3.5 Solving Polynomial Equations

Essential Question How can you determine whether a polynomial equation has a repeated solution?

EXPLORATION 1 Cubic Equations and Repeated Solutions

Work with a partner. Some cubic equations have three distinct solutions. Others have repeated solutions. Match each cubic polynomial equation with the graph of its related polynomial function. Then solve each equation. For those equations that have repeated solutions, describe the behavior of the related function near the repeated zero using the graph or a table of values.

USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools to explore and deepen your understanding of concepts.

a. $x^3 - 6x^2 + 12x - 8 = 0$

b. $x^3 + 3x^2 + 3x + 1 = 0$

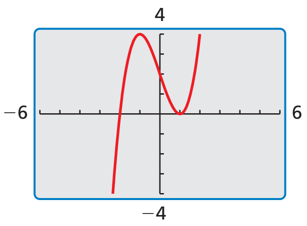
c. $x^3 - 3x + 2 = 0$

d. $x^3 + x^2 - 2x = 0$

e. $x^3 - 3x - 2 = 0$

f. $x^3 - 3x^2 + 2x = 0$

A.



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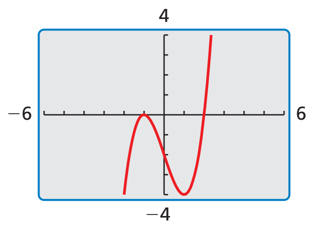

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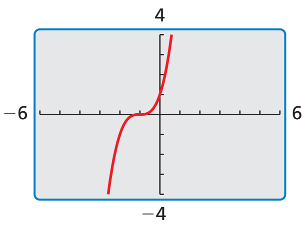

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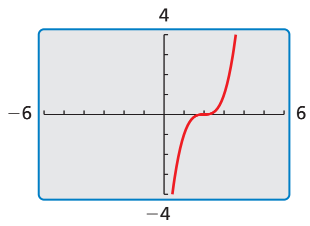

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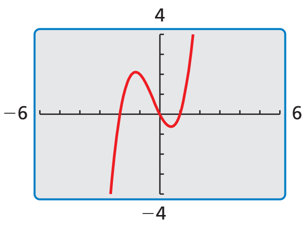

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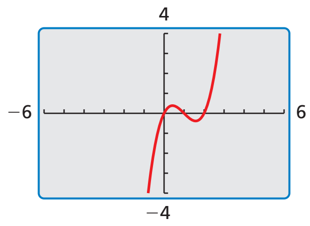


EXPLORATION 2 Quartic Equations and Repeated Solutions

Work with a partner. Determine whether each quartic equation has repeated solutions using the graph of the related quartic function or a table of values. Explain your reasoning. Then solve each equation.

a. $x^4 - 4x^3 + 5x^2 - 2x = 0$

b. $x^4 - 2x^3 - x^2 + 2x = 0$

c. $x^4 - 4x^3 + 4x^2 = 0$

d. $x^4 + 3x^3 = 0$

Communicate Your Answer

3. How can you determine whether a polynomial equation has a repeated solution?

4. Write a cubic or a quartic polynomial equation that is different from the equations in Explorations 1 and 2 and has a repeated solution.

3.5 Lesson

Core Vocabulary

repeated solution, *p. 146*

Previous
roots of an equation
real numbers
conjugates

What You Will Learn

- Find solutions of polynomial equations and zeros of polynomial functions.
- Use the Rational Root Theorem.
- Use the Irrational Conjugates Theorem.

Finding Solutions and Zeros

You have used the Zero-Product Property to solve factorable quadratic equations. You can extend this technique to solve some higher-degree polynomial equations.

EXAMPLE 1 Solving a Polynomial Equation by Factoring

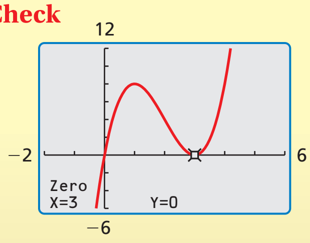
Solve $2x^3 - 12x^2 + 18x = 0$.

