Essential Question   What are some common characteristics of the graphs of cubic and quartic polynomial functions?

A *polynomial function* of the form
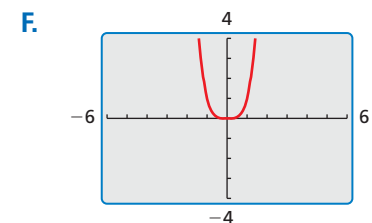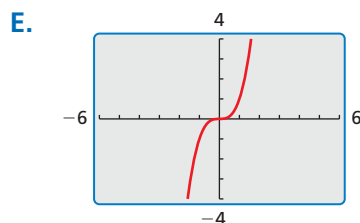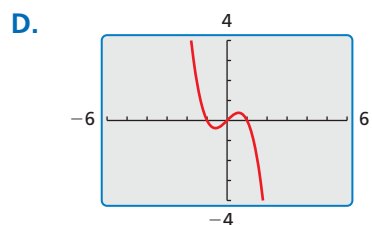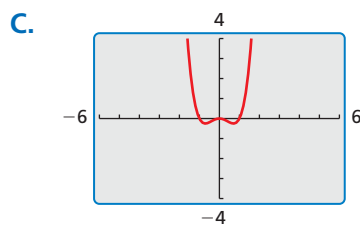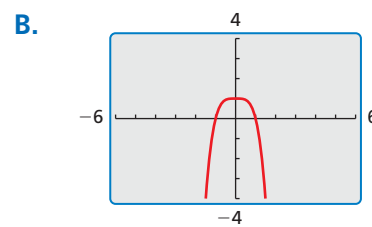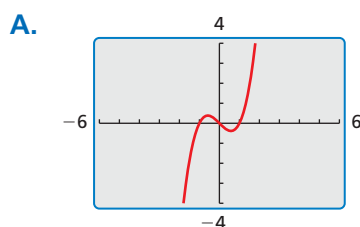
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, is *cubic* when $n = 3$ and *quartic* when $n = 4$.

EXPLORATION 1   Identifying Graphs of Polynomial Functions

Work with a partner. Match each polynomial function with its graph. Explain your reasoning. Use a graphing calculator to verify your answers.

a. $f(x) = x^3 - x$ **b.** $f(x) = -x^3 + x$ **c.** $f(x) = -x^4 + 1$

d. $f(x) = x^4$ **e.** $f(x) = x^3$ **f.** $f(x) = x^4 - x^2$

A.

B.

C.

D.

E.

F.

EXPLORATION 2   Identifying *x*-Intercepts of Polynomial Graphs

Work with a partner. Each of the polynomial graphs in Exploration 1 has x-intercept(s) of $-1, 0,$ or 1. Identify the x-intercept(s) of each graph. Explain how you can verify your answers.

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to justify your conclusions and communicate them to others.

Communicate Your Answer

3. What are some common characteristics of the graphs of cubic and quartic polynomial functions?

4. Determine whether each statement is *true* or *false*. Justify your answer.

    a. When the graph of a cubic polynomial function rises to the left, it falls to the right.

    b. When the graph of a quartic polynomial function falls to the left, it rises to the right.

What You Will Learn

▶ Identify polynomial functions.

▶ Graph polynomial functions using tables and end behavior.

Polynomial Functions

Recall that a monomial is a number, a variable, or the product of a number and one or more variables with whole number exponents. A **polynomial** is a monomial or a sum of monomials. A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, the exponents are all whole numbers, and the coefficients are all real numbers. For this function, a_n is the leading coefficient, n is the degree, and a_0 is the constant term. A polynomial function is in *standard form* when its terms are written in descending order of exponents from left to right.

You are already familiar with some types of polynomial functions, such as linear and quadratic. Here is a summary of common types of polynomial functions.

