

# 5.1 The Natural Base $e$

## Essential Question What is the natural base $e$ ?

So far in your study of mathematics, you have worked with special numbers such as  $\pi$  and  $i$ . Another special number is called the *natural base* and is denoted by  $e$ . The natural base  $e$  is irrational, so you cannot find its exact value.

### EXPLORATION 1 Approximating the Natural Base $e$

**Work with a partner.** One way to approximate the natural base  $e$  is to approximate the sum

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

Use a spreadsheet or a graphing calculator to approximate this sum. Explain the steps you used. How many decimal places did you use in your approximation?

### USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools to explore and deepen your understanding of concepts.

### EXPLORATION 2 Approximating the Natural Base $e$

**Work with a partner.** Another way to approximate the natural base  $e$  is to consider the expression

$$\left(1 + \frac{1}{x}\right)^x$$

As  $x$  increases, the value of this expression approaches the value of  $e$ . Copy and complete the table. Then use the results in the table to approximate  $e$ . Compare this approximation to the one you obtained in Exploration 1.

$x$	$10^1$	$10^2$	$10^3$	$10^4$	$10^5$	$10^6$
$\left(1 + \frac{1}{x}\right)^x$						

### EXPLORATION 3 Graphing a Natural Base Function

**Work with a partner.** Use your approximate value of  $e$  in Exploration 1 or 2 to complete the table. Then sketch the graph of the *natural base exponential function*  $y = e^x$ . You can use a graphing calculator and the  $e^x$  key to check your graph. What are the domain and range of  $y = e^x$ ? Justify your answers.

$x$	-2	-1	0	1	2
$y = e^x$					

## Communicate Your Answer

- What is the natural base  $e$ ?
- Repeat Exploration 3 for the natural base exponential function  $y = e^{-x}$ . Then compare the graph of  $y = e^x$  to the graph of  $y = e^{-x}$ .
- The natural base  $e$  is used in a wide variety of real-life applications. Use the Internet or some other reference to research some of the real-life applications of  $e$ .

# 5.1 Lesson

## Core Vocabulary

natural base  $e$ , p. 252

### Previous

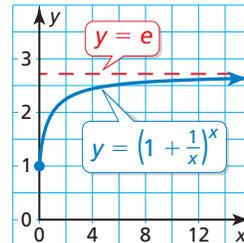
irrational number  
 properties of exponents  
 exponential growth function  
 exponential decay function  
 asymptote  
 percent increase  
 percent decrease  
 compound interest

## What You Will Learn

- ▶ Define and use the natural base  $e$ .
- ▶ Graph natural base functions.
- ▶ Solve real-life problems.

## The Natural Base $e$

The history of mathematics is marked by the discovery of special numbers, such as  $\pi$  and  $i$ . Another special number is denoted by the letter  $e$ . The number is called the **natural base  $e$** . The expression  $\left(1 + \frac{1}{x}\right)^x$  approaches  $e$  as  $x$  increases, as shown in the graph and table.



$x$	$10^1$	$10^2$	$10^3$	$10^4$	$10^5$	$10^6$
$\left(1 + \frac{1}{x}\right)^x$	2.59374	2.70481	2.71692	2.71815	2.71827	2.71828

## Core Concept

### The Natural Base $e$

The natural base  $e$  is irrational. It is defined as follows:

As  $x$  approaches  $+\infty$ ,  $\left(1 + \frac{1}{x}\right)^x$  approaches  $e \approx 2.71828182846$ .

### EXAMPLE 1

### Simplifying Natural Base Expressions

Simplify each expression.

- a.  $e^3 \cdot e^6$                       b.  $\frac{16e^5}{4e^4}$                       c.  $(3e^{-4x})^2$

### SOLUTION

- a.  $e^3 \cdot e^6 = e^{3+6}$   
 $= e^9$
- b.  $\frac{16e^5}{4e^4} = 4e^{5-4}$   
 $= 4e$
- c.  $(3e^{-4x})^2 = 3^2(e^{-4x})^2$   
 $= 9e^{-8x}$   
 $= \frac{9}{e^{8x}}$

### Check

You can use a calculator to check the equivalence of numerical expressions involving  $e$ .

```
e^(3)*e^(6)
8103.083928
e^(9)
8103.083928
```

## Monitoring Progress



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Simplify the expression.

1.  $e^7 \cdot e^4$                       2.  $\frac{24e^8}{8e^5}$                       3.  $(10e^{-3x})^3$

# Graphing Natural Base Functions

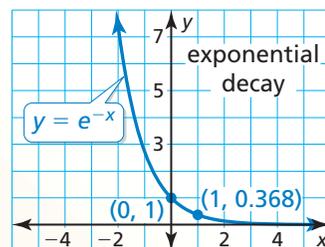
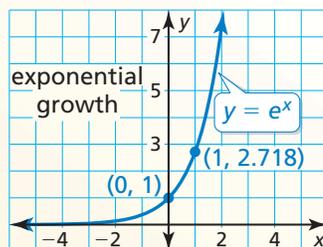
## Core Concept

### Natural Base Functions

A function of the form  $y = ae^{rx}$  is called a *natural base exponential function*.

