5.2 Logarithms and Logarithmic Functions

Essential Question What are some of the characteristics of the graph of a logarithmic function?

Every exponential function of the form $f(x) = b^x$, where b is a positive real number other than 1, has an inverse function that you can denote by $g(x) = \log_b x$. This inverse function is called a *logarithmic function with base b*.

EXPLORATION 1 Rewriting Exponential Equations

Work with a partner. Find the value of x in each exponential equation. Explain your reasoning. Then use the value of x to rewrite the exponential equation in its equivalent logarithmic form, $x = \log_b y$.

a. $2^x = 8$ **b.** $3^x = 9$ **c.** $4^x = 2$

d. $5^x = 1$ **e.** $5^x = \frac{1}{5}$ **f.** $8^x = 4$

EXPLORATION 2 Graphing Exponential and Logarithmic Functions

Work with a partner. Complete each table for the given exponential function. Use the results to complete the table for the given logarithmic function. Explain your reasoning. Then sketch the graphs of f and g in the same coordinate plane.

a.

x	−2	−1	0	1	2
$f(x) = 2^x$					

x					
$g(x) = \log_2 x$	−2	−1	0	1	2

b.

x	−2	−1	0	1	2
$f(x) = 10^x$					

x					
$g(x) = \log_{10} x$	−2	−1	0	1	2

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to justify your conclusions and communicate them to others.

EXPLORATION 3 Characteristics of Graphs of Logarithmic Functions

Work with a partner. Use the graphs you sketched in Exploration 2 to determine the domain, range, x-intercept, and asymptote of the graph of $g(x) = \log_b x$, where b is a positive real number other than 1. Explain your reasoning.

Communicate Your Answer

4. What are some of the characteristics of the graph of a logarithmic function?

5. How can you use the graph of an exponential function to obtain the graph of a logarithmic function?

5.2 Lesson

Core Vocabulary

logarithm of y with base b, *p. 258*
common logarithm, *p. 259*
natural logarithm, *p. 259*

Previous
inverse functions

What You Will Learn

- Define and evaluate logarithms.
- Use inverse properties of logarithmic and exponential functions.
- Graph logarithmic functions.

Logarithms

You know that $2^2 = 4$ and $2^3 = 8$. However, for what value of x does $2^x = 6$? Mathematicians define this x-value using a *logarithm* and write $x = \log_2 6$. The definition of a logarithm can be generalized as follows.

Core Concept

Definition of Logarithm with Base *b*

Let b and y be positive real numbers with $b \neq 1$. The **logarithm of *y* with base *b*** is denoted by $\log_b y$ and is defined as

$$\log_b y = x \quad \text{if and only if} \quad b^x = y.$$

The expression $\log_b y$ is read as "log base b of y."

This definition tells you that the equations $\log_b y = x$ and $b^x = y$ are equivalent. The first is in *logarithmic form*, and the second is in *exponential form*.

EXAMPLE 1 Rewriting Logarithmic Equations

Rewrite each equation in exponential form.

a. $\log_2 16 = 4$ **b.** $\log_4 1 = 0$ **c.** $\log_{12} 12 = 1$ **d.** $\log_{1/4} 4 = -1$

SOLUTION

Logarithmic Form	Exponential Form
a. $\log_2 16 = 4$	$2^4 = 16$
b. $\log_4 1 = 0$	$4^0 = 1$
c. $\log_{12} 12 = 1$	$12^1 = 12$
d. $\log_{1/4} 4 = -1$	$\left(\frac{1}{4}\right)^{-1} = 4$

EXAMPLE 2 Rewriting Exponential Equations

Rewrite each equation in logarithmic form.

a. $5^2 = 25$ **b.** $10^{-1} = 0.1$ **c.** $8^{2/3} = 4$ **d.** $6^{-3} = \frac{1}{216}$

SOLUTION

Exponential Form	Logarithmic Form
a. $5^2 = 25$	$\log_5 25 = 2$
b. $10^{-1} = 0.1$	$\log_{10} 0.1 = -1$
c. $8^{2/3} = 4$	$\log_8 4 = \frac{2}{3}$
d. $6^{-3} = \frac{1}{216}$	$\log_6 \frac{1}{216} = -3$

Parts (b) and (c) of Example 1 illustrate two special logarithm values that you should learn to recognize. Let b be a positive real number such that $b \neq 1$.

Logarithm of 1

$\log_b 1 = 0$ because $b^0 = 1$.

Logarithm of b with Base b

$\log_b b = 1$ because $b^1 = b$.

