can reveal stars with a magnitude of 12?

45. **ANALYZING RELATIONSHIPS** Approximate the solution of each equation using the graph.

a. $1 - 5^{5-x} = -9$ **b.** $\log_2 5x = 2$





46. **MAKING AN ARGUMENT** Your friend states that a logarithmic equation cannot have a negative solution because logarithmic functions are not defined for negative numbers. Is your friend correct? Justify your answer.

In Exercises 47–54, solve the inequality. *(See Examples 5 and 6.)*

47. $9^x > 54$

48. $4^x \le 36$

49. $\ln x \ge 3$

50. $\log_4 x < 4$

51. $3^{4x-5} < 8$

52. $e^{3x+4} > 11$

53. $-3 \log_5 x + 6 \le 9$

54. $-4 \log_5 x - 5 \ge 3$

55. **COMPARING METHODS** Solve $\log_5 x < 2$ algebraically and graphically. Which method do you prefer? Explain your reasoning.

56. **PROBLEM SOLVING** You deposit \$1000 in an account that pays 3.5% annual interest compounded monthly. When is your balance at least \$1200? \$3500?

57. **PROBLEM SOLVING** An investment that earns a rate of return r doubles in value in t years, where $t = \dfrac{\ln 2}{\ln(1 + r)}$ and r is expressed as a decimal. What rates of return will double the value of an investment in less than 10 years?

58. **PROBLEM SOLVING** Your family purchases a new car for \$20,000. Its value decreases by 15% each year. During what interval does the car's value exceed \$10,000?

USING TOOLS In Exercises 59–62, use a graphing calculator to solve the equation.

59. $\ln 2x = 3^{-x+2}$

60. $\log x = 7^{-x}$

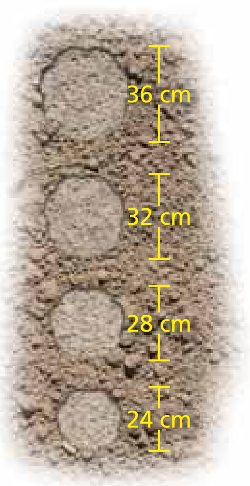
61. $\log x = 3^{x-3}$

62. $\ln 2x = e^{x-3}$

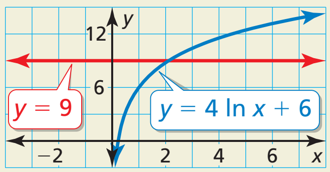
63. **REWRITING A FORMULA** A biologist can estimate the age of an African elephant by measuring the length of its footprint and using the equation $\ell = 45 - 25.7e^{-0.09a}$, where ℓ is the length (in centimeters) of the footprint and a is the age (in years).

a. Rewrite the equation, solving for a in terms of ℓ.

b. Use the equation in part (a) to find the ages of the elephants whose footprints are shown.



64. **HOW DO YOU SEE IT?** Use the graph to approximate the solution of the inequality $4 \ln x + 6 > 9$. Explain your reasoning.



65. **OPEN-ENDED** Write an exponential equation that has a solution of $x = 4$. Then write a logarithmic equation that has a solution of $x = -3$.

66. **THOUGHT PROVOKING** Give examples of logarithmic or exponential equations that have one solution, two solutions, and no solutions.

CRITICAL THINKING **In Exercises 67–72, solve the equation.**

67. $2^{x+3} = 5^{3x-1}$

68. $10^{3x-8} = 2^{5-x}$

69. $\log_3(x - 6) = \log_9 2x$

70. $\log_4 x = \log_8 4x$

71. $2^{2x} - 12 \cdot 2^x + 32 = 0$

72. $5^{2x} + 20 \cdot 5^x - 125 = 0$

73. **WRITING** In Exercises 67–70, you solved exponential and logarithmic equations with different bases. Describe general methods for solving such equations.

74. **PROBLEM SOLVING** When X-rays of a fixed wavelength strike a material x centimeters thick, the intensity $I(x)$ of the X-rays transmitted through the material is given by $I(x) = I_0e^{-\mu x}$, where I_0 is the initial intensity and μ is a value that depends on the type of material and the wavelength of the X-rays. The table shows the values of μ for various materials and X-rays of medium wavelength.

Material	Aluminum	Copper	Lead
Value of μ	0.43	3.2	43

a. Find the thickness of aluminum shielding that reduces the intensity of X-rays to 30% of their initial intensity. (*Hint*: Find the value of x for which $I(x) = 0.3I_0$.)

b. Repeat part (a) for the copper shielding.

c. Repeat part (a) for the lead shielding.

d. Your dentist puts a lead apron on you before taking X-rays of your teeth to protect you from harmful radiation. Based on your results from parts (a)–(c), explain why lead is a better material to use than aluminum or copper.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write an equation in point-slope form of the line that passes through the given point and has the given slope. *(Skills Review Handbook)*

75. $(1, -2);\ m = 4$

76. $(3, 2);\ m = -2$

77. $(3, -8);\ m = -\frac{1}{3}$

78. $(2, 5);\ m = 2$

Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function. *(Section 3.9)*

79. $(-3, -50), (-2, -13), (-1, 0), (0, 1), (1, 2), (2, 15), (3, 52), (4, 125)$

80. $(-3, 139), (-2, 32), (-1, 1), (0, -2), (1, -1), (2, 4), (3, 37), (4, 146)$

81. $(-3, -327), (-2, -84), (-1, -17), (0, -6), (1, -3), (2, -32), (3, -189), (4, -642)$