- When  $a > 0$  and  $r > 0$ , the function is an exponential growth function.
- When  $a > 0$  and  $r < 0$ , the function is an exponential decay function.

The graphs of the basic functions  $y = e^x$  and  $y = e^{-x}$  are shown.



### REMEMBER

An *asymptote* is a line that a graph approaches more and more closely. In the graphs of these natural base exponential functions, the line  $y = 0$  is an asymptote.

### EXAMPLE 2

### Graphing Natural Base Functions

Tell whether each function represents *exponential growth* or *exponential decay*. Then graph the function.

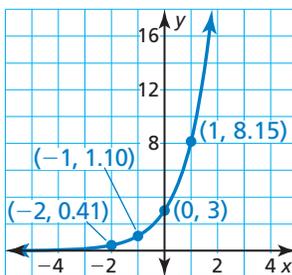
a.  $y = 3e^x$

b.  $f(x) = e^{-0.5x}$

### SOLUTION

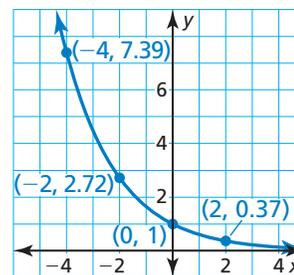
a. Because  $a = 3$  is positive and  $r = 1$  is positive, the function is an exponential growth function. Use a table to graph the function.

x	-2	-1	0	1
y	0.41	1.10	3	8.15



b. Because  $a = 1$  is positive and  $r = -0.5$  is negative, the function is an exponential decay function. Use a table to graph the function.

x	-4	-2	0	2
y	7.39	2.72	1	0.37



### LOOKING FOR STRUCTURE

You can rewrite natural base exponential functions to find percent rates of change. In Example 2(b),

$$\begin{aligned} f(x) &= e^{-0.5x} \\ &= (e^{-0.5})^x \\ &\approx (0.6065)^x \\ &= (1 - 0.3935)^x. \end{aligned}$$

So, the percent decrease is about 39.35%.

### Monitoring Progress



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Tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function.

4.  $y = \frac{1}{2}e^{-x}$

5.  $y = 4e^{-x}$

6.  $f(x) = 2e^{2x}$

## Solving Real-Life Problems

You have learned that the balance of an account earning compound interest is given by  $A = P\left(1 + \frac{r}{n}\right)^{nt}$ . As the frequency  $n$  of compounding approaches positive infinity, the compound interest formula approximates the following formula.

### Core Concept

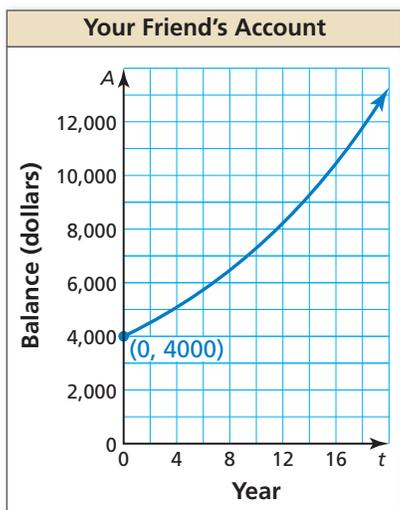
#### Continuously Compounded Interest

When interest is compounded *continuously*, the amount  $A$  in an account after  $t$  years is given by the formula

$$A = Pe^{rt}$$

where  $P$  is the principal and  $r$  is the annual interest rate expressed as a decimal.

### EXAMPLE 3 Modeling with Mathematics



You and your friend each have accounts that earn annual interest compounded continuously. The balance  $A$  (in dollars) of your account after  $t$  years can be modeled by  $A = 4500e^{0.04t}$ . The graph shows the balance of your friend's account over time. Which account has a greater principal? Which has a greater balance after 10 years?

#### SOLUTION

- Understand the Problem** You are given a graph and an equation that represent account balances. You are asked to identify the account with the greater principal and the account with the greater balance after 10 years.
- Make a Plan** Use the equation to find your principal and account balance after 10 years. Then compare these values to the graph of your friend's account.
- Solve the Problem** The equation  $A = 4500e^{0.04t}$  is of the form  $A = Pe^{rt}$ , where  $P = 4500$ . So, your principal is \$4500. Your balance  $A$  when  $t = 10$  is

$$A = 4500e^{0.04(10)} = \$6713.21.$$

Because the graph passes through  $(0, 4000)$ , your friend's principal is \$4000. The graph also shows that the balance is about \$7250 when  $t = 10$ .