EXAMPLE 3 Evaluating Logarithmic Expressions

Evaluate each logarithm.

a. $\log_4 64$ **b.** $\log_5 0.2$ **c.** $\log_{1/5} 125$ **d.** $\log_{36} 6$

SOLUTION

To help you find the value of $\log_b y$, ask yourself what power of b gives you y.

a. What power of 4 gives you 64? $4^3 = 64$, so $\log_4 64 = 3$.

b. What power of 5 gives you 0.2? $5^{-1} = 0.2$, so $\log_5 0.2 = -1$.

c. What power of $\frac{1}{5}$ gives you 125? $\left(\frac{1}{5}\right)^{-3} = 125$, so $\log_{1/5} 125 = -3$.

d. What power of 36 gives you 6? $36^{1/2} = 6$, so $\log_{36} 6 = \frac{1}{2}$.

A **common logarithm** is a logarithm with base 10. It is denoted by $\log_{10}$ or simply by log. A **natural logarithm** is a logarithm with base e. It can be denoted by $\log_e$ but is usually denoted by ln.

Common Logarithm

$\log_{10} x = \log x$

Natural Logarithm

$\log_e x = \ln x$

EXAMPLE 4 Evaluating Common and Natural Logarithms

Evaluate (a) log 8 and (b) ln 0.3 using a calculator. Round your answer to three decimal places.

SOLUTION

Most calculators have keys for evaluating common and natural logarithms.

```
log(8)
          .903089987
ln(0.3)
         -1.203972804
```

a. $\log 8 \approx 0.903$

b. $\ln 0.3 \approx -1.204$

Check your answers by rewriting each logarithm in exponential form and evaluating.

Check

```
10^(0.903)
           7.99834255
e^(-1.204)
          .2999918414
```

Monitoring Progress

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Rewrite the equation in exponential form.

1. $\log_3 81 = 4$ **2.** $\log_7 7 = 1$ **3.** $\log_{14} 1 = 0$ **4.** $\log_{1/2} 32 = -5$

Rewrite the equation in logarithmic form.

5. $7^2 = 49$ **6.** $50^0 = 1$ **7.** $4^{-1} = \frac{1}{4}$ **8.** $256^{1/8} = 2$

Evaluate the logarithm. If necessary, use a calculator and round your answer to three decimal places.

9. $\log_2 32$ **10.** $\log_{27} 3$ **11.** $\log 12$ **12.** $\ln 0.75$

Using Inverse Properties

By the definition of a logarithm, it follows that the logarithmic function $g(x) = \log_b x$ is the inverse of the exponential function $f(x) = b^x$. This means that

$$g(f(x)) = \log_b b^x = x \qquad \text{and} \qquad f(g(x)) = b^{\log_b x} = x.$$

In other words, exponential functions and logarithmic functions "undo" each other.

EXAMPLE 5 Using Inverse Properties

Simplify (a) $10^{\log 4}$ and (b) $\log_5 25^x$.

SOLUTION

a. $10^{\log 4} = 4$ — $b^{\log_b x} = x$

b. $\log_5 25^x = \log_5(5^2)^x$ — Express 25 as a power with base 5.

$= \log_5 5^{2x}$ — Power of a Power Property

$= 2x$ — $\log_b b^x = x$

EXAMPLE 6 Finding Inverse Functions

Find the inverse of each function.

a. $f(x) = 6^x$

b. $y = \ln(x + 3)$

SOLUTION

a. From the definition of logarithm, the inverse of $f(x) = 6^x$ is $g(x) = \log_6 x$.

b.

$y = \ln(x + 3)$ — Write original function.

$x = \ln(y + 3)$ — Switch x and y.

$e^x = y + 3$ — Write in exponential form.

$e^x - 3 = y$ — Subtract 3 from each side.

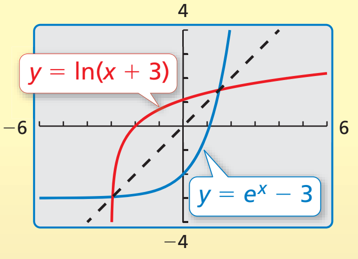
▶ The inverse of $y = \ln(x + 3)$ is $y = e^x - 3$.

Check

a. $f(g(x)) = 6^{\log_6 x} = x$ ✓

$g(f(x)) = \log_6 6^x = x$ ✓

b.



The graphs appear to be reflections of each other in the line $y = x$. ✓

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Simplify the expression.

