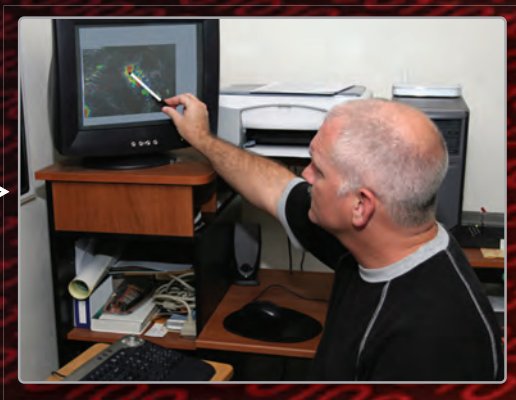


2 Linear and Quadratic Functions

- 2.1 Parent Functions and Transformations
- 2.2 Transformations of Linear and Absolute Value Functions
- 2.3 Modeling with Linear Functions
- 2.4 Solving Linear Systems
- 2.5 Transformations of Quadratic Functions
- 2.6 Characteristics of Quadratic Functions
- 2.7 Modeling with Quadratic Functions



Meteorologist (p. 93)



Soccer (p. 89)



Kangaroo (p. 79)



Dirt Bike (p. 43)



Prom (p. 57)

Maintaining Mathematical Proficiency

Evaluating Expressions

Example 1 Evaluate the expression $36 \div (3^2 \times 2) - 3$.

$$\begin{aligned} 36 \div (3^2 \times 2) - 3 &= 36 \div (9 \times 2) - 3 \\ &= 36 \div 18 - 3 \\ &= 2 - 3 \\ &= -1 \end{aligned}$$

Evaluate the power within parentheses.

Multiply within parentheses.

Divide.

Subtract.

Evaluate.

1. $5 \cdot 2^3 + 7$

2. $4 - 2(3 + 2)^2$

3. $48 \div 4^2 + \frac{3}{5}$

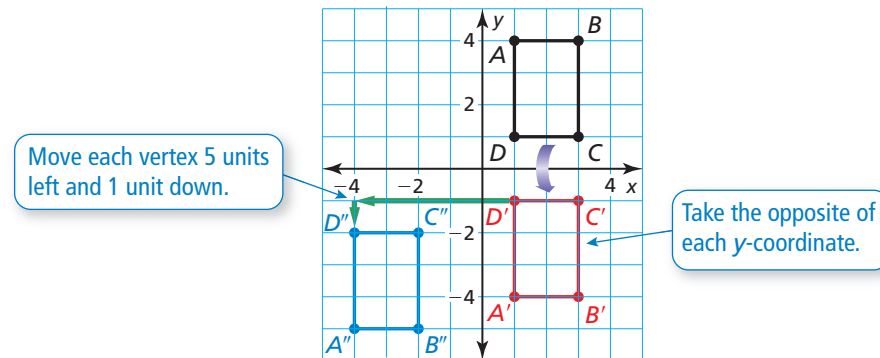
4. $50 \div 5^2 \cdot 2$

5. $\frac{1}{2}(2^2 + 22)$

6. $\frac{1}{6}(6 + 18) - 2^2$

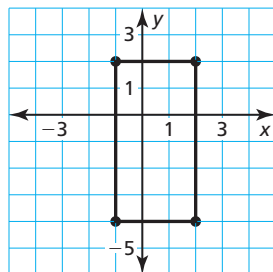
Transformations of Figures

Example 2 Reflect the black rectangle in the x -axis. Then translate the new rectangle 5 units to the left and 1 unit down.

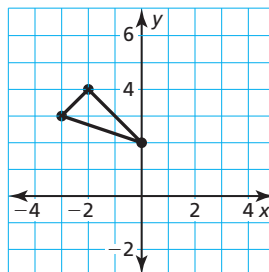


Graph the transformation of the figure.

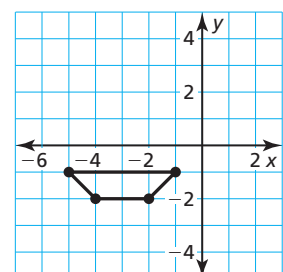
7. Translate the rectangle 1 unit right and 4 units up.



8. Reflect the triangle in the y -axis. Then translate 2 units left.



9. Translate the trapezoid 3 units down. Then reflect in the x -axis.



10. **ABSTRACT REASONING** Give an example to show why the order of operations is important when evaluating a numerical expression. Is the order of transformations of figures important? Justify your answer.

Using a Graphing Calculator

Core Concept

Standard and Square Viewing Windows

A typical screen on a graphing calculator has a height-to-width ratio of 2 to 3. This means that when you view a graph using the *standard viewing window* of -10 to 10 (on each axis), the graph will not be shown in its true perspective.

To view a graph in its true perspective, you need to change to a *square viewing window*, where the tick marks on the x -axis are spaced the same as the tick marks on the y -axis.

```
WINDOW
Xmin=-10
Xmax=10
Xscl=1
Ymin=-10
Ymax=10
Yscl=1
```

This is the standard viewing window.

```
WINDOW
Xmin=-9
Xmax=9
Xscl=1
Ymin=-6
Ymax=6
Yscl=1
```

This is a square viewing window.

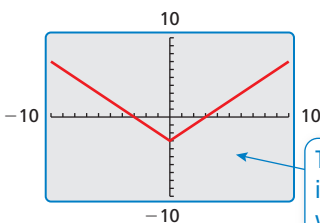
EXAMPLE 1 Using a Graphing Calculator

Use a graphing calculator to graph $y = |x| - 3$.

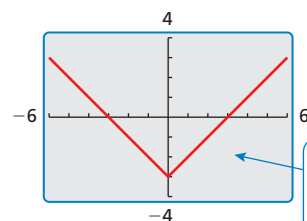
SOLUTION

In the standard viewing window, notice that the tick marks on the y -axis are closer together than those on the x -axis. This implies that the graph is not shown in its true perspective.

In a square viewing window, notice that the tick marks on both axes have the same spacing. This implies that the graph is shown in its true perspective.



This is the graph in the standard viewing window.



This is the graph in a square viewing window.

Monitoring Progress

Use a graphing calculator to graph the equation using the standard viewing window and a square viewing window. Describe any differences in the graphs.

- $y = 2x - 3$
- $y = -x + 1$
- $y = -|x - 4|$
- $y = |x + 2|$
- $y = x^2 - 2$
- $y = -x^2 + 1$

Determine whether the viewing window is square. Explain.

- $-8 \leq x \leq 8, -2 \leq y \leq 8$
- $-7 \leq x \leq 8, -2 \leq y \leq 8$
- $-6 \leq x \leq 9, -2 \leq y \leq 8$
- $-2 \leq x \leq 2, -3 \leq y \leq 3$
- $-4 \leq x \leq 5, -3 \leq y \leq 3$
- $-4 \leq x \leq 4, -3 \leq y \leq 3$

2.1 Parent Functions and Transformations

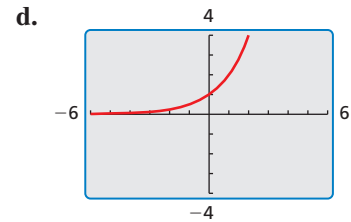
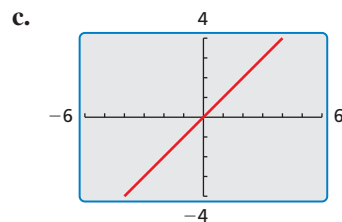
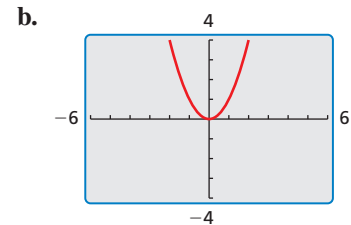
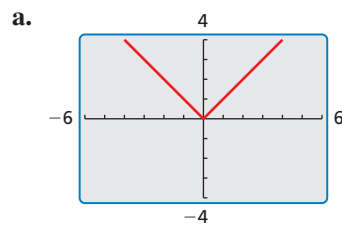
Essential Question What are the characteristics of some of the basic parent functions?

EXPLORATION 1 Identifying Basic Parent Functions

Work with a partner. Graphs of four basic parent functions are shown below. Classify each function as *linear*, *absolute value*, *quadratic*, or *exponential*. Justify your reasoning.

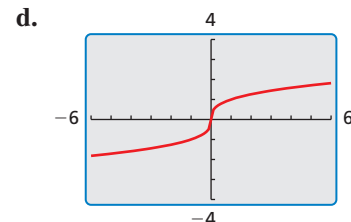
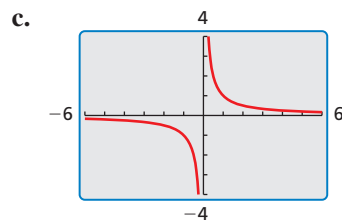
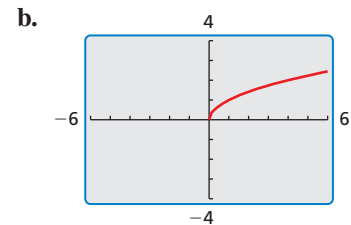
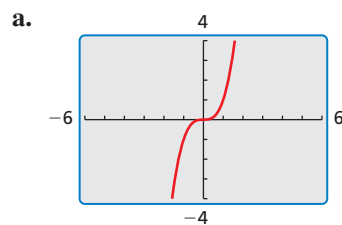
JUSTIFYING CONCLUSIONS

To be proficient in math, you need to justify your conclusions and communicate them clearly to others.



EXPLORATION 2 Identifying Basic Parent Functions

Work with a partner. Graphs of four basic parent functions that you will study later in this course are shown below. Classify each function as *square root*, *cube root*, *cubic*, or *reciprocal*. Justify your reasoning.