- So, your account has a greater principal, but your friend's account has a greater balance after 10 years.

- Look Back** Because your friend's account has a lesser principal but a greater balance after 10 years, the average rate of change from  $t = 0$  to  $t = 10$  should be greater for your friend's account than for your account.

#### MAKING CONJECTURES

You can also use this reasoning to conclude that your friend's account has a greater annual interest rate than your account.

$$\text{Your account: } \frac{A(10) - A(0)}{10 - 0} = \frac{6713.21 - 4500}{10} = 221.321$$

$$\text{Your friend's account: } \frac{A(10) - A(0)}{10 - 0} \approx \frac{7250 - 4000}{10} = 325 \quad \checkmark$$

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- You deposit \$4250 in an account that earns 5% annual interest compounded continuously. Compare the balance after 10 years with the accounts in Example 3.

# 5.1 Exercises

## Vocabulary and Core Concept Check

- VOCABULARY** What is the Euler number?
- WRITING** Tell whether the function  $f(x) = \frac{1}{3}e^{4x}$  represents exponential growth or exponential decay. Explain.

## Monitoring Progress and Modeling with Mathematics

In Exercises 3–12, simplify the expression.  
(See Example 1.)

- |                                   |                                   |
|-----------------------------------|-----------------------------------|
| 3. $e^3 \cdot e^5$                | 4. $e^{-4} \cdot e^6$             |
| 5. $\frac{11e^9}{22e^{10}}$       | 6. $\frac{27e^7}{3e^4}$           |
| 7. $(5e^{7x})^4$                  | 8. $(4e^{-2x})^3$                 |
| 9. $\sqrt{9e^{6x}}$               | 10. $\sqrt[3]{8e^{12x}}$          |
| 11. $e^x \cdot e^{-6x} \cdot e^8$ | 12. $e^x \cdot e^4 \cdot e^{x+3}$ |

**ERROR ANALYSIS** In Exercises 13 and 14, describe and correct the error in simplifying the expression.

13.   $(4e^{3x})^2 = 4e^{(3x)(2)}$   
 $= 4e^{6x}$

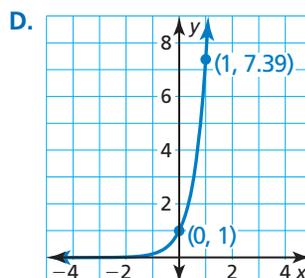
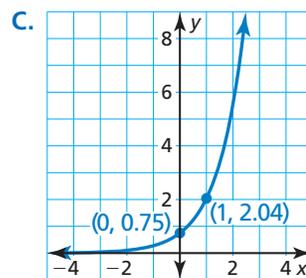
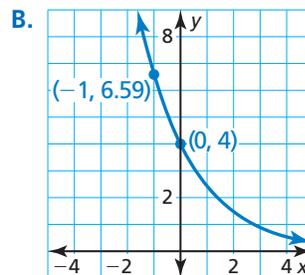
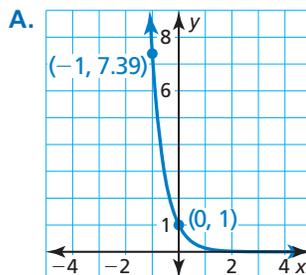
14.   $\frac{e^{5x}}{e^{-2x}} = e^{5x-2x}$   
 $= e^{3x}$

In Exercises 15–22, tell whether the function represents exponential growth or exponential decay. Then graph the function. (See Example 2.)

- |                         |                       |
|-------------------------|-----------------------|
| 15. $y = e^{3x}$        | 16. $y = e^{-2x}$     |
| 17. $y = 2e^{-x}$       | 18. $y = 3e^{2x}$     |
| 19. $y = 0.5e^x$        | 20. $y = 0.25e^{-3x}$ |
| 21. $y = 0.4e^{-0.25x}$ | 22. $y = 0.6e^{0.5x}$ |

**ANALYZING EQUATIONS** In Exercises 23–26, match the function with its graph. Explain your reasoning.

- |                      |                   |
|----------------------|-------------------|
| 23. $y = e^{2x}$     | 24. $y = e^{-2x}$ |
| 25. $y = 4e^{-0.5x}$ | 26. $y = 0.75e^x$ |



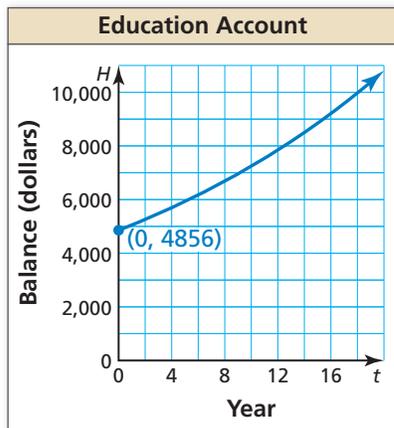
**USING STRUCTURE** In Exercises 27–30, use the properties of exponents to rewrite the function in the form  $y = a(1+r)^t$  or  $y = a(1-r)^t$ . Then find the percent rate of change.