SOLUTION

$2x^3 - 12x^2 + 18x = 0$	Write the equation.
$2x(x^2 - 6x + 9) = 0$	Factor common monomial.
$2x(x - 3)^2 = 0$	Perfect Square Trinomial Pattern
$2x = 0$ or $(x - 3)^2 = 0$	Zero-Product Property
$x = 0$ or $x = 3$	Solve for x.

▶ The solutions, or roots, are $x = 0$ and $x = 3$.

Check



In Example 1, the factor $x - 3$ appears more than once. This creates a **repeated solution** of $x = 3$. Note that the graph of the related function touches the x-axis (but does not cross the x-axis) at the repeated zero $x = 3$, and crosses the x-axis at the zero $x = 0$. This concept can be generalized as follows.

- When a factor $x - k$ of $f(x)$ is raised to an odd power, the graph of f *crosses* the x-axis at $x = k$.
- When a factor $x - k$ of $f(x)$ is raised to an even power, the graph of f *touches* the x-axis (but does not cross the x-axis) at $x = k$.

STUDY TIP

Because the factor $x - 3$ appears twice, the root $x = 3$ has a *multiplicity* of 2.

EXAMPLE 2 Finding Zeros of a Polynomial Function

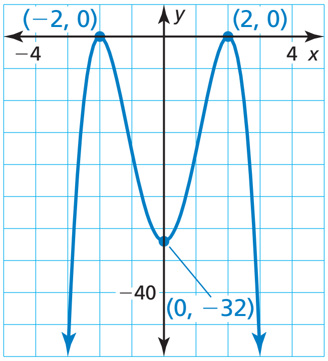
Find the zeros of $f(x) = -2x^4 + 16x^2 - 32$. Then sketch a graph of the function.

SOLUTION

$0 = -2x^4 + 16x^2 - 32$	Set $f(x)$ equal to 0.
$0 = -2(x^4 - 8x^2 + 16)$	Factor out -2.
$0 = -2(x^2 - 4)(x^2 - 4)$	Factor trinomial in quadratic form.
$0 = -2(x + 2)(x - 2)(x + 2)(x - 2)$	Difference of Two Squares Pattern
$0 = -2(x + 2)^2(x - 2)^2$	Rewrite using exponents.

Because both factors $x + 2$ and $x - 2$ are raised to an even power, the graph of f touches the x-axis at the zeros $x = -2$ and $x = 2$.

By analyzing the original function, you can determine that the y-intercept is -32. Because the degree is even and the leading coefficient is negative, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$. Use these characteristics to sketch a graph of the function.



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Solve the equation.

1. $4x^4 - 40x^2 + 36 = 0$

2. $2x^5 + 24x = 14x^3$

Find the zeros of the function. Then sketch a graph of the function.

3. $f(x) = 3x^4 - 6x^2 + 3$

4. $f(x) = x^3 + x^2 - 6x$

The Rational Root Theorem

The solutions of the equation $64x^3 + 152x^2 - 62x - 105 = 0$ are $-\frac{5}{2}$, $-\frac{3}{4}$, and $\frac{7}{8}$. Notice that the numerators (5, 3, and 7) of the zeros are factors of the constant term, −105. Also notice that the denominators (2, 4, and 8) are factors of the leading coefficient, 64. These observations are generalized by the *Rational Root Theorem*.

Core Concept

The Rational Root Theorem

If $f(x) = a_n x^n + \cdots + a_1 x + a_0$ has *integer* coefficients, then every rational solution of $f(x) = 0$ has the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

STUDY TIP

Notice that you can use the Rational Root Theorem to list possible zeros of polynomial functions.

The Rational Root Theorem can be a starting point for finding solutions of polynomial equations. However, the theorem lists only *possible* solutions. In order to find the *actual* solutions, you must test values from the list of possible solutions.