Communicate Your Answer

3. What are the characteristics of some of the basic parent functions?
4. Write an equation for each function whose graph is shown in Exploration 1. Then use a graphing calculator to verify that your equations are correct.

2.1 Lesson

Core Vocabulary

parent function, p. 40
 transformation, p. 41
 translation, p. 41
 reflection, p. 41
 vertical stretch, p. 42
 vertical shrink, p. 42

Previous

function
 domain
 range
 slope
 scatter plot

What You Will Learn

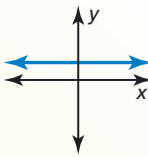
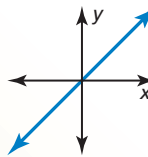
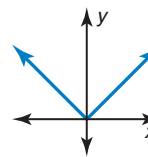
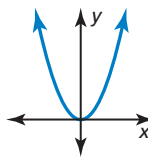
- ▶ Identify families of functions.
- ▶ Describe transformations of parent functions.
- ▶ Describe combinations of transformations.

Identifying Function Families

Functions that belong to the same *family* share key characteristics. The **parent function** is the most basic function in a family. Functions in the same family are *transformations* of their parent function.

Core Concept

Parent Functions

Family	Constant	Linear	Absolute Value	Quadratic
Rule	$f(x) = 1$	$f(x) = x$	$f(x) = x $	$f(x) = x^2$
Graph				
Domain	All real numbers	All real numbers	All real numbers	All real numbers
Range	$y = 1$	All real numbers	$y \geq 0$	$y \geq 0$

LOOKING FOR STRUCTURE

You can also use function rules to identify functions. The only variable term in f is an $|x|$ -term, so it is an absolute value function.

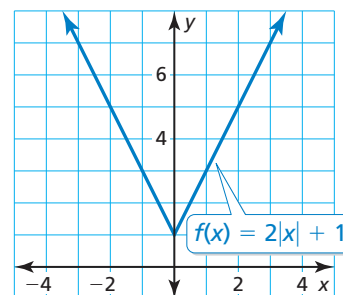
EXAMPLE 1 Identifying a Function Family

Identify the function family to which f belongs. Compare the graph of f to the graph of its parent function.

SOLUTION

The graph of f is V-shaped, so f is an absolute value function.

The graph is shifted up and is narrower than the graph of the parent absolute value function. The domain of each function is all real numbers, but the range of f is $y \geq 1$ and the range of the parent absolute value function is $y \geq 0$.

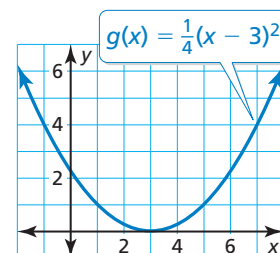


Monitoring Progress



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1. Identify the function family to which g belongs. Compare the graph of g to the graph of its parent function.



Describing Transformations

A **transformation** changes the size, shape, position, or orientation of a graph.

A **translation** is a transformation that shifts a graph horizontally and/or vertically but does not change its size, shape, or orientation.

REMEMBER

The slope-intercept form of a linear equation is $y = mx + b$, where m is the slope and b is the y -intercept.

EXAMPLE 2

Graphing and Describing Translations

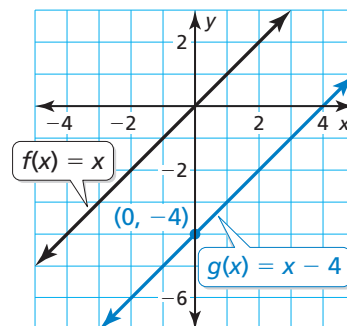
Graph $g(x) = x - 4$ and its parent function. Then describe the transformation.

SOLUTION

The function g is a linear function with a slope of 1 and a y -intercept of -4 . So, draw a line through the point $(0, -4)$ with a slope of 1.

The graph of g is 4 units below the graph of the parent linear function f .

► So, the graph of $g(x) = x - 4$ is a vertical translation 4 units down of the graph of the parent linear function.



A **reflection** is a transformation that flips a graph over a line called the *line of reflection*. A reflected point is the same distance from the line of reflection as the original point but on the opposite side of the line.

EXAMPLE 3

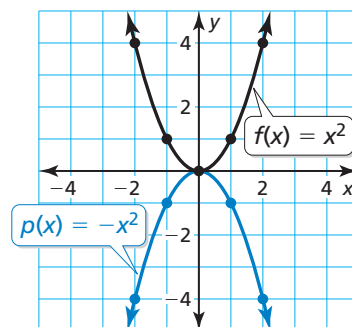
Graphing and Describing Reflections

Graph $p(x) = -x^2$ and its parent function. Then describe the transformation.

SOLUTION

The function p is a quadratic function. Use a table of values to graph each function.

x	$y = x^2$	$y = -x^2$
-2	4	-4
-1	1	-1
0	0	0
1	1	-1
2	4	-4



The graph of p is the graph of the parent function flipped over the x -axis.

► So, the graph of $p(x) = -x^2$ is a reflection in the x -axis of the graph of the parent quadratic function.

REMEMBER

The function $p(x) = -x^2$ is written in *function notation*, where $p(x)$ is another name for y .

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Graph the function and its parent function. Then describe the transformation.

2. $g(x) = x + 3$
3. $h(x) = (x - 2)^2$
4. $n(x) = -|x|$

Another way to transform the graph of a function is to multiply all of the y -coordinates by the same positive factor (other than 1). When the factor is greater than 1, the transformation is a **vertical stretch**. When the factor is greater than 0 and less than 1, it is a **vertical shrink**.

EXAMPLE 4 Graphing and Describing Stretches and Shrinks

Graph each function and its parent function. Then describe the transformation.

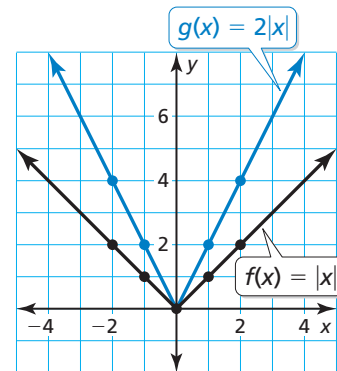
a. $g(x) = 2|x|$

b. $h(x) = \frac{1}{2}x^2$

SOLUTION

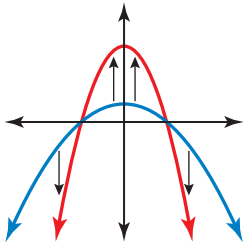
- a. The function g is an absolute value function. Use a table of values to graph the functions.

x	$y = x $	$y = 2 x $
-2	2	4
-1	1	2
0	0	0
1	1	2
2	2	4

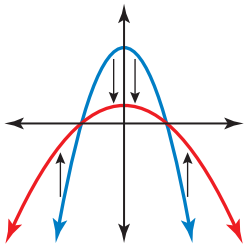


REASONING ABSTRACTLY

To visualize a vertical stretch, imagine *pulling* the points away from the x -axis.



To visualize a vertical shrink, imagine *pushing* the points toward the x -axis.

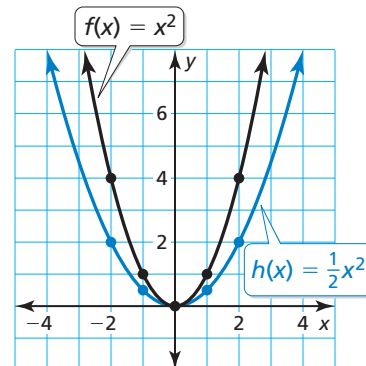


The y -coordinate of each point on g is two times the y -coordinate of the corresponding point on the parent function.

- So, the graph of $g(x) = 2|x|$ is a vertical stretch of the graph of the parent absolute value function.

- b. The function h is a quadratic function. Use a table of values to graph the functions.

x	$y = x^2$	$y = \frac{1}{2}x^2$
-2	4	2
-1	1	$\frac{1}{2}$
0	0	0
1	1	$\frac{1}{2}$
2	4	2



The y -coordinate of each point on h is one-half of the y -coordinate of the corresponding point on the parent function.

- So, the graph of $h(x) = \frac{1}{2}x^2$ is a vertical shrink of the graph of the parent quadratic function.

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Graph the function and its parent function. Then describe the transformation.

5. $g(x) = 3x$

6. $h(x) = \frac{3}{2}x^2$

7. $c(x) = 0.2|x|$

Combinations of Transformations

You can use more than one transformation to change the graph of a function.

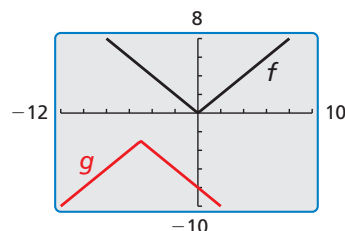
EXAMPLE 5 Describing Combinations of Transformations

Use a graphing calculator to graph $g(x) = -|x + 5| - 3$ and its parent function. Then describe the transformations.

SOLUTION

The function g is an absolute value function.

- The graph shows that $g(x) = -|x + 5| - 3$ is a reflection in the x -axis followed by a translation 5 units left and 3 units down of the graph of the parent absolute value function.



EXAMPLE 6 Modeling with Mathematics

Time (seconds), x	Height (feet), y
0	8
0.5	20
1	24
1.5	20
2	8

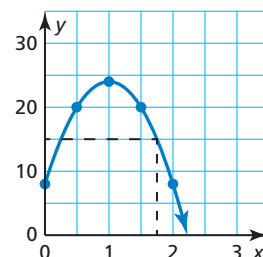
The table shows the height y of a dirt bike x seconds after jumping off a ramp. What type of function can you use to model the data? Estimate the height after 1.75 seconds.