13. $8^{\log_8 x}$

14. $\log_7 7^{-3x}$

15. $\log_2 64^x$

16. $e^{\ln 20}$

17. Find the inverse of $y = 4^x$.

18. Find the inverse of $y = \ln(x - 5)$.

Graphing Logarithmic Functions

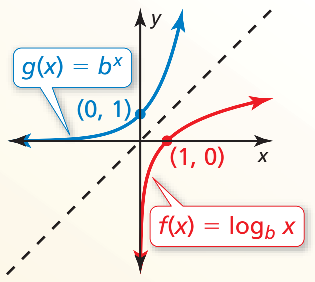
You can use the inverse relationship between exponential and logarithmic functions to graph logarithmic functions.

Core Concept

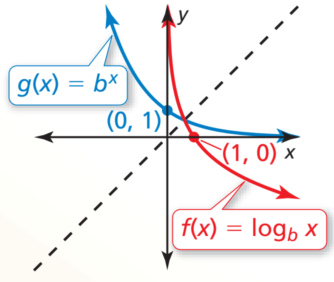
Parent Graphs for Logarithmic Functions

The graph of $f(x) = \log_b x$ is shown below for $b > 1$ and for $0 < b < 1$. Because $f(x) = \log_b x$ and $g(x) = b^x$ are inverse functions, the graph of $f(x) = \log_b x$ is the reflection of the graph of $g(x) = b^x$ in the line $y = x$.

Graph of $f(x) = \log_b x$ for $b > 1$



Graph of $f(x) = \log_b x$ for $0 < b < 1$



Note that the y-axis is a vertical asymptote of the graph of $f(x) = \log_b x$. The domain of $f(x) = \log_b x$ is $x > 0$, and the range is all real numbers.

EXAMPLE 7 Graphing a Logarithmic Function

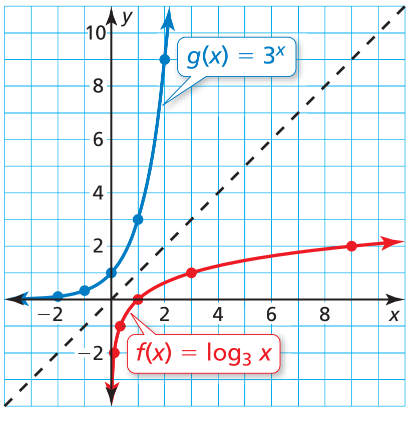
Graph $f(x) = \log_3 x$.

SOLUTION

Step 1 Find the inverse of f. From the definition of logarithm, the inverse of $f(x) = \log_3 x$ is $g(x) = 3^x$.

Step 2 Make a table of values for $g(x) = 3^x$.

x	-2	-1	0	1	2
$g(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



Step 3 Plot the points from the table and connect them with a smooth curve.

Step 4 Because $f(x) = \log_3 x$ and $g(x) = 3^x$ are inverse functions, the graph of f is obtained by reflecting the graph of g in the line $y = x$. To do this, reverse the coordinates of the points on g and plot these new points on the graph of f.

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Graph the function.

19. $y = \log_2 x$

20. $f(x) = \log_5 x$

21. $y = \log_{1/2} x$

5.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A logarithm with base 10 is called a(n) ___________ logarithm.
2. **COMPLETE THE SENTENCE** The expression $\log_3 9$ is read as ______________.
3. **WRITING** Describe the relationship between $y = 7^x$ and $y = \log_7 x$.
4. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

What power of 4 gives you 16?	What is log base 4 of 16?
Evaluate 4^2.	Evaluate $\log_4 16$.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, rewrite the equation in exponential form. *(See Example 1.)*

5. $\log_3 9 = 2$
6. $\log_4 4 = 1$
7. $\log_6 1 = 0$
8. $\log_7 343 = 3$
9. $\log_{1/2} 16 = -4$
10. $\log_3 \frac{1}{3} = -1$

In Exercises 11–16, rewrite the equation in logarithmic form. *(See Example 2.)*

11. $6^2 = 36$
12. $12^0 = 1$
13. $16^{-1} = \frac{1}{16}$
14. $5^{-2} = \frac{1}{25}$
15. $125^{2/3} = 25$
16. $49^{1/2} = 7$

In Exercises 17–24, evaluate the logarithm. *(See Example 3.)*

17. $\log_3 81$
18. $\log_7 49$
19. $\log_3 3$
20. $\log_{1/2} 1$
21. $\log_5 \frac{1}{625}$
22. $\log_8 \frac{1}{512}$
23. $\log_4 0.25$
24. $\log_{10} 0.001$

25. **NUMBER SENSE** Order the logarithms from least value to greatest value.