EXAMPLE 3 Using the Rational Root Theorem

Find all real solutions of $x^3 - 8x^2 + 11x + 20 = 0$.

SOLUTION

ANOTHER WAY

You can use direct substitution to test possible solutions, but synthetic division helps you identify other factors of the polynomial.

The polynomial $f(x) = x^3 - 8x^2 + 11x + 20$ is not easily factorable. Begin by using the Rational Root Theorem.

Step 1 List the possible rational solutions. The leading coefficient of $f(x)$ is 1 and the constant term is 20. So, the possible rational solutions of $f(x) = 0$ are

$$x = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{4}{1}, \pm\frac{5}{1}, \pm\frac{10}{1}, \pm\frac{20}{1}.$$

Step 2 Test possible solutions using synthetic division until a solution is found.

Test $x = 1$:

1	1	−8	11	20
		1	−7	4
	1	−7	4	24

$f(1) \neq 0$, so $x - 1$ is not a factor of $f(x)$.

Test $x = -1$:

−1	1	−8	11	20
		−1	9	−20
	1	−9	20	0

$f(-1) = 0$, so $x + 1$ is a factor of $f(x)$.

Step 3 Factor completely using the result of the synthetic division.

$(x + 1)(x^2 - 9x + 20) = 0$ Write as a product of factors.

$(x + 1)(x - 4)(x - 5) = 0$ Factor the trinomial.

So, the solutions are $x = -1$, $x = 4$, and $x = 5$.

In Example 3, the leading coefficient of the polynomial is 1. When the leading coefficient is not 1, the list of possible rational solutions or zeros can increase dramatically. In such cases, the search can be shortened by using a graph.

EXAMPLE 4 Finding Zeros of a Polynomial Function

Find all real zeros of $f(x) = 10x^4 - 11x^3 - 42x^2 + 7x + 12$.

SOLUTION

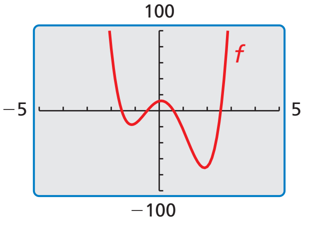
Step 1 List the possible rational zeros of f: $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}, \pm\frac{12}{1}$,

$\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{4}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}, \pm\frac{1}{10}, \pm\frac{3}{10}$

Step 2 Choose reasonable values from the list above to test using the graph of the function. For f, the values

$$x = -\frac{3}{2},\ x = -\frac{1}{2},\ x = \frac{3}{5}, \text{ and } x = \frac{12}{5}$$

are reasonable based on the graph shown at the right.



Step 3 Test the values using synthetic division until a zero is found.

$$\begin{array}{r|rrrrr} -\frac{3}{2} & 10 & -11 & -42 & 7 & 12 \\ & & -15 & 39 & \frac{9}{2} & -\frac{69}{4} \\ \hline & 10 & -26 & -3 & \frac{23}{2} & -\frac{21}{4} \end{array} \qquad \begin{array}{r|rrrrr} -\frac{1}{2} & 10 & -11 & -42 & 7 & 12 \\ & & -5 & 8 & 17 & -12 \\ \hline & 10 & -16 & -34 & 24 & 0 \end{array}$$

$-\frac{1}{2}$ is a zero.

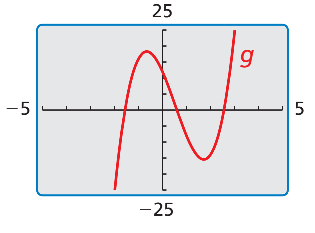
Step 4 Factor out a binomial using the result of the synthetic division.

$f(x) = \left(x + \frac{1}{2}\right)(10x^3 - 16x^2 - 34x + 24)$ Write as a product of factors.

$= \left(x + \frac{1}{2}\right)(2)(5x^3 - 8x^2 - 17x + 12)$ Factor 2 out of the second factor.