SOLUTION

- Understand the Problem** You are asked to identify the type of function that can model the table of values and then to find the height at a specific time.
- Make a Plan** Create a scatter plot of the data. Then use the relationship shown in the scatter plot to estimate the height after 1.75 seconds.
- Solve the Problem** Create a scatter plot.

The data appear to lie on a curve that resembles a quadratic function. Sketch the curve.

- So, you can model the data with a quadratic function. The graph shows that the height is about 15 feet after 1.75 seconds.



- Look Back** To check that your solution is reasonable, analyze the values in the table. Notice that the heights decrease after 1 second. Because 1.75 is between 1.5 and 2, the height must be between 20 feet and 8 feet.

$$8 < 15 < 20 \quad \checkmark$$



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Use a graphing calculator to graph the function and its parent function. Then describe the transformations.

8. $h(x) = -\frac{1}{4}x + 5$

9. $d(x) = 3(x - 5)^2 - 1$

10. The table shows the amount of fuel in a chainsaw over time. What type of function can you use to model the data? When will the tank be empty?

Time (minutes), x	0	10	20	30	40
Fuel remaining (fluid ounces), y	15	12	9	6	3

Vocabulary and Core Concept Check

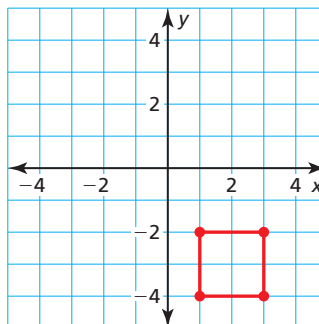
- COMPLETE THE SENTENCE** The function $f(x) = x^2$ is the _____ of $f(x) = 2x^2 - 3$.
- DIFFERENT WORDS, SAME QUESTION** Which is different? Find “both” answers.

What are the vertices of the figure after a reflection in the x -axis, followed by a translation 2 units right?

What are the vertices of the figure after a translation 6 units up and 2 units right?

What are the vertices of the figure after a translation 2 units right, followed by a reflection in the x -axis?

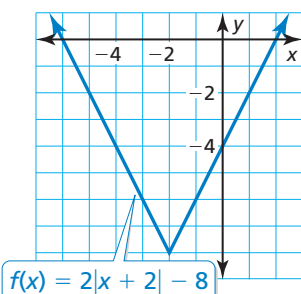
What are the vertices of the figure after a translation 6 units up, followed by a reflection in the x -axis?



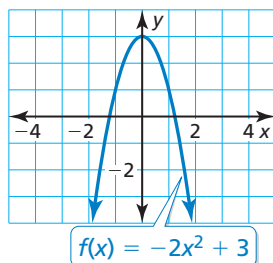
Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, identify the function family to which f belongs. Compare the graph of f to the graph of its parent function. (See Example 1.)

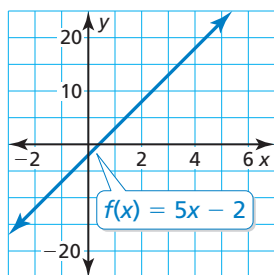
3.



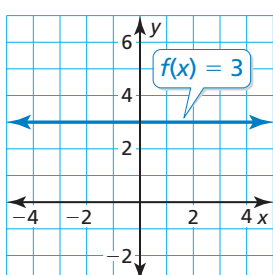
4.



5.



6.



- MODELING WITH MATHEMATICS** At 8:00 A.M., the temperature is 43°F . The temperature increases 2°F each hour for the next 7 hours. Graph the temperatures over time t ($t = 0$ represents 8:00 A.M.). What type of function can you use to model the data? Explain.
- MODELING WITH MATHEMATICS** You purchase a car from a dealership for \$10,000. The trade-in value of the car each year after the purchase is given by the function $f(x) = 10,000 - 250x^2$. Identify the function family to which f belongs.

In Exercises 9–18, graph the function and its parent function. Then describe the transformation. (See Examples 2 and 3.)

- $g(x) = x + 4$
- $f(x) = x - 6$
- $f(x) = x^2 - 1$
- $h(x) = (x + 4)^2$
- $g(x) = |x - 5|$
- $f(x) = 4 + |x|$
- $h(x) = -x^2$
- $g(x) = -x$
- $f(x) = 3$
- $f(x) = -2$

In Exercises 19–26, graph the function and its parent function. Then describe the transformation. (See Example 4.)

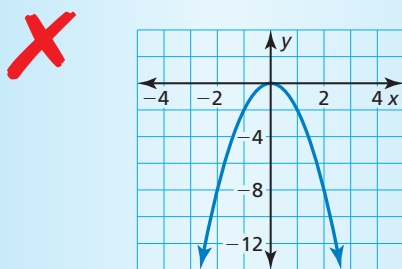
19. $f(x) = \frac{1}{3}x$ 20. $g(x) = 4x$
 21. $f(x) = 2x^2$ 22. $h(x) = \frac{1}{3}x^2$
 23. $h(x) = \frac{3}{4}x$ 24. $g(x) = \frac{4}{3}x$
 25. $h(x) = 3|x|$ 26. $f(x) = \frac{1}{2}|x|$

In Exercises 27–34, use a graphing calculator to graph the function and its parent function. Then describe the transformations. (See Example 5.)

27. $f(x) = 3x + 2$ 28. $h(x) = -x + 5$
 29. $h(x) = -3|x| - 1$ 30. $f(x) = \frac{3}{4}|x| + 1$
 31. $g(x) = \frac{1}{2}x^2 - 6$ 32. $f(x) = 4x^2 - 3$
 33. $f(x) = -(x + 3)^2 + \frac{1}{4}$
 34. $g(x) = -|x - 1| - \frac{1}{2}$

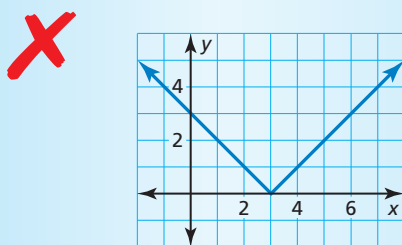
ERROR ANALYSIS In Exercises 35 and 36, identify and correct the error in describing the transformation of the parent function.

35.



The graph is a reflection in the x -axis and a vertical shrink of the parent quadratic function.

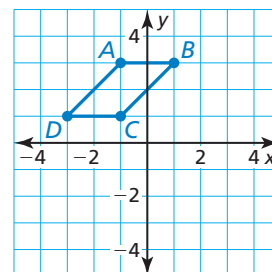
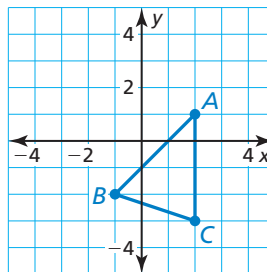
36.



The graph is a translation 3 units right of the parent absolute value function, so the function is $f(x) = |x + 3|$.

MATHEMATICAL CONNECTIONS In Exercises 37 and 38, find the coordinates of the figure after the transformation.

37. Translate 2 units down. 38. Reflect in the x -axis.



USING TOOLS In Exercises 39–44, identify the function family and describe the domain and range. Use a graphing calculator to verify your answer.

39. $g(x) = |x + 2| - 1$ 40. $h(x) = |x - 3| + 2$
 41. $g(x) = 3x + 4$ 42. $f(x) = -4x + 11$
 43. $f(x) = 5x^2 - 2$ 44. $f(x) = -2x^2 + 6$

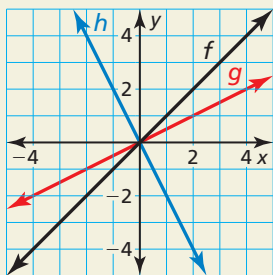
45. **MODELING WITH MATHEMATICS** The table shows the speeds of a car as it travels through an intersection with a stop sign. What type of function can you use to model the data? Estimate the speed of the car when it is 20 yards past the intersection. (See Example 6.)

Displacement from sign (yards), x	Speed (miles per hour), y
-100	40
-50	20
-10	4
0	0
10	4
50	20
100	40

46. **THOUGHT PROVOKING** In the same coordinate plane, sketch the graph of the parent quadratic function and the graph of a quadratic function that has no x -intercepts. Describe the transformation(s) of the parent function.

47. **USING STRUCTURE** Graph the functions $f(x) = |x - 4|$ and $g(x) = |x| - 4$. Are they equivalent? Explain.

48. **HOW DO YOU SEE IT?** Consider the graphs of f , g , and h .



- a. Does the graph of g represent a vertical stretch or a vertical shrink of the graph of f ? Explain your reasoning.
- b. Describe how to transform the graph of f to obtain the graph of h .
49. **MAKING AN ARGUMENT** Your friend says two different translations of the graph of the parent linear function can result in the graph of $f(x) = x - 2$. Is your friend correct? Explain.
50. **DRAWING CONCLUSIONS** A person swims at a constant speed of 1 meter per second. What type of function can be used to model the distance the swimmer travels? If the person has a 10-meter head start, what type of transformation does this represent? Explain.



51. **PROBLEM SOLVING** You are playing basketball with your friends. The height (in feet) of the ball above the ground t seconds after a shot is released from your hand is modeled by the function $f(t) = -16t^2 + 32t + 5.2$.

- a. Without graphing, identify the type of function that models the height of the basketball.
- b. What is the value of t when the ball is released from your hand? Explain your reasoning.
- c. How many feet above the ground is the ball when it is released from your hand? Explain.

52. **MODELING WITH MATHEMATICS** The table shows the battery lives of a computer over time. What type of function can you use to model the data? Interpret the meaning of the x -intercept in this situation.

Time (hours), x	Battery life remaining, y
1	80%
3	40%
5	0%
6	20%
8	60%

53. **REASONING** Compare each function with its parent function. State whether it contains a *horizontal translation*, *vertical translation*, *both*, or *neither*. Explain your reasoning.