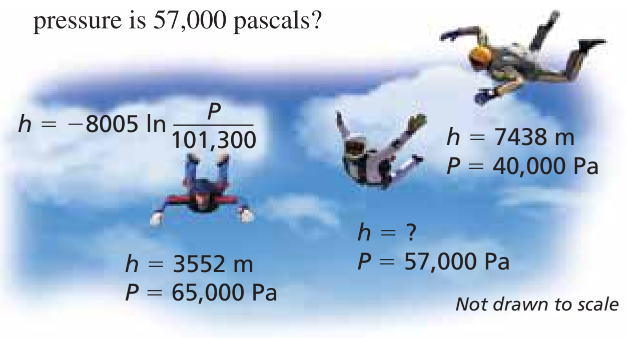
$\log_5 23$	$\log_6 38$	$\log_7 8$	$\log_2 10$

26. **WRITING** Explain why the expressions $\log_2(-1)$ and $\log_1 1$ are not defined.

In Exercises 27–32, evaluate the logarithm using a calculator. Round your answer to three decimal places. *(See Example 4.)*

27. $\log 6$
28. $\ln 12$
29. $\ln \frac{1}{3}$
30. $\log \frac{2}{7}$
31. $3 \ln 0.5$
32. $\log 0.6 + 1$

33. **MODELING WITH MATHEMATICS** Skydivers use an instrument called an *altimeter* to track their altitude as they fall. The altimeter determines altitude by measuring air pressure. The altitude h (in meters) above sea level is related to the air pressure P (in pascals) by the function shown in the diagram. What is the altitude above sea level when the air pressure is 57,000 pascals?



34. **MODELING WITH MATHEMATICS** The pH value for a substance measures how acidic or alkaline the substance is. It is given by the formula $\text{pH} = -\log[H^+]$, where H^+ is the hydrogen ion concentration (in moles per liter). Find the pH of each substance.

    a. baking soda: $[H^+] = 10^{-8}$ moles per liter

    b. vinegar: $[H^+] = 10^{-3}$ moles per liter

In Exercises 35–40, simplify the expression. *(See Example 5.)*

35. $7^{\log_7 x}$ **36.** $3^{\log_3 5x}$

37. $e^{\ln 4}$ **38.** $10^{\log 15}$

39. $\log_3 3^{2x}$ **40.** $\ln e^{x+1}$

41. **ERROR ANALYSIS** Describe and correct the error in rewriting $4^{-3} = \frac{1}{64}$ in logarithmic form.

✗ $\log_4(-3) = \frac{1}{64}$

42. **ERROR ANALYSIS** Describe and correct the error in simplifying the expression $\log_4 64^x$.

✗
$$\begin{aligned}\log_4 64^x &= \log_4(16 \cdot 4^x)\\ &= \log_4(4^2 \cdot 4^x)\\ &= \log_4 4^{2+x}\\ &= 2 + x\end{aligned}$$

In Exercises 43–52, find the inverse of the function. *(See Example 6.)*

43. $y = 0.3^x$ **44.** $y = 11^x$

45. $y = \log_2 x$ **46.** $y = \log_{1/5} x$

47. $y = \ln(x - 1)$ **48.** $y = \ln 2x$

49. $y = e^{3x}$ **50.** $y = e^{x-4}$

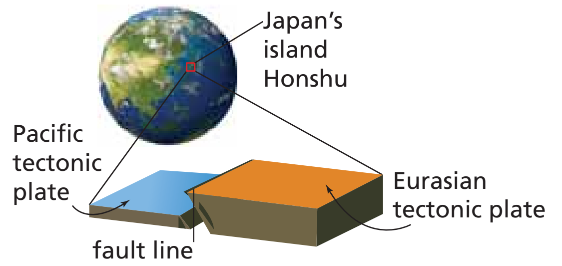
51. $y = 5^x - 9$ **52.** $y = 13 + \log x$

53. **PROBLEM SOLVING** The wind speed s (in miles per hour) near the center of a tornado can be modeled by $s = 93 \log d + 65$, where d is the distance (in miles) that the tornado travels.

a. In 1925, a tornado traveled 220 miles through three states. Estimate the wind speed near the center of the tornado.

b. Find the inverse of the given function. Describe what the inverse represents.

54. **MODELING WITH MATHEMATICS** The energy magnitude M of an earthquake can be modeled by $M = \frac{2}{3} \log E - 9.9$, where E is the amount of energy released (in ergs).



a. In 2011, a powerful earthquake in Japan, caused by the slippage of two tectonic plates along a fault, released 2.24×10^{28} ergs. What was the energy magnitude of the earthquake?

b. Find the inverse of the given function. Describe what the inverse represents.

In Exercises 55–60, graph the function. *(See Example 7.)*

55. $y = \log_4 x$ **56.** $y = \log_6 x$

57. $y = \log_{1/3} x$ **58.** $y = \log_{1/4} x$

59. $y = \log_2 x - 1$ **60.** $y = \log_3(x + 2)$

USING TOOLS **In Exercises 61–64, use a graphing calculator to graph the function. Determine the domain, range, and asymptote of the function.**

61. $y = \log(x + 2)$ **62.** $y = -\ln x$

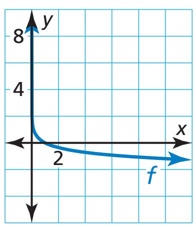
63. $y = \ln(-x)$ **64.** $y = 3 - \log x$

65. **MAKING AN ARGUMENT** Your friend states that every logarithmic function of the form $y = \log_b x$ will pass through the point (1, 0). Is your friend correct? Explain your reasoning.

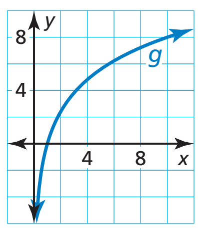
66. **ANALYZING RELATIONSHIPS** Rank the functions in order from the least average rate of change to the greatest average rate of change over the interval $1 \le x \le 10$.

a. $y = \log_6 x$ **b.** $y = \log_{3/5} x$

c.



d.



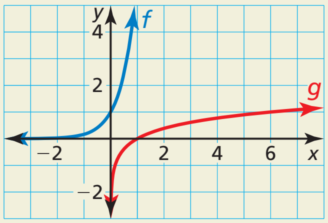
67. **PROBLEM SOLVING** Biologists have found that the length ℓ (in inches) of an alligator and its weight w (in pounds) are related by the function $\ell = 27.1 \ln w - 32.8$.

a. Use a graphing calculator to graph the function.

b. Use your graph to estimate the weight of an alligator that is 10 feet long.

c. Use the *zero* feature to find the x-intercept of the graph of the function. Does this x-value make sense in the context of the situation? Explain.

68. **HOW DO YOU SEE IT?** The figure shows the graphs of the two functions f and g.



a. Compare the end behavior of the logarithmic function g to that of the exponential function f.

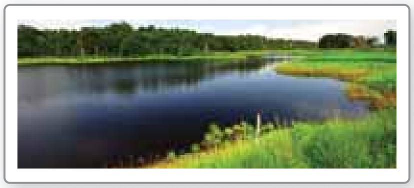
b. Determine whether the functions are inverse functions. Explain.

c. What is the base of each function? Explain.

69. **PROBLEM SOLVING** A study in Florida found that the number s of fish species in a pool or lake can be modeled by the function

$$s = 30.6 - 20.5 \log A + 3.8(\log A)^2$$

where A is the area (in square meters) of the pool or lake.



a. Use a graphing calculator to graph the function on the domain $200 \leq A \leq 35{,}000$.

b. Use your graph to estimate the number of species in a lake with an area of 30,000 square meters.

c. Use your graph to estimate the area of a lake that contains six species of fish.

d. Describe what happens to the number of fish species as the area of a pool or lake increases. Explain why your answer makes sense.

70. **THOUGHT PROVOKING** Write a logarithmic function that has an output of -4. Then sketch the graph of your function.

71. **CRITICAL THINKING** Evaluate each logarithm. (*Hint*: For each logarithm $\log_b x$, rewrite b and x as powers of the same base.)

a. $\log_{125} 25$ **b.** $\log_8 32$

c. $\log_{27} 81$ **d.** $\log_4 128$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Let $f(x) = \sqrt[3]{x}$. Write a rule for g that represents the indicated transformation of the graph of f. *(Section 4.3)*

72. $g(x) = -f(x)$

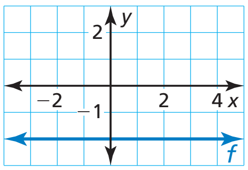
73. $g(x) = f\left(\frac{1}{2}x\right)$

74. $g(x) = f(-x) + 3$

75. $g(x) = f(x + 2)$

Identify the function family to which f belongs. Compare the graph of f to the graph of its parent function. *(Section 2.1)*

76.



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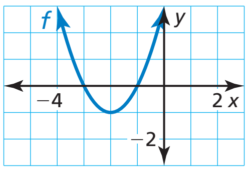

78.