$= (2x + 1)(5x^3 - 8x^2 - 17x + 12)$ Multiply the first factor by 2.

Step 5 Repeat the steps above for $g(x) = 5x^3 - 8x^2 - 17x + 12$. Any zero of g will also be a zero of f. The possible rational zeros of g are:



$$x = \pm1, \pm2, \pm3, \pm4, \pm6, \pm12, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{4}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}$$

The graph of g shows that $\frac{3}{5}$ may be a zero. Synthetic division shows that $\frac{3}{5}$ *is* a zero and $g(x) = \left(x - \frac{3}{5}\right)(5x^2 - 5x - 20) = (5x - 3)(x^2 - x - 4)$. It follows that:

$$f(x) = (2x + 1) \cdot g(x) = (2x + 1)(5x - 3)(x^2 - x - 4)$$

Step 6 Find the remaining zeros of f by solving $x^2 - x - 4 = 0$.

$x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2(1)}$ Substitute 1 for *a*, −1 for *b*, and −4 for *c* in the Quadratic Formula.

$x = \dfrac{1 \pm \sqrt{17}}{2}$ Simplify.

▶ The real zeros of f are $-\frac{1}{2}, \frac{3}{5}, \frac{1 + \sqrt{17}}{2} \approx 2.56$, and $\frac{1 - \sqrt{17}}{2} \approx -1.56$.

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5. Find all real solutions of $x^3 - 5x^2 - 2x + 24 = 0$.

6. Find all real zeros of $f(x) = 3x^4 - 2x^3 - 37x^2 + 24x + 12$.

The Irrational Conjugates Theorem

In Example 4, notice that the irrational zeros are *conjugates* of the form $a + \sqrt{b}$ and $a - \sqrt{b}$. This illustrates the theorem below.

Core Concept

The Irrational Conjugates Theorem

Let f be a polynomial function with rational coefficients, and let a and b be rational numbers such that $\sqrt{b}$ is irrational. If $a + \sqrt{b}$ is a zero of f, then $a - \sqrt{b}$ is also a zero of f.

EXAMPLE 5 Using Zeros to Write a Polynomial Function

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 3 and $2 + \sqrt{5}$.

SOLUTION

Because the coefficients are rational and $2 + \sqrt{5}$ is a zero, $2 - \sqrt{5}$ must also be a zero by the Irrational Conjugates Theorem. Use the three zeros and the Factor Theorem to write $f(x)$ as a product of three factors.

$$f(x) = (x - 3)\left[x - \left(2 + \sqrt{5}\right)\right]\left[x - \left(2 - \sqrt{5}\right)\right] \quad \text{Write } f(x) \text{ in factored form.}$$

$$= (x - 3)\left[(x - 2) - \sqrt{5}\right]\left[(x - 2) + \sqrt{5}\right] \quad \text{Regroup terms.}$$

$$= (x - 3)\left[(x - 2)^2 - 5\right] \quad \text{Multiply.}$$

$$= (x - 3)\left[(x^2 - 4x + 4) - 5\right] \quad \text{Expand binomial.}$$

$$= (x - 3)(x^2 - 4x - 1) \quad \text{Simplify.}$$

$$= x^3 - 4x^2 - x - 3x^2 + 12x + 3 \quad \text{Multiply.}$$

$$= x^3 - 7x^2 + 11x + 3 \quad \text{Combine like terms.}$$

Check

You can check this result by evaluating f at each of its three zeros.

$$f(3) = 3^3 - 7(3)^2 + 11(3) + 3 = 27 - 63 + 33 + 3 = 0 \ \checkmark$$

$$f\left(2 + \sqrt{5}\right) = \left(2 + \sqrt{5}\right)^3 - 7\left(2 + \sqrt{5}\right)^2 + 11\left(2 + \sqrt{5}\right) + 3$$

$$= 38 + 17\sqrt{5} - 63 - 28\sqrt{5} + 22 + 11\sqrt{5} + 3$$

$$= 0 \ \checkmark$$

Because $f\left(2 + \sqrt{5}\right) = 0$, by the Irrational Conjugates Theorem $f\left(2 - \sqrt{5}\right) = 0$. ✓

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7. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 4 and $1 - \sqrt{5}$.