- a. $f(x) = 2|x| - 3$ b. $f(x) = (x - 8)^2$
 c. $f(x) = |x + 2| + 4$ d. $f(x) = 4x^2$

54. **CRITICAL THINKING** Use the values -1 , 0 , 1 , and 2 in the correct box so the graph of each function intersects the x -axis. Explain your reasoning.

- a. $f(x) = 3x \square + 1$ b. $f(x) = |2x - 6| - \square$
 c. $f(x) = \square x^2 + 1$ d. $f(x) = \square$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Determine whether the ordered pair is a solution of the equation. (*Skills Review Handbook*)

55. $f(x) = |x + 2|$; $(1, -3)$ 56. $f(x) = |x| - 3$; $(-2, -5)$
 57. $f(x) = x - 3$; $(5, 2)$ 58. $f(x) = x - 4$; $(12, 8)$

Find the x -intercept and the y -intercept of the graph of the equation. (*Skills Review Handbook*)

59. $y = x$ 60. $y = x + 2$
 61. $3x + y = 1$ 62. $x - 2y = 8$

2.2 Transformations of Linear and Absolute Value Functions

Essential Question How do the graphs of $y = f(x) + k$, $y = f(x - h)$, and $y = -f(x)$ compare to the graph of the parent function f ?

USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools to visualize results and explore consequences.

EXPLORATION 1

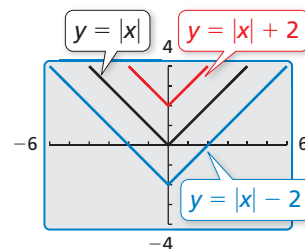
Transformations of the Parent Absolute Value Function

Work with a partner. Compare the graph of the function

$$y = |x| + k \quad \text{Transformation}$$

to the graph of the parent function

$$f(x) = |x|. \quad \text{Parent function}$$



EXPLORATION 2

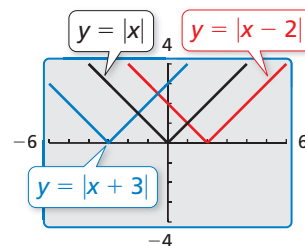
Transformations of the Parent Absolute Value Function

Work with a partner. Compare the graph of the function

$$y = |x - h| \quad \text{Transformation}$$

to the graph of the parent function

$$f(x) = |x|. \quad \text{Parent function}$$



EXPLORATION 3

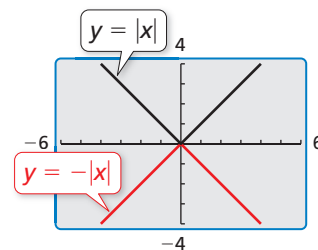
Transformation of the Parent Absolute Value Function

Work with a partner. Compare the graph of the function

$$y = -|x| \quad \text{Transformation}$$

to the graph of the parent function

$$f(x) = |x|. \quad \text{Parent function}$$



Communicate Your Answer

- How do the graphs of $y = f(x) + k$, $y = f(x - h)$, and $y = -f(x)$ compare to the graph of the parent function f ?
- Compare the graph of each function to the graph of its parent function f . Use a graphing calculator to verify your answers are correct.

a. $y = 2^x - 4$	b. $y = 2^{x+4}$	c. $y = -2^x$
d. $y = x^2 + 1$	e. $y = (x - 1)^2$	f. $y = -x^2$

2.2 Lesson

What You Will Learn

- ▶ Write functions representing translations and reflections.
- ▶ Write functions representing stretches and shrinks.
- ▶ Write functions representing combinations of transformations.

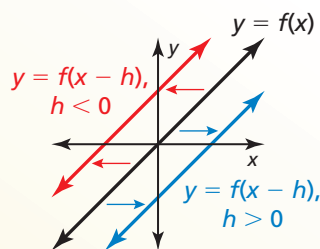
Translations and Reflections

You can use function notation to represent transformations of graphs of functions.

Core Concept

Horizontal Translations

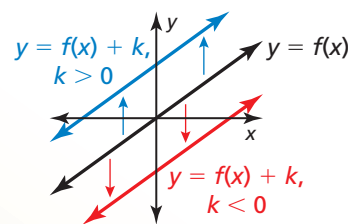
The graph of $y = f(x - h)$ is a horizontal translation of the graph of $y = f(x)$, where $h \neq 0$.



Subtracting h from the **inputs** before evaluating the function shifts the graph left when $h < 0$ and right when $h > 0$.

Vertical Translations

The graph of $y = f(x) + k$ is a vertical translation of the graph of $y = f(x)$, where $k \neq 0$.



Adding k to the **outputs** shifts the graph down when $k < 0$ and up when $k > 0$.

EXAMPLE 1 Writing Translations of Functions

Let $f(x) = 2x + 1$.

- Write a function g whose graph is a translation 3 units down of the graph of f .
- Write a function h whose graph is a translation 2 units to the left of the graph of f .

SOLUTION

- A translation 3 units down is a vertical translation that adds -3 to each output value.

$$\begin{aligned} g(x) &= f(x) + (-3) && \text{Add } -3 \text{ to the output.} \\ &= 2x + 1 + (-3) && \text{Substitute } 2x + 1 \text{ for } f(x). \\ &= 2x - 2 && \text{Simplify.} \end{aligned}$$

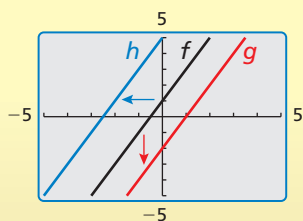
- ▶ The translated function is $g(x) = 2x - 2$.

- A translation 2 units to the left is a horizontal translation that subtracts -2 from each input value.

$$\begin{aligned} h(x) &= f(x - (-2)) && \text{Subtract } -2 \text{ from the input.} \\ &= f(x + 2) && \text{Add the opposite.} \\ &= 2(x + 2) + 1 && \text{Replace } x \text{ with } x + 2 \text{ in } f(x). \\ &= 2x + 5 && \text{Simplify.} \end{aligned}$$

- ▶ The translated function is $h(x) = 2x + 5$.

Check



Core Concept

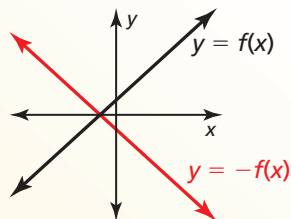
STUDY TIP

When you reflect a function in a line, the graphs are symmetric about that line.



Reflections in the x -Axis

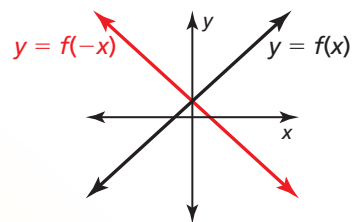
The graph of $y = -f(x)$ is a reflection in the x -axis of the graph of $y = f(x)$.



Multiplying the **outputs** by -1 changes their signs.

Reflections in the y -Axis

The graph of $y = f(-x)$ is a reflection in the y -axis of the graph of $y = f(x)$.



Multiplying the **inputs** by -1 changes their signs.

EXAMPLE 2 Writing Reflections of Functions

Let $f(x) = |x + 3| + 1$.

- Write a function g whose graph is a reflection in the x -axis of the graph of f .
- Write a function h whose graph is a reflection in the y -axis of the graph of f .

SOLUTION

- A reflection in the x -axis changes the sign of each output value.

$$\begin{aligned} g(x) &= -f(x) && \text{Multiply the output by } -1. \\ &= -(|x + 3| + 1) && \text{Substitute } |x + 3| + 1 \text{ for } f(x). \\ &= -|x + 3| - 1 && \text{Distributive Property} \end{aligned}$$

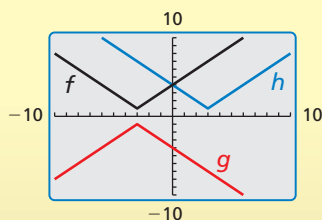
- ▶ The reflected function is $g(x) = -|x + 3| - 1$.

- A reflection in the y -axis changes the sign of each input value.

$$\begin{aligned} h(x) &= f(-x) && \text{Multiply the input by } -1. \\ &= |-x + 3| + 1 && \text{Replace } x \text{ with } -x \text{ in } f(x). \\ &= |-(x - 3)| + 1 && \text{Factor out } -1. \\ &= |-1| \cdot |x - 3| + 1 && \text{Product Property of Absolute Value} \\ &= |x - 3| + 1 && \text{Simplify.} \end{aligned}$$

- ▶ The reflected function is $h(x) = |x - 3| + 1$.

Check



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Write a function g whose graph represents the indicated transformation of the graph of f . Use a graphing calculator to check your answer.

- $f(x) = 3x$; translation 5 units up
- $f(x) = |x| - 3$; translation 4 units to the right
- $f(x) = -|x + 2| - 1$; reflection in the x -axis
- $f(x) = \frac{1}{2}x + 1$; reflection in the y -axis

Stretches and Shrinks

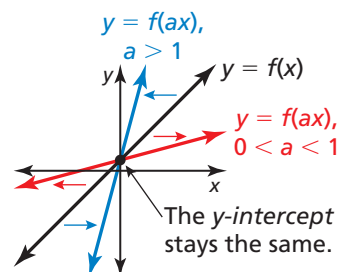
In the previous section, you learned that vertical stretches and shrinks transform graphs. You can also use *horizontal* stretches and shrinks to transform graphs.

Core Concept

Horizontal Stretches and Shrinks

The graph of $y = f(ax)$ is a horizontal stretch or shrink by a factor of $\frac{1}{a}$ of the graph of $y = f(x)$, where $a > 0$ and $a \neq 1$.