Common Polynomial Functions			
Degree	**Type**	**Standard Form**	**Example**
0	Constant	$f(x) = a_0$	$f(x) = -14$
1	Linear	$f(x) = a_1 x + a_0$	$f(x) = 5x - 7$
2	Quadratic	$f(x) = a_2 x^2 + a_1 x + a_0$	$f(x) = 2x^2 + x - 9$
3	Cubic	$f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$	$f(x) = x^3 - x^2 + 3x$
4	Quartic	$f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$	$f(x) = x^4 + 2x - 1$

EXAMPLE 1 Identifying Polynomial Functions

Decide whether each function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

a. $f(x) = -2x^3 + 5x + 8$
b. $g(x) = -0.8x^3 + \sqrt{2}x^4 - 12$
c. $h(x) = -x^2 + 7x^{-1} + 4x$
d. $k(x) = x^2 + 3^x$

SOLUTION

a. The function is a polynomial function that is already written in standard form. It has degree 3 (cubic) and a leading coefficient of -2.

b. The function is a polynomial function written as $g(x) = \sqrt{2}x^4 - 0.8x^3 - 12$ in standard form. It has degree 4 (quartic) and a leading coefficient of $\sqrt{2}$.

c. The function is not a polynomial function because the term $7x^{-1}$ has an exponent that is not a whole number.

d. The function is not a polynomial function because the term 3^x does not have a variable base and an exponent that is a whole number.

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Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

1. $f(x) = 7 - 1.6x^2 - 5x$ **2.** $p(x) = x + 2x^{-2} + 9.5$ **3.** $q(x) = x^3 - 6x + 3x^4$

EXAMPLE 2 **Evaluating a Polynomial Function**

Evaluate $f(x) = 2x^4 - 8x^2 + 5x - 7$ when $x = 3$.

SOLUTION

$f(x) = 2x^4 - 8x^2 + 5x - 7$	Write original equation.
$f(3) = 2(3)^4 - 8(3)^2 + 5(3) - 7$	Substitute 3 for *x*.
$= 162 - 72 + 15 - 7$	Evaluate powers and multiply.
$= 98$	Simplify.

The **end behavior** of a function's graph is the behavior of the graph as x approaches positive infinity $(+\infty)$ or negative infinity $(-\infty)$. For the graph of a polynomial function, the end behavior is determined by the function's degree and the sign of its leading coefficient.
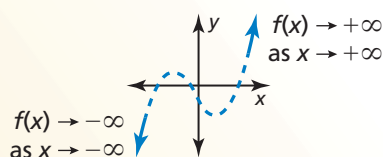
Core Concept

READING

The expression "$x \to +\infty$" is read as "x approaches positive infinity."
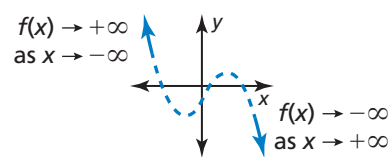
End Behavior of Polynomial Functions
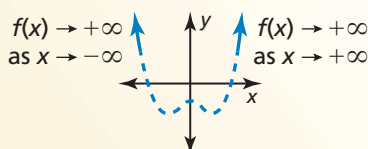
Degree: odd
Leading coefficient: positive

$f(x) \to +\infty$
as $x \to +\infty$
$f(x) \to -\infty$
as $x \to -\infty$

Degree: odd
Leading coefficient: negative

$f(x) \to +\infty$
as $x \to -\infty$
$f(x) \to -\infty$
as $x \to +\infty$

Degree: even
Leading coefficient: positive

$f(x) \to +\infty$
as $x \to -\infty$
$f(x) \to +\infty$
as $x \to +\infty$

Degree: even
Leading coefficient: negative

$f(x) \to -\infty$
as $x \to -\infty$
$f(x) \to -\infty$
as $x \to +\infty$
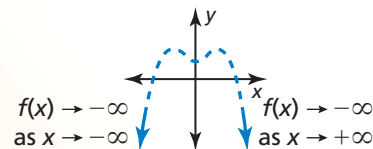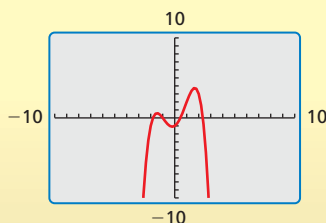
EXAMPLE 3 **Describing End Behavior**

Describe the end behavior of the graph of $f(x) = -0.5x^4 + 2.5x^2 + x - 1$.

Check

SOLUTION

The function has degree 4 and leading coefficient -0.5. Because the degree is even and the leading coefficient is negative, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$. Check this by graphing the function on a graphing calculator, as shown.

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Evaluate the function for the given value of *x*.