3.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. COMPLETE THE SENTENCE If a polynomial function f has integer coefficients, then every rational solution of $f(x) = 0$ has the form $\frac{p}{q}$, where p is a factor of the ____________ and q is a factor of the ____________.

2. DIFFERENT WORDS, SAME QUESTION Which is different? Find "both" answers.

Find the y-intercept of the graph of $y = x^3 - 2x^2 - x + 2$.	Find the x-intercepts of the graph of $y = x^3 - 2x^2 - x + 2$.
Find all the real solutions of $x^3 - 2x^2 - x + 2 = 0$.	Find the real zeros of $f(x) = x^3 - 2x^2 - x + 2$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–12, solve the equation. *(See Example 1.)*

3. $z^3 - z^2 - 12z = 0$

4. $a^3 - 4a^2 + 4a = 0$

5. $2x^4 - 4x^3 = -2x^2$

6. $v^3 - 2v^2 - 16v = -32$

7. $5w^3 = 50w$

8. $9m^5 = 27m^3$

9. $2c^4 - 6c^3 = 12c^2 - 36c$

10. $p^4 + 40 = 14p^2$

11. $12n^2 + 48n = -n^3 - 64$

12. $y^3 - 27 = 9y^2 - 27y$

In Exercises 13–20, find the zeros of the function. Then sketch a graph of the function. *(See Example 2.)*

13. $h(x) = x^4 + x^3 - 6x^2$

14. $f(x) = x^4 - 18x^2 + 81$

15. $p(x) = x^6 - 11x^5 + 30x^4$

16. $g(x) = -2x^5 + 2x^4 + 40x^3$

17. $g(x) = -4x^4 + 8x^3 + 60x^2$

18. $h(x) = -x^3 - 2x^2 + 15x$

19. $h(x) = -x^3 - x^2 + 9x + 9$

20. $p(x) = x^3 - 5x^2 - 4x + 20$

21. USING EQUATIONS According to the Rational Root Theorem, which is *not* a possible solution of the equation $2x^4 - 5x^3 + 10x^2 - 9 = 0$?

(A) -9 (B) $-\frac{1}{2}$ (C) $\frac{5}{2}$ (D) 3

22. USING EQUATIONS According to the Rational Root Theorem, which is *not* a possible zero of the function $f(x) = 40x^5 - 42x^4 - 107x^3 + 107x^2 + 33x - 36$?

(A) $-\frac{2}{3}$ (B) $-\frac{3}{8}$ (C) $\frac{3}{4}$ (D) $\frac{4}{5}$

ERROR ANALYSIS In Exercises 23 and 24, describe and correct the error in listing the possible rational zeros of the function.

23. ✗

$f(x) = x^3 + 5x^2 - 9x - 45$

Possible rational zeros of f:

1, 3, 5, 9, 15, 45

24. ✗

$f(x) = 3x^3 + 13x^2 - 41x + 8$

Possible rational zeros of f:

$\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{3}{8}$

In Exercises 25–32, find all the real solutions of the equation. *(See Example 3.)*

25. $x^3 + x^2 - 17x + 15 = 0$

26. $x^3 - 2x^2 - 5x + 6 = 0$

27. $x^3 - 10x^2 + 19x + 30 = 0$

28. $x^3 + 4x^2 - 11x - 30 = 0$

29. $x^3 - 6x^2 - 7x + 60 = 0$

30. $x^3 - 16x^2 + 55x + 72 = 0$

31. $2x^3 - 3x^2 - 50x - 24 = 0$

32. $3x^3 + x^2 - 38x + 24 = 0$

In Exercises 33–38, find all the real zeros of the function. *(See Example 4.)*

33. $f(x) = x^3 - 2x^2 - 23x + 60$

34. $g(x) = x^3 - 28x - 48$

35. $h(x) = x^3 + 10x^2 + 31x + 30$

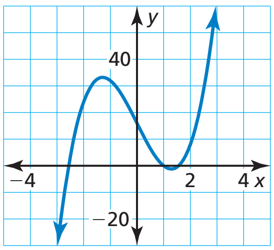
36. $f(x) = x^3 - 14x^2 + 55x - 42$

37. $p(x) = 2x^3 - x^2 - 27x + 36$

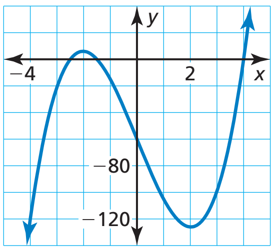
38. $g(x) = 3x^3 - 25x^2 + 58x - 40$

USING TOOLS In Exercises 39 and 40, use the graph to shorten the list of possible rational zeros of the function. Then find all real zeros of the function.

39. $f(x) = 4x^3 - 20x + 16$



40. $f(x) = 4x^3 - 49x - 60$



In Exercises 41–46, write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros. *(See Example 5.)*

41. $-2, 3, 6$

42. $-4, -2, 5$

43. $-2, 1 + \sqrt{7}$

44. $4, 6 - \sqrt{7}$

45. $-6, 0, 3 - \sqrt{5}$

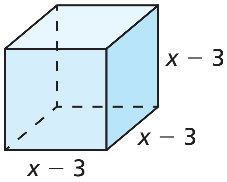
46. $0, 5, -5 + \sqrt{8}$

47. **COMPARING METHODS** Solve the equation $x^3 - 4x^2 - 9x + 36 = 0$ using two different methods. Which method do you prefer? Explain your reasoning.

48. **REASONING** Is it possible for a cubic function to have more than three real zeros? Explain.

49. **PROBLEM SOLVING** At a factory, molten glass is poured into molds to make paperweights. Each mold is a rectangular prism with a height 3 centimeters greater than the length of each side of its square base. Each mold holds 112 cubic centimeters of glass. What are the dimensions of the mold?

50. **MATHEMATICAL CONNECTIONS** The volume of the cube shown is 8 cubic centimeters.



a. Write a polynomial equation that you can use to find the value of x.

b. Identify the possible rational solutions of the equation in part (a).

c. Use synthetic division to find a rational solution of the equation. Show that no other real solutions exist.

d. What are the dimensions of the cube?

51. **PROBLEM SOLVING** Archaeologists discovered a huge hydraulic concrete block at the ruins of Caesarea with a volume of 945 cubic meters. The block is x meters high by $12x - 15$ meters long by $12x - 21$ meters wide. What are the dimensions of the block?

52. **MAKING AN ARGUMENT** Your friend claims that when a polynomial function has a leading coefficient of 1 and the coefficients are all integers, every possible rational zero is an integer. Is your friend correct? Explain your reasoning.

53. **MODELING WITH MATHEMATICS** During a 10-year period, the amount (in millions of dollars) of athletic equipment E sold domestically can be modeled by $E(t) = -20t^3 + 252t^2 - 280t + 21{,}614$, where t is in years.

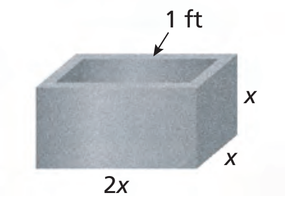
a. Write a polynomial equation to find the year when about \$24,014,000,000 of athletic equipment is sold.

b. List the possible whole-number solutions of the equation in part (a). Consider the domain when making your list of possible solutions.