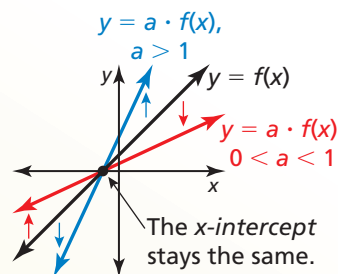
Multiplying the **inputs** by a before evaluating the function stretches the graph horizontally (away from the y -axis) when $0 < a < 1$, and shrinks the graph horizontally (toward the y -axis) when $a > 1$.



Vertical Stretches and Shrinks

The graph of $y = a \cdot f(x)$ is a vertical stretch or shrink by a factor of a of the graph of $y = f(x)$, where $a > 0$ and $a \neq 1$.

Multiplying the **outputs** by a stretches the graph vertically (away from the x -axis) when $a > 1$, and shrinks the graph vertically (toward the x -axis) when $0 < a < 1$.



STUDY TIP

The graphs of $y = f(-ax)$ and $y = -a \cdot f(x)$ represent a stretch or shrink and a reflection in the x - or y -axis of the graph of $y = f(x)$.

EXAMPLE 3 Writing Stretches and Shrinks of Functions

Let $f(x) = |x - 3| - 5$. Write (a) a function g whose graph is a horizontal shrink of the graph of f by a factor of $\frac{1}{3}$, and (b) a function h whose graph is a vertical stretch of the graph of f by a factor of 2.

SOLUTION

- a. A horizontal shrink by a factor of $\frac{1}{3}$ multiplies each input value by 3.

$$\begin{aligned} g(x) &= f(3x) && \text{Multiply the input by 3.} \\ &= |3x - 3| - 5 && \text{Replace } x \text{ with } 3x \text{ in } f(x). \end{aligned}$$

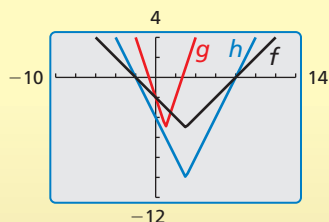
- ▶ The transformed function is $g(x) = |3x - 3| - 5$.

- b. A vertical stretch by a factor of 2 multiplies each output value by 2.

$$\begin{aligned} h(x) &= 2 \cdot f(x) && \text{Multiply the output by 2.} \\ &= 2 \cdot (|x - 3| - 5) && \text{Substitute } |x - 3| - 5 \text{ for } f(x). \\ &= 2|x - 3| - 10 && \text{Distributive Property} \end{aligned}$$

- ▶ The transformed function is $h(x) = 2|x - 3| - 10$.

Check



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Write a function g whose graph represents the indicated transformation of the graph of f . Use a graphing calculator to check your answer.

- $f(x) = 4x + 2$; horizontal stretch by a factor of 2
- $f(x) = |x| - 3$; vertical shrink by a factor of $\frac{1}{3}$

Combinations of Transformations

You can write a function that represents a series of transformations on the graph of another function by applying the transformations one at a time in the stated order.

EXAMPLE 4 Combining Transformations

Let the graph of g be a vertical shrink by a factor of 0.25 followed by a translation 3 units up of the graph of $f(x) = x$. Write a rule for g .

SOLUTION

Step 1 First write a function h that represents the vertical shrink of f .

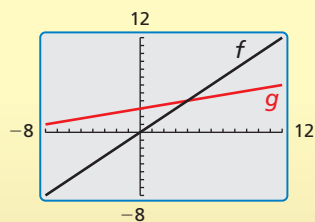
$$\begin{aligned} h(x) &= 0.25 \cdot f(x) && \text{Multiply the output by 0.25.} \\ &= 0.25x && \text{Substitute } x \text{ for } f(x). \end{aligned}$$

Step 2 Then write a function g that represents the translation of h .

$$\begin{aligned} g(x) &= h(x) + 3 && \text{Add 3 to the output.} \\ &= 0.25x + 3 && \text{Substitute } 0.25x \text{ for } h(x). \end{aligned}$$

► The transformed function is $g(x) = 0.25x + 3$.

Check



EXAMPLE 5 Modeling with Mathematics

You design a computer game. Your revenue for x downloads is given by $f(x) = 2x$. Your profit is \$50 less than 90% of the revenue for x downloads. Describe how to transform the graph of f to model the profit. What is your profit for 100 downloads?

SOLUTION

- 1. Understand the Problem** You are given a function that represents your revenue and a verbal statement that represents your profit. You are asked to find the profit for 100 downloads.
- 2. Make a Plan** Write a function p that represents your profit. Then use this function to find the profit for 100 downloads.
- 3. Solve the Problem** profit = 90% • revenue – 50

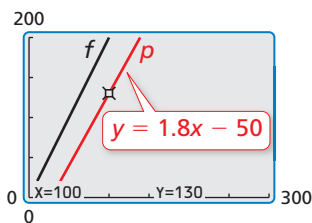
$$\begin{aligned} p(x) &= 0.9 \cdot f(x) - 50 && \begin{array}{l} \text{Vertical shrink by a factor of 0.9} \\ \text{Translation 50 units down} \end{array} \\ &= 0.9 \cdot 2x - 50 && \text{Substitute } 2x \text{ for } f(x). \\ &= 1.8x - 50 && \text{Simplify.} \end{aligned}$$

To find the profit for 100 downloads, evaluate p when $x = 100$.

$$p(100) = 1.8(100) - 50 = 130$$

► Your profit is \$130 for 100 downloads.

- 4. Look Back** The vertical shrink decreases the slope, and the translation shifts the graph 50 units down. So, the graph of p is below and not as steep as the graph of f .



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- Let the graph of g be a translation 6 units down followed by a reflection in the x -axis of the graph of $f(x) = |x|$. Write a rule for g . Use a graphing calculator to check your answer.
- WHAT IF?** In Example 5, your revenue function is $f(x) = 3x$. How does this affect your profit for 100 downloads?

Vocabulary and Core Concept Check

- COMPLETE THE SENTENCE** The function $g(x) = |5x| - 4$ is a horizontal _____ of the function $f(x) = |x| - 4$.
- WHICH ONE DOESN'T BELONG?** Which transformation does *not* belong with the other three? Explain your reasoning.

Translate the graph of $f(x) = 2x + 3$ up 2 units.

Shrink the graph of $f(x) = x + 5$ horizontally by a factor of $\frac{1}{2}$.

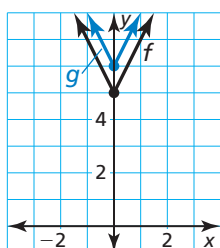
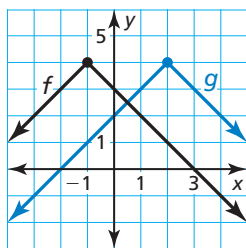
Stretch the graph of $f(x) = x + 3$ vertically by a factor of 2.

Translate the graph of $f(x) = 2x + 3$ left 1 unit.

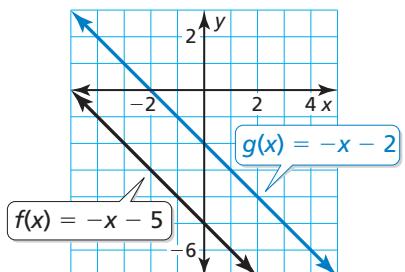
Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, write a function g whose graph represents the indicated transformation of the graph of f . Use a graphing calculator to check your answer. (See Example 1.)

- $f(x) = x - 5$; translation 4 units to the left
- $f(x) = x + 2$; translation 2 units to the right
- $f(x) = |4x + 3| + 2$; translation 2 units down
- $f(x) = 2x - 9$; translation 6 units up
- $f(x) = 4 - |x + 1|$ 8. $f(x) = |4x| + 5$



- WRITING** Describe two different translations of the graph of f that result in the graph of g .



- PROBLEM SOLVING** You open a café. The function $f(x) = 4000x$ represents your expected net income (in dollars) after being open x weeks. Before you open, you incur an extra expense of \$12,000. What transformation of f is necessary to model this situation? How many weeks will it take to pay off the extra expense?

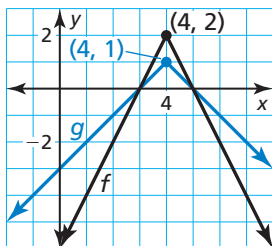


In Exercises 11–16, write a function g whose graph represents the indicated transformation of the graph of f . Use a graphing calculator to check your answer. (See Example 2.)

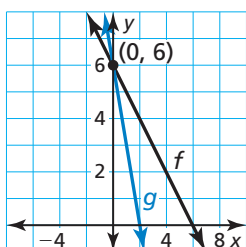
- $f(x) = -5x + 2$; reflection in the x -axis
- $f(x) = \frac{1}{2}x - 3$; reflection in the x -axis
- $f(x) = |6x| - 2$; reflection in the y -axis
- $f(x) = |2x - 1| + 3$; reflection in the y -axis
- $f(x) = -3 + |x - 11|$; reflection in the y -axis
- $f(x) = -x + 1$; reflection in the y -axis

In Exercises 17–22, write a function g whose graph represents the indicated transformation of the graph of f . Use a graphing calculator to check your answer. (See Example 3.)