4. $f(x) = -x^3 + 3x^2 + 9$; $x = 4$

5. $f(x) = 3x^5 - x^4 - 6x + 10$; $x = -2$

6. Describe the end behavior of the graph of $f(x) = 0.25x^3 - x^2 - 1$.

Graphing Polynomial Functions

To graph a polynomial function, first plot points to determine the shape of the graph's middle portion. Then connect the points with a smooth continuous curve and use what you know about end behavior to sketch the graph.

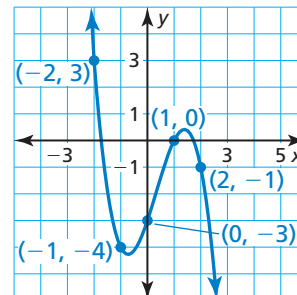### EXAMPLE 4 Graphing Polynomial Functions

Graph (a) $f(x) = -x^3 + x^2 + 3x - 3$ and (b) $f(x) = x^4 - x^3 - 4x^2 + 4$.

SOLUTION

a. To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.

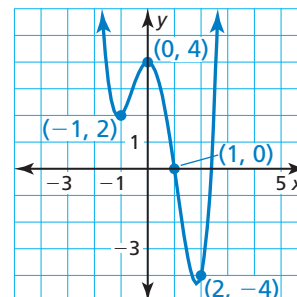| x	−2	−1	0	1	2
f(x)	3	−4	−3	0	−1

The degree is odd and the leading coefficient is negative. So, $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$.

b. To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.

x	−2	−1	0	1	2
f(x)	12	2	4	0	−4

The degree is even and the leading coefficient is positive. So, $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.
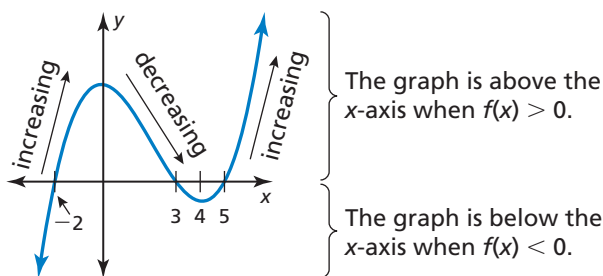

EXAMPLE 5 Sketching a Graph

Sketch a graph of the polynomial function f having these characteristics.

- f is increasing when $x < 0$ and $x > 4$.
- f is decreasing when $0 < x < 4$.
- $f(x) > 0$ when $-2 < x < 3$ and $x > 5$.
- $f(x) < 0$ when $x < -2$ and $3 < x < 5$.

Use the graph to describe the degree and leading coefficient of f.

SOLUTION

The graph is above the x-axis when $f(x) > 0$.

The graph is below the x-axis when $f(x) < 0$.

▶ From the graph, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.
So, the degree is odd and the leading coefficient is positive.

EXAMPLE 6 **Solving a Real-Life Problem**

The estimated number V (in thousands) of electric vehicles in use in the United States can be modeled by the polynomial function

$$V(t) = 0.151280t^3 - 3.28234t^2 + 23.7565t - 2.041$$

where t represents the year, with $t = 1$ corresponding to 2001.

a. Use a graphing calculator to graph the function for the interval $1 \le t \le 10$. Describe the behavior of the graph on this interval.

b. What was the average rate of change in the number of electric vehicles in use from 2001 to 2010?

c. Do you think this model can be used for years before 2001 or after 2010? Explain your reasoning.

SOLUTION

a. Using a graphing calculator and a viewing window of $1 \le x \le 10$ and $0 \le y \le 65$, you obtain the graph shown.
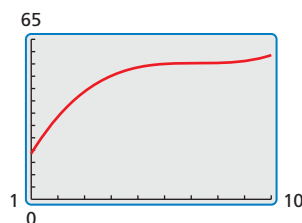
▶ From 2001 to 2004, the numbers of electric vehicles in use increased. Around 2005, the growth in the numbers in use slowed and started to level off. Then the numbers in use started to increase again in 2009 and 2010.

b. The years 2001 and 2010 correspond to $t = 1$ and $t = 10$.