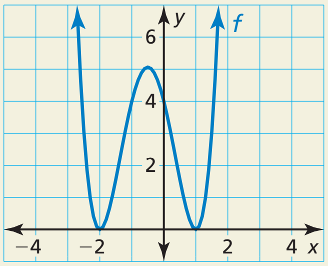
c. Use synthetic division to find when \$24,014,000,000 of athletic equipment is sold.

54. THOUGHT PROVOKING Write a third or fourth degree polynomial function that has zeros at $\pm\frac{3}{4}$. Justify your answer.

55. MODELING WITH MATHEMATICS You are designing a marble basin that will hold a fountain for a city park. The sides and bottom of the basin should be 1 foot thick. Its outer length should be twice its outer width and outer height. What should the outer dimensions of the basin be if it is to hold 36 cubic feet of water?



56. HOW DO YOU SEE IT? Use the information in the graph to answer the questions.



a. What are the real zeros of the function f?

b. Write an equation of the quartic function in factored form.

57. REASONING Determine the value of k for each equation so that the given x-value is a solution.

a. $x^3 - 6x^2 - 7x + k = 0;\ x = 4$

b. $2x^3 + 7x^2 - kx - 18 = 0;\ x = -6$

c. $kx^3 - 35x^2 + 19x + 30 = 0;\ x = 5$

58. WRITING EQUATIONS Write a polynomial function g of least degree that has rational coefficients, a leading coefficient of 1, and the zeros $-2 + \sqrt{7}$ and $3 + \sqrt{2}$.

In Exercises 59–62, solve $f(x) = g(x)$ by graphing and algebraic methods.

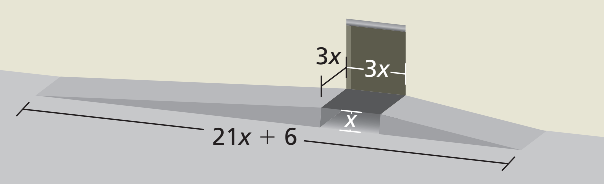
59. $f(x) = x^3 + x^2 - x - 1;\ g(x) = -x + 1$

60. $f(x) = x^4 - 5x^3 + 2x^2 + 8x;\ g(x) = -x^2 + 6x - 8$

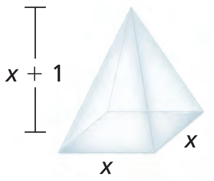
61. $f(x) = x^3 - 4x^2 + 4x;\ g(x) = -2x + 4$

62. $f(x) = x^4 + 2x^3 - 11x^2 - 12x + 36;$
$g(x) = -x^2 - 6x - 9$

63. MODELING WITH MATHEMATICS You are building a pair of ramps for a loading platform. The left ramp is twice as long as the right ramp. If 150 cubic feet of concrete are used to build the ramps, what are the dimensions of each ramp?



64. MODELING WITH MATHEMATICS Some ice sculptures are made by filling a mold and then freezing it. You are making an ice mold for a school dance. It is to be shaped like a pyramid with a height 1 foot greater than the length of each side of its square base. The volume of the ice sculpture is 4 cubic feet. What are the dimensions of the mold?



65. ABSTRACT REASONING Let a_n be the leading coefficient of a polynomial function f and a_0 be the constant term. If a_n has r factors and a_0 has s factors, what is the greatest number of possible rational zeros of f that can be generated by the Rational Zero Theorem? Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient. *(Section 3.1)*

66. $h(x) = -3x^2 + 2x - 9 + \sqrt{4}x^3$

67. $g(x) = 2x^3 - 7x^2 - 3x^{-1} + x$

68. $f(x) = \frac{1}{3}x^2 + 2x^3 - 4x^4 - \sqrt{3}$

69. $p(x) = 2x - 5x^3 + 9x^2 + \sqrt[4]{x} + 1$

Find the zeros of the function. *(Skills Review Handbook)*

70. $f(x) = 7x^2 + 42$

71. $g(x) = 9x^2 + 81$

72. $h(x) = 5x^2 + 40$

73. $f(x) = 8x^2 - 1$