17. $f(x) = x + 2$; vertical stretch by a factor of 5
18. $f(x) = 2x + 6$; vertical shrink by a factor of $\frac{1}{2}$
19. $f(x) = |2x| + 4$; horizontal shrink by a factor of $\frac{1}{2}$
20. $f(x) = |x + 3|$; horizontal stretch by a factor of 4
21. $f(x) = -2|x - 4| + 2$

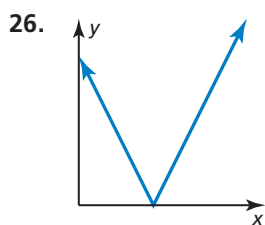
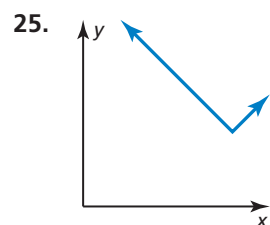
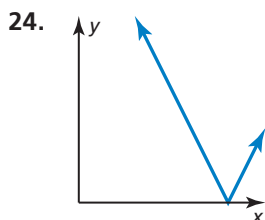
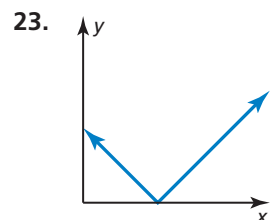
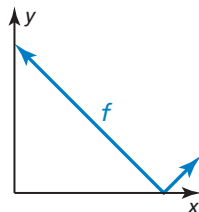


22. $f(x) = 6 - x$



ANALYZING RELATIONSHIPS

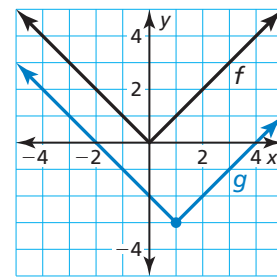
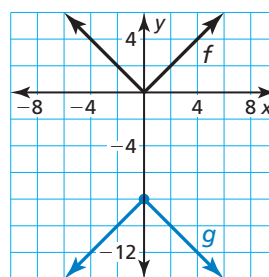
In Exercises 23–26, match the graph of the transformation of f with the correct equation shown. Explain your reasoning.



- | | |
|-------------------|-------------------|
| A. $y = 2f(x)$ | B. $y = f(2x)$ |
| C. $y = f(x + 2)$ | D. $y = f(x) + 2$ |

In Exercises 27–32, write a function g whose graph represents the indicated transformations of the graph of f . (See Example 4.)

27. $f(x) = x$; vertical stretch by a factor of 2 followed by a translation 1 unit up
28. $f(x) = x$; translation 3 units down followed by a vertical shrink by a factor of $\frac{1}{3}$
29. $f(x) = |x|$; translation 2 units to the right followed by a horizontal stretch by a factor of 2
30. $f(x) = |x|$; reflection in the y -axis followed by a translation 3 units to the right
31. $f(x) = |x|$
32. $f(x) = |x|$



ERROR ANALYSIS In Exercises 33 and 34, identify and correct the error in writing the function g whose graph represents the indicated transformations of the graph of f .

33. $f(x) = |x|$; translation 3 units to the right followed by a translation 2 units up
 $g(x) = |x + 3| + 2$

34. $f(x) = x$; translation 6 units down followed by a vertical stretch by a factor of 5
 $g(x) = 5x - 6$

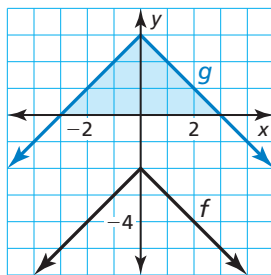
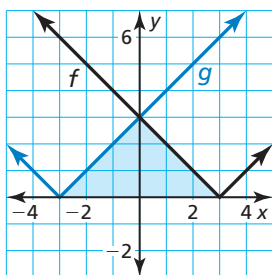
35. **MAKING AN ARGUMENT** Your friend claims that when writing a function whose graph represents a combination of transformations, the order is not important. Is your friend correct? Justify your answer.

- 36. MODELING WITH MATHEMATICS** During a recent period of time, bookstore sales have been declining. The sales (in billions of dollars) can be modeled by the function $f(t) = -\frac{7}{5}t + 17.2$, where t is the number of years since 2006. Suppose sales decreased at twice the rate. How can you transform the graph of f to model the sales? Explain how the sales in 2010 are affected by this change. (See Example 5.)

MATHEMATICAL CONNECTIONS For Exercises 37–40, describe the transformation of the graph of f to the graph of g . Then find the area of the shaded triangle.

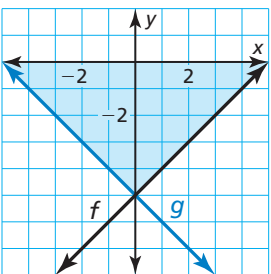
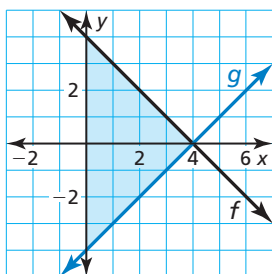
37. $f(x) = |x - 3|$

38. $f(x) = -|x| - 2$



39. $f(x) = -x + 4$

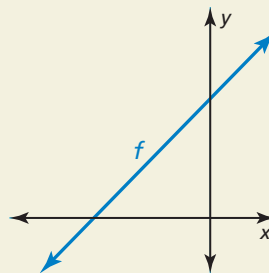
40. $f(x) = x - 5$



- 41. ABSTRACT REASONING** The functions $f(x) = mx + b$ and $g(x) = mx + c$ represent two parallel lines.

- Write an expression for the vertical translation of the graph of f to the graph of g .
- Use the definition of slope to write an expression for the horizontal translation of the graph of f to the graph of g .

- 42. HOW DO YOU SEE IT?** Consider the graph of $f(x) = mx + b$. Describe the effect each transformation has on the slope of the line and the intercepts of the graph.



- Reflect the graph of f in the y -axis.
- Shrink the graph of f vertically by a factor of $\frac{1}{3}$.
- Stretch the graph of f horizontally by a factor of 2.

- 43. REASONING** The graph of $g(x) = -4|x| + 2$ is a reflection in the x -axis, vertical stretch by a factor of 4, and a translation 2 units down of the graph of its parent function. Choose the correct order for the transformations of the graph of the parent function to obtain the graph of g . Explain your reasoning.

- 44. THOUGHT PROVOKING** You are planning a cross-country bicycle trip of 4320 miles. Your distance d (in miles) from the halfway point can be modeled by $d = 72|x - 30|$, where x is the time (in days) and $x = 0$ represents June 1. Your plans are altered so that the model is now a right shift of the original model. Give an example of how this can happen. Sketch both the original model and the shifted model.

- 45. CRITICAL THINKING** Use the correct value 0, -2 , or 1 with a , b , and c so the graph of $g(x) = a|x - b| + c$ is a reflection in the x -axis followed by a translation one unit to the left and one unit up of the graph of $f(x) = 2|x - 2| + 1$. Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the function for the given value of x . (Skills Review Handbook)

46. $f(x) = x + 4$; $x = 3$

47. $f(x) = 4x - 1$; $x = -1$

48. $f(x) = -x + 3$; $x = 5$

49. $f(x) = -2x - 2$; $x = -1$

Create a scatter plot of the data. (Skills Review Handbook)

50.

x	8	10	11	12	15
$f(x)$	4	9	10	12	12

51.

x	2	5	6	10	13
$f(x)$	22	13	15	12	6

2.3 Modeling with Linear Functions

Essential Question How can you use a linear function to model and analyze a real-life situation?

EXPLORATION 1 Modeling with a Linear Function

Work with a partner. A company purchases a copier for \$12,000. The spreadsheet shows how the copier depreciates over an 8-year period.

	A	B
1	Year, t	Value, V
2	0	\$12,000
3	1	\$10,750
4	2	\$9,500
5	3	\$8,250
6	4	\$7,000
7	5	\$5,750
8	6	\$4,500
9	7	\$3,250
10	8	\$2,000
11		

- Write a linear function to represent the value V of the copier as a function of the number t of years.
- Sketch a graph of the function. Explain why this type of depreciation is called *straight line depreciation*.
- Interpret the slope of the graph in the context of the problem.

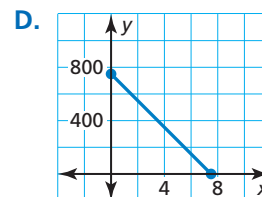
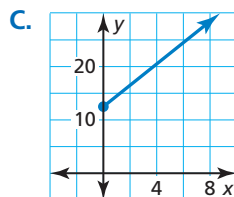
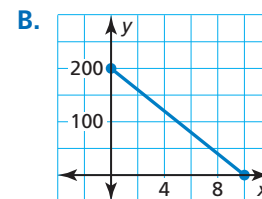
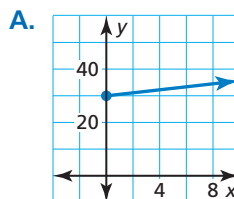
MODELING WITH MATHEMATICS

To be proficient in math, you need to routinely interpret your results in the context of the situation.

EXPLORATION 2 Modeling with Linear Functions

Work with a partner. Match each description of the situation with its corresponding graph. Explain your reasoning.

- A person gives \$20 per week to a friend to repay a \$200 loan.
- An employee receives \$12.50 per hour plus \$2 for each unit produced per hour.
- A sales representative receives \$30 per day for food plus \$0.565 for each mile driven.
- A computer that was purchased for \$750 depreciates \$100 per year.



Communicate Your Answer

- How can you use a linear function to model and analyze a real-life situation?
- Use the Internet or some other reference to find a real-life example of straight line depreciation.
 - Use a spreadsheet to show the depreciation.
 - Write a function that models the depreciation.
 - Sketch a graph of the function.

2.3 Lesson

Core Vocabulary

line of fit, p. 58
 line of best fit, p. 59
 correlation coefficient, p. 59

Previous

slope
 slope-intercept form
 point-slope form
 scatter plot

What You Will Learn

- Write equations of linear functions using points and slopes.
- Find lines of fit and lines of best fit.