- |                      |                       |
|----------------------|-----------------------|
| 27. $y = e^{-0.25t}$ | 28. $y = e^{-0.75t}$  |
| 29. $y = 2e^{0.4t}$  | 30. $y = 0.5e^{0.8t}$ |

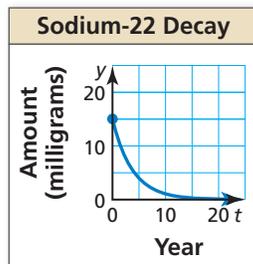
**USING TOOLS** In Exercises 31–34, use a table of values or a graphing calculator to graph the function. Then identify the domain and range.

- |                    |                    |
|--------------------|--------------------|
| 31. $y = e^{x-2}$  | 32. $y = e^{x+1}$  |
| 33. $y = 2e^x + 1$ | 34. $y = 3e^x - 5$ |

- 35. MODELING WITH MATHEMATICS** Investment accounts for a house and education earn annual interest compounded continuously. The balance  $H$  (in dollars) of the house fund after  $t$  years can be modeled by  $H = 3224e^{0.05t}$ . The graph shows the balance in the education fund over time. Which account has the greater principal? Which account has a greater balance after 10 years? (See Example 3.)



- 36. MODELING WITH MATHEMATICS** Tritium is an isotope of hydrogen that can be used to illuminate watches. Tritium and sodium-22 decay over time. In a sample of tritium, the amount  $y$  (in milligrams) remaining after  $t$  years is given by  $y = 10e^{-0.0562t}$ . The graph shows the amount of sodium-22 in a sample over time. Which sample started with a greater amount? Which has a greater amount after 10 years?



- 37. OPEN-ENDED** Find values of  $a$ ,  $b$ ,  $r$ , and  $q$  such that  $f(x) = ae^{rx}$  and  $g(x) = be^{qx}$  are exponential decay functions, but  $\frac{f(x)}{g(x)}$  represents exponential growth.

- 38. THOUGHT PROVOKING** Explain why  $A = P\left(1 + \frac{r}{n}\right)^{nt}$  approximates  $A = Pe^{rt}$  as  $n$  approaches positive infinity.

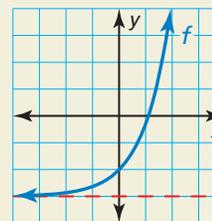
- 39. WRITING** Can the natural base  $e$  be written as a ratio of two integers? Explain.

- 40. MAKING AN ARGUMENT** Your friend evaluates  $f(x) = e^{-x}$  when  $x = 1000$  and concludes that the graph of  $y = f(x)$  has an  $x$ -intercept at  $(1000, 0)$ . Is your friend correct? Explain your reasoning.

- 41. DRAWING CONCLUSIONS** You invest \$2500 in an account to save for college. Account 1 pays 6% annual interest compounded quarterly. Account 2 pays 4% annual interest compounded continuously. Which account should you choose to obtain the greater amount in 10 years? Justify your answer.

- 42. HOW DO YOU SEE IT?** Use the graph to complete each statement.

- a.  $f(x)$  approaches \_\_\_\_\_ as  $x$  approaches  $+\infty$ .  
 b.  $f(x)$  approaches \_\_\_\_\_ as  $x$  approaches  $-\infty$ .



- 43. PROBLEM SOLVING** The growth of *Mycobacterium tuberculosis* bacteria can be modeled by the function  $N(t) = ae^{0.166t}$ , where  $N$  is the number of cells after  $t$  hours and  $a$  is the number of cells when  $t = 0$ .

- a. At 1:00 P.M., there are 30 *M. tuberculosis* bacteria in a sample. Write a function that gives the number of bacteria after 1:00 P.M.  
 b. Use a graphing calculator to graph the function in part (a).  
 c. Describe how to find the number of cells in the sample at 3:45 P.M.

## Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write the number in scientific notation. (Skills Review Handbook)

44. 0.006

45. 5000

46. 26,000,000

47. 0.000000047

Find the inverse of the function. Then graph the function and its inverse. (Section 4.6)

48.  $y = 3x + 5$

49.  $y = x^2 - 1, x \leq 0$

50.  $y = \sqrt{x + 6}$

51.  $y = x^3 - 2$