Average rate of change over $1 \le t \le 10$:

$$\frac{V(10) - V(1)}{10 - 1} = \frac{58.57 - 18.58444}{9} \approx 4.443$$

▶ The average rate of change from 2001 to 2010 is about 4.4 thousand electric vehicles per year.

c. Because the degree is odd and the leading coefficient is positive, $V(t) \to -\infty$ as $t \to -\infty$ and $V(t) \to +\infty$ as $t \to +\infty$. The end behavior indicates that the model has unlimited growth as t increases. While the model may be valid for a few years after 2010, in the long run, unlimited growth is not reasonable. Notice in 2000 that $V(0) = -2.041$. Because negative values of $V(t)$ do not make sense given the context (electric vehicles in use), the model should not be used for years before 2001.

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Graph the polynomial function.

7. $f(x) = x^4 + x^2 - 3$

8. $f(x) = 4 - x^3$

9. $f(x) = x^3 - x^2 + x - 1$

10. Sketch a graph of the polynomial function f having these characteristics.

- f is decreasing when $x < -1.5$ and $x > 2.5$; f is increasing when $-1.5 < x < 2.5$.
- $f(x) > 0$ when $x < -3$ and $1 < x < 4$; $f(x) < 0$ when $-3 < x < 1$ and $x > 4$.

Use the graph to describe the degree and leading coefficient of f.

11. WHAT IF? Repeat Example 6 using the alternative model for electric vehicles of

$$V(t) = -0.0290900t^4 + 0.791260t^3 - 7.96583t^2 + 36.5561t - 12.025.$$

Vocabulary and Core Concept Check

1. **WRITING** Explain what is meant by the end behavior of a polynomial function.

2. **WHICH ONE DOESN'T BELONG?** Which function does *not* belong with the other three? Explain your reasoning.

   $$f(x) = 7x^5 + 3x^2 - 2x$$

   $$g(x) = 3x^3 - 2x^8 + \frac{3}{4}$$

   $$h(x) = -3x^4 + 5x^{-1} - 3x^2$$

   $$k(x) = \sqrt{3}x + 8x^4 + 2x + 1$$

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient. *(See Example 1.)*

3. $f(x) = -3x + 5x^3 - 6x^2 + 2$

4. $p(x) = \frac{1}{2}x^2 + 3x - 4x^3 + 6x^4 - 1$

5. $f(x) = 9x^4 + 8x^3 - 6x^{-2} + 2x$

6. $g(x) = \sqrt{3} - 12x + 13x^2$

7. $h(x) = \frac{5}{3}x^2 - \sqrt{7}x^4 + 8x^3 - \frac{1}{2} + x$

8. $h(x) = 3x^4 + 2x - \dfrac{5}{x} + 9x^3 - 7$

ERROR ANALYSIS **In Exercises 9 and 10, describe and correct the error in analyzing the function.**

9. $f(x) = 8x^3 - 7x^4 - 9x - 3x^2 + 11$

> ✗ *f* is a polynomial function.
> The degree is 3 and *f* is a cubic function.
> The leading coefficient is 8.

10. $f(x) = 2x^4 + 4x - 9\sqrt{x} + 3x^2 - 8$

> ✗ *f* is a polynomial function.
> The degree is 4 and *f* is a quartic function.
> The leading coefficient is 2.

**In Exercises 11–16, evaluate the function for the given value of *x*.** *(See Example 2.)*

11. $h(x) = -3x^4 + 2x^3 - 12x - 6; x = -2$

12. $f(x) = 7x^4 - 10x^2 + 14x - 26; x = -7$

13. $g(x) = x^6 - 64x^4 + x^2 - 7x - 51; x = 8$

14. $g(x) = -x^3 + 3x^2 + 5x + 1; x = -12$

15. $p(x) = 2x^3 + 4x^2 + 6x + 7; x = \frac{1}{2}$

16. $h(x) = 5x^3 - 3x^2 + 2x + 4; x = -\frac{1}{3}$

In Exercises 17–20, describe the end behavior of the graph of the function. *(See Example 3.)*
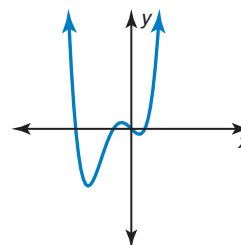
17. $h(x) = -5x^4 + 7x^3 - 6x^2 + 9x + 2$

18. $g(x) = 7x^7 + 12x^5 - 6x^3 - 2x - 18$

19. $f(x) = -2x^4 + 12x^8 + 17 + 15x^2$

20. $f(x) = 11 - 18x^2 - 5x^5 - 12x^4 - 2x$

In Exercises 21 and 22, describe the degree and leading coefficient of the polynomial function using the graph.

21.
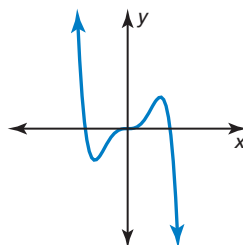
22.

23. USING STRUCTURE Determine whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

$$f(x) = 5x^3x + \tfrac{5}{2}x^3 - 9x^4 + \sqrt{2}x^2 + 4x - 1 - x^{-5}x^5 - 4$$

24. WRITING Let $f(x) = 13$. State the degree, type, and leading coefficient. Describe the end behavior of the function. Explain your reasoning.

In Exercises 25–32, graph the polynomial function. *(See Example 4.)*

25. $p(x) = 3 - x^4$

26. $g(x) = x^3 + x + 3$

27. $f(x) = 4x - 9 - x^3$

28. $p(x) = x^5 - 3x^3 + 2$

29. $h(x) = x^4 - 2x^3 + 3x$

30. $h(x) = 5 + 3x^2 - x^4$

31. $g(x) = x^5 - 3x^4 + 2x - 4$

32. $p(x) = x^6 - 2x^5 - 2x^3 + x + 5$

ANALYZING RELATIONSHIPS In Exercises 33–36, describe the x-values for which (a) f is increasing or decreasing, (b) f(x) > 0, and (c) f(x) < 0.

33.

34.

35.

36.
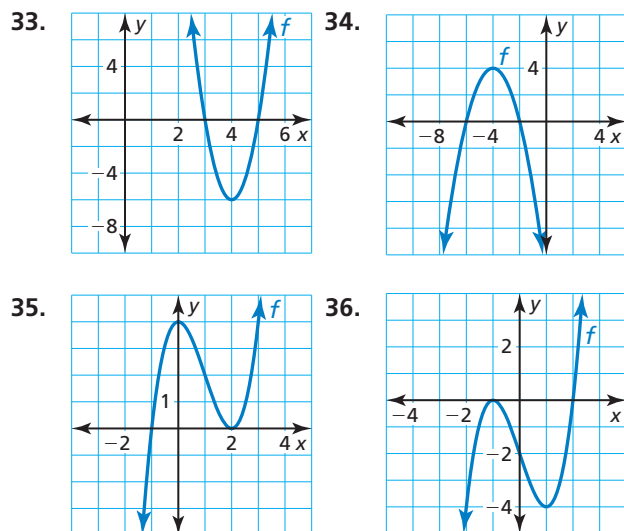
In Exercises 37–40, sketch a graph of the polynomial function f having the given characteristics. Use the graph to describe the degree and leading coefficient of the function f. *(See Example 5.)*

37. • f is increasing when $x > 0.5$; f is decreasing when $x < 0.5$.

• $f(x) > 0$ when $x < -2$ and $x > 3$; $f(x) < 0$ when $-2 < x < 3$.

38. • f is increasing when $-2 < x < 3$; f is decreasing when $x < -2$ and $x > 3$.

• $f(x) > 0$ when $x < -4$ and $1 < x < 5$; $f(x) < 0$ when $-4 < x < 1$ and $x > 5$.

39. • f is increasing when $-2 < x < 0$ and $x > 2$; f is decreasing when $x < -2$ and $0 < x < 2$.

• $f(x) > 0$ when $x < -3$, $-1 < x < 1$, and $x > 3$; $f(x) < 0$ when $-3 < x < -1$ and $1 < x < 3$.

40. • f is increasing when $x < -1$ and $x > 1$; f is decreasing when $-1 < x < 1$.

• $f(x) > 0$ when $-1.5 < x < 0$ and $x > 1.5$; $f(x) < 0$ when $x < -1.5$ and $0 < x < 1.5$.

41. MODELING WITH MATHEMATICS From 1980 to 2007 the number of drive-in theaters in the United States can be modeled by the function

$$d(t) = -0.141t^3 + 9.64t^2 - 232.5t + 2421$$

where $d(t)$ is the number of open theaters and t is the number of years after 1980. *(See Example 6.)*

a. Use a graphing calculator to graph the function for the interval $0 \le t \le 27$. Describe the behavior of the graph on this interval.

b. What is the average rate of change in the number of drive-in movie theaters from 1980 to 1995 and from 1995 to 2007? Interpret the average rates of change.

c. Do you think this model can be used for years before 1980 or after 2007? Explain.

42. PROBLEM SOLVING The weight of an ideal round-cut diamond can be modeled by

$$w = 0.00583d^3 - 0.0125d^2 + 0.022d - 0.01$$

where w is the weight of the diamond (in carats) and d is the diameter (in millimeters). According to the model, what is the weight of a diamond with a diameter of 12 millimeters?

43. ABSTRACT REASONING Suppose $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$. Describe the end behavior of $g(x) = -f(x)$. Justify your answer.

44. THOUGHT PROVOKING Write an even degree polynomial function such that the end behavior of f is given by $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$. Justify your answer by drawing the graph of your function.

45. USING TOOLS In Section 1.2 Exercise 12, the function $V = 4r^3(\pi + 4)$ represents the volume V of the tank. Use a graphing calculator to graph the function. Estimate the percent change in volume when r increases from 1 foot to 1 foot 1 inch. Is the percent change greater than you expected? Explain.

46. MAKING AN ARGUMENT Your friend uses the table to speculate that the function f is an even degree polynomial and the function g is an odd degree polynomial. Is your friend correct? Explain your reasoning.

x	f(x)	g(x)
−8	4113	497
−2	21	5
0	1	1
2	13	−3
8	4081	−495

47. DRAWING CONCLUSIONS The graph of a function is symmetric with respect to the y-axis if for each point (a, b) on the graph, $(-a, b)$ is also a point on the graph. The graph of a function is symmetric with respect to the origin if for each point (a, b) on the graph, $(-a, -b)$ is also a point on the graph.

a. Use a graphing calculator to graph the function $y = x^n$ when $n = 1, 2, 3, 4, 5,$ and 6. In each case, identify the symmetry of the graph.

b. Predict what symmetry the graphs of $y = x^{10}$ and $y = x^{11}$ each have. Explain your reasoning and then confirm your predictions by graphing.

48. HOW DO YOU SEE IT? The graph of a polynomial function is shown.
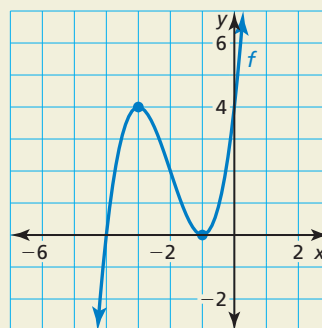
a. Describe the degree and leading coefficient of f.

b. Describe the intervals where the function is increasing and decreasing.

c. What is the constant term of the polynomial function?

49. REASONING A cubic polynomial function f has a leading coefficient of 2 and a constant term of -5. When $f(1) = 0$ and $f(2) = 3$, what is $f(-5)$? Explain your reasoning.

50. CRITICAL THINKING The weight y (in pounds) of a rainbow trout can be modeled by $y = 0.000304x^3$, where x is the length (in inches) of the trout.

a. Write a function that relates the weight y and length x of a rainbow trout when y is measured in kilograms and x is measured in centimeters. Use the fact that 1 kilogram $\approx$ 2.20 pounds and 1 centimeter $\approx$ 0.394 inch.

b. Graph the original function and the function from part (a) in the same coordinate plane. What type of transformation can you apply to the graph of $y = 0.000304x^3$ to produce the graph from part (a)?

Maintaining Mathematical Proficiency
Reviewing what you learned in previous grades and lessons

Simplify the expression. *(Skills Review Handbook)*

51. $xy + x^2 + 2xy + y^2 - 3x^2$

52. $2h^3g + 3hg^3 + 7h^2g^2 + 5h^3g + 2hg^3$

53. $-wk + 3kz - 2kw + 9zk - kw$

54. $a^2(m - 7a^3) - m(a^2 - 10)$

55. $3x(xy - 4) + 3(4xy + 3) - xy(x^2y - 1)$

56. $cv(9 - 3c) + 2c(v - 4c) + 6c$