Writing Linear Equations

Core Concept

Writing an Equation of a Line

Given slope m and y -intercept b

Use slope-intercept form:

$$y = mx + b$$

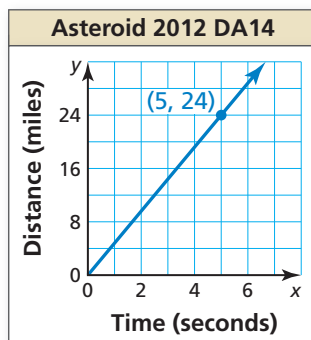
Given slope m and a point (x_1, y_1)

Use point-slope form:

$$y - y_1 = m(x - x_1)$$

Given points (x_1, y_1) and (x_2, y_2)

First use the slope formula to find m .
 Then use point-slope form with either given point.



EXAMPLE 1 Writing a Linear Equation from a Graph

The graph shows the distance Asteroid 2012 DA14 travels in x seconds. Write an equation of the line and interpret the slope. The asteroid came within 17,200 miles of Earth in February, 2013. About how long does it take the asteroid to travel that distance?

SOLUTION

From the graph, you can see the slope is $m = \frac{24}{5} = 4.8$ and the y -intercept is $b = 0$. Use slope-intercept form to write an equation of the line.

$$\begin{aligned} y &= mx + b && \text{Slope-intercept form} \\ &= 4.8x + 0 && \text{Substitute 4.8 for } m \text{ and 0 for } b. \end{aligned}$$

The equation is $y = 4.8x$. The slope indicates that the asteroid travels 4.8 miles per second. Use the equation to find how long it takes the asteroid to travel 17,200 miles.

$$\begin{aligned} 17,200 &= 4.8x && \text{Substitute 17,200 for } y. \\ 3583 &\approx x && \text{Divide each side by 4.8.} \end{aligned}$$

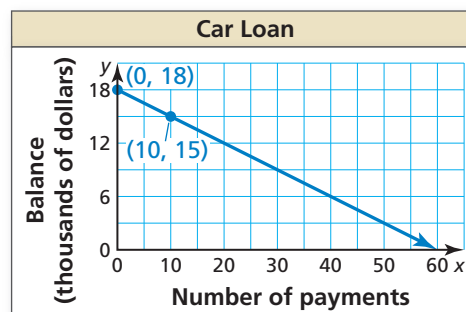
- Because there are 3600 seconds in 1 hour, it takes the asteroid about 1 hour to travel 17,200 miles.

REMEMBER

An equation of the form $y = mx$ indicates that x and y are in a proportional relationship.

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- The graph shows the remaining balance y on a car loan after making x monthly payments. Write an equation of the line and interpret the slope and y -intercept. What is the remaining balance after 36 payments?



EXAMPLE 2 Modeling with Mathematics

Lakeside Inn	
Number of students, x	Total cost, y
100	\$1500
125	\$1800
150	\$2100
175	\$2400
200	\$2700



Two prom venues charge a rental fee plus a fee per student. The table shows the total costs for different numbers of students at Lakeside Inn. The total cost y (in dollars) for x students at Sunview Resort is represented by the equation

$$y = 10x + 600.$$

Which venue charges less per student? How many students must attend for the total costs to be the same?

SOLUTION

- 1. Understand the Problem** You are given an equation that represents the total cost at one venue and a table of values showing total costs at another venue. You need to compare the costs.
- 2. Make a Plan** Write an equation that models the total cost at Lakeside Inn. Then compare the slopes to determine which venue charges less per student. Finally, equate the cost expressions and solve to determine the number of students for which the total costs are equal.
- 3. Solve the Problem** First find the slope using any two points from the table. Use $(x_1, y_1) = (100, 1500)$ and $(x_2, y_2) = (125, 1800)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1800 - 1500}{125 - 100} = \frac{300}{25} = 12$$

Write an equation that represents the total cost at Lakeside Inn using the slope of 12 and a point from the table. Use $(x_1, y_1) = (100, 1500)$.

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$

$$y - 1500 = 12(x - 100) \quad \text{Substitute for } m, x_1, \text{ and } y_1.$$

$$y - 1500 = 12x - 1200 \quad \text{Distributive Property}$$

$$y = 12x + 300 \quad \text{Add 1500 to each side.}$$

Equate the cost expressions and solve.

$$10x + 600 = 12x + 300 \quad \text{Set cost expressions equal.}$$

$$300 = 2x \quad \text{Combine like terms.}$$

$$150 = x \quad \text{Divide each side by 2.}$$

- ▶ Comparing the slopes of the equations, Sunview Resort charges \$10 per student, which is less than the \$12 per student that Lakeside Inn charges. The total costs are the same for 150 students.

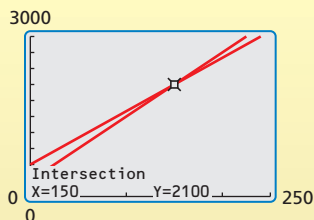
- 4. Look Back** Notice that the table shows the total cost for 150 students at Lakeside Inn is \$2100. To check that your solution is correct, verify that the total cost at Sunview Resort is also \$2100 for 150 students.

$$y = 10(150) + 600 \quad \text{Substitute 150 for } x.$$

$$= 2100 \quad \checkmark \quad \text{Simplify.}$$

Check

Another way to check your solution is to graph each equation and find the point of intersection. The x -value of the point of intersection is 150.



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- 2. WHAT IF?** Maple Ridge charges a rental fee plus a \$10 fee per student. The total cost is \$1900 for 140 students. Describe the number of students that must attend for the total cost at Maple Ridge to be less than the total costs at the other two venues. Use a graph to justify your answer.

Finding Lines of Fit and Lines of Best Fit

Data do not always show an *exact* linear relationship. When the data in a scatter plot show an approximately linear relationship, you can model the data with a **line of fit**.

Core Concept

Finding a Line of Fit

- Step 1** Create a scatter plot of the data.
- Step 2** Sketch the line that most closely appears to follow the trend given by the data points. There should be about as many points above the line as below it.
- Step 3** Choose two points on the line and estimate the coordinates of each point. These points do not have to be original data points.
- Step 4** Write an equation of the line that passes through the two points from Step 3. This equation is a model for the data.

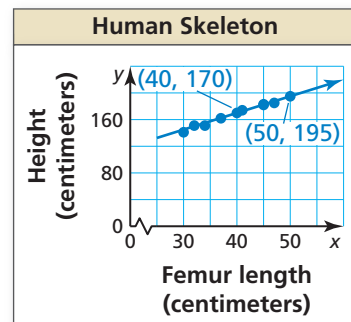
EXAMPLE 3 Finding a Line of Fit

Femur length, x	Height, y
40	170
45	183
32	151
50	195
37	162
41	174
30	141
34	151
47	185
45	182

The table shows the femur lengths (in centimeters) and heights (in centimeters) of several people. Do the data show a linear relationship? If so, write an equation of a line of fit and use it to estimate the height of a person whose femur is 35 centimeters long.

SOLUTION

- Step 1** Create a scatter plot of the data.
The data show a linear relationship.
- Step 2** Sketch the line that most closely appears to fit the data. One possibility is shown.
- Step 3** Choose two points on the line.
For the line shown, you might choose (40, 170) and (50, 195).
- Step 4** Write an equation of the line.



First, find the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{195 - 170}{50 - 40} = \frac{25}{10} = 2.5$$

Use point-slope form to write an equation. Use $(x_1, y_1) = (40, 170)$.

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$

$$y - 170 = 2.5(x - 40) \quad \text{Substitute for } m, x_1, \text{ and } y_1.$$

$$y - 170 = 2.5x - 100 \quad \text{Distributive Property}$$

$$y = 2.5x + 70 \quad \text{Add 170 to each side.}$$

Use the equation to estimate the height of the person.

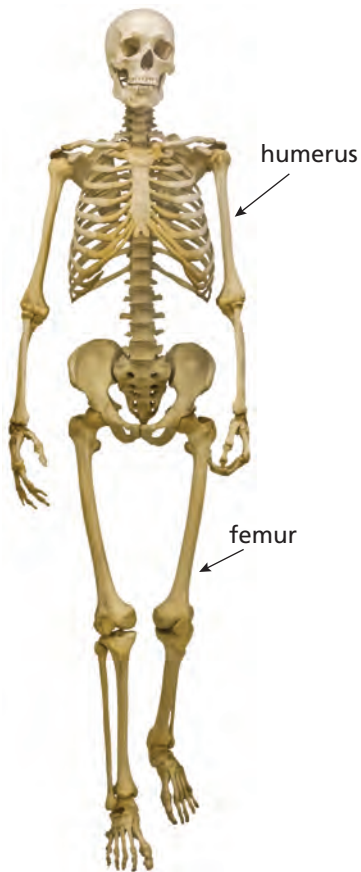
$$y = 2.5(35) + 70 \quad \text{Substitute 35 for } x.$$

$$= 157.5 \quad \text{Simplify.}$$

- ▶ The approximate height of a person with a 35-centimeter femur is 157.5 centimeters.

The **line of best fit** is the line that lies as close as possible to all of the data points. Many technology tools have a *linear regression* feature that you can use to find the line of best fit for a set of data.

The **correlation coefficient**, denoted by r , is a number from -1 to 1 that measures how well a line fits a set of data pairs (x, y) . When r is near 1 , the points lie close to a line with a positive slope. When r is near -1 , the points lie close to a line with a negative slope. When r is near 0 , the points do not lie close to any line.



EXAMPLE 4 Using a Graphing Calculator

Use the *linear regression* feature on a graphing calculator to find an equation of the line of best fit for the data in Example 3. Estimate the height of a person whose femur is 35 centimeters long. Compare this height to your estimate in Example 3.

SOLUTION

Step 1 Enter the data into two lists.

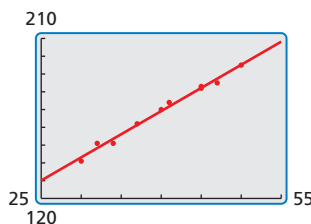
L1	L2	L3
40	170	---
45	183	---
32	151	---
50	195	---
37	162	---
41	174	---
30	141	---
L1(1)=40		

Step 2 Use the *linear regression* feature. The line of best fit is $y = 2.6x + 65$.

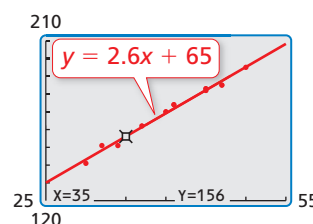
```
LinReg
y=ax+b
a=2.603570555
b=64.99682074
r^2=.9890669473
r=.9945184499
```

The value of r is close to 1.

Step 3 Graph the regression equation with the scatter plot.



Step 4 Use the *trace* feature to find the value of y when $x = 35$.



ATTENDING TO PRECISION

Be sure to analyze the data values to help you select an appropriate viewing window for your graph.

The approximate height of a person with a 35-centimeter femur is 156 centimeters. This is less than the estimate found in Example 3.

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3. The table shows the humerus lengths (in centimeters) and heights (in centimeters) of several females.

Humerus length, x	33	25	22	30	28	32	26	27
Height, y	166	142	130	154	152	159	141	145

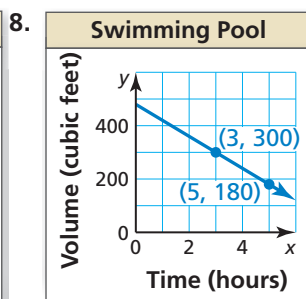
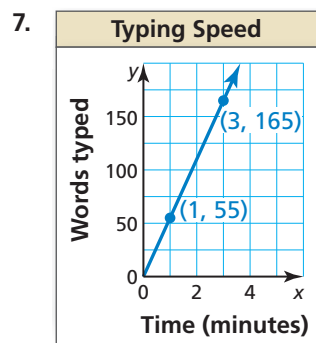
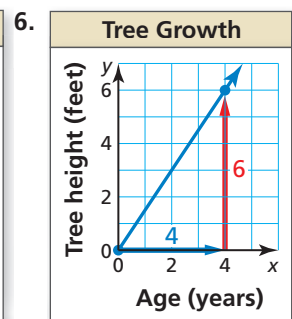
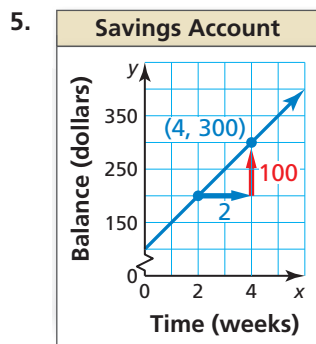
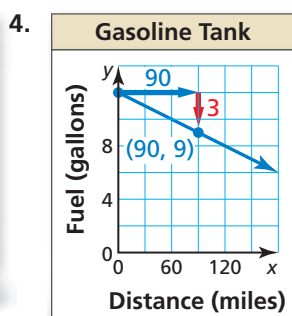
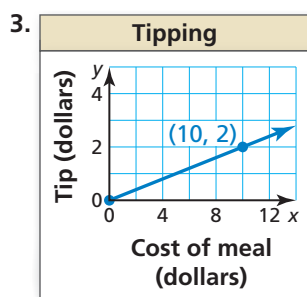
- Do the data show a linear relationship? If so, write an equation of a line of fit and use it to estimate the height of a female whose humerus is 40 centimeters long.
- Use the *linear regression* feature on a graphing calculator to find an equation of the line of best fit for the data. Estimate the height of a female whose humerus is 40 centimeters long. Compare this height to your estimate in part (a).

Vocabulary and Core Concept Check

- COMPLETE THE SENTENCE** The linear equation $y = \frac{1}{2}x + 3$ is written in _____ form.
- VOCABULARY** A line of best fit has a correlation coefficient of -0.98 . What can you conclude about the slope of the line?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, use the graph to write an equation of the line and interpret the slope. (See Example 1.)



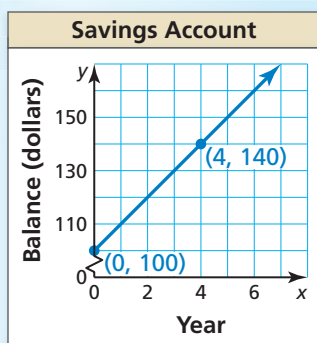
9. **MODELING WITH MATHEMATICS** Two newspapers charge a fee for placing an advertisement in their paper plus a fee based on the number of lines in the advertisement. The table shows the total costs for different length advertisements at the Daily Times. The total cost y (in dollars) for an advertisement that is x lines long at the Greenville Journal is represented by the equation $y = 2x + 20$. Which newspaper charges less per line? How many lines must be in an advertisement for the total costs to be the same? (See Example 2.)

Daily Times	
Number of lines, x	Total cost, y
4	27
5	30
6	33
7	36
8	39

10. **PROBLEM SOLVING** While on vacation in Canada, you notice that temperatures are reported in degrees Celsius. You know there is a linear relationship between Fahrenheit and Celsius, but you forget the formula. From science class, you remember the freezing point of water is 0°C or 32°F , and its boiling point is 100°C or 212°F .
- Write an equation that represents degrees Fahrenheit in terms of degrees Celsius.
 - The temperature outside is 22°C . What is this temperature in degrees Fahrenheit?
 - Rewrite your equation in part (a) to represent degrees Celsius in terms of degrees Fahrenheit.
 - The temperature of the hotel pool water is 83°F . What is this temperature in degrees Celsius?

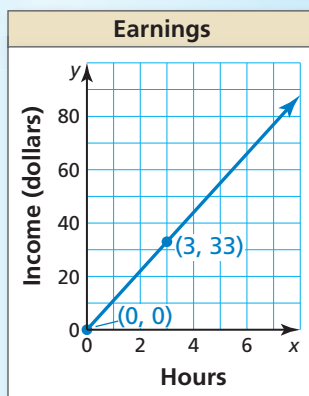
ERROR ANALYSIS In Exercises 11 and 12, describe and correct the error in interpreting the slope in the context of the situation.

11.



The slope of the line is 10, so after 7 years, the balance is \$70.

12.



The slope is 3, so the income is \$3 per hour.

In Exercises 13–16, determine whether the data show a linear relationship. If so, write an equation of a line of fit. Estimate y when $x = 15$ and explain its meaning in the context of the situation. (See Example 3.)

13.	Minutes walking, x	1	6	11	13	16
	Calories burned, y	6	27	50	56	70

14.	Months, x	9	13	18	22	23
	Hair length (in.), y	3	5	7	10	11

15.	Hours, x	3	7	9	17	20
	Battery life (%), y	86	61	50	26	0

16.	Shoe size, x	6	8	8.5	10	13
	Heart rate (bpm), y	112	94	100	132	87

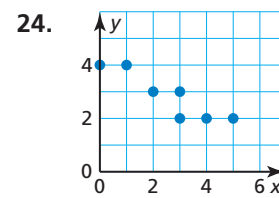
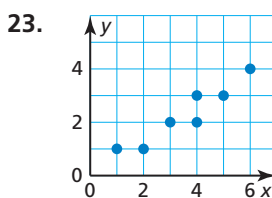
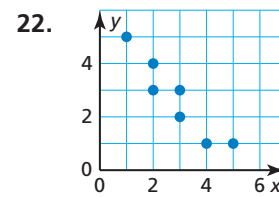
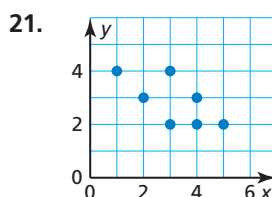
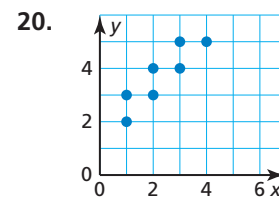
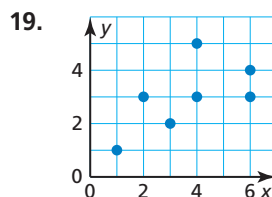
17. MODELING WITH MATHEMATICS The data pairs (x, y) represent the average annual tuition y (in dollars) for public colleges in the United States x years after 2005. Use the *linear regression* feature on a graphing calculator to find an equation of the line of best fit. Estimate the average annual tuition in 2020. Interpret the slope and y -intercept in this situation. (See Example 4.)

- $(0, 11,386), (1, 11,731), (2, 11,848)$
 $(3, 12,375), (4, 12,804), (5, 13,297)$

18. MODELING WITH MATHEMATICS The table shows the numbers of tickets sold for a concert when different prices are charged. Write an equation of a line of fit for the data. Does it seem reasonable to use your model to predict the number of tickets sold when the ticket price is \$85? Explain.

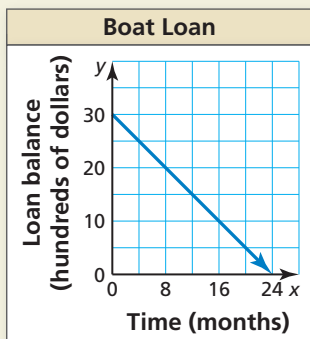
Ticket price (dollars), x	17	20	22	26
Tickets sold, y	450	423	400	395

USING TOOLS In Exercises 19–24, use the *linear regression* feature on a graphing calculator to find an equation of the line of best fit for the data. Find and interpret the correlation coefficient.



25. OPEN-ENDED Give two real-life quantities that have (a) a positive correlation, (b) a negative correlation, and (c) approximately no correlation. Explain.

26. **HOW DO YOU SEE IT?** You secure an interest-free loan to purchase a boat. You agree to make equal monthly payments for the next two years. The graph shows the amount of money you still owe.



- What is the slope of the line? What does the slope represent?
- What is the domain and range of the function? What does each represent?
- How much do you still owe after making payments for 12 months?

27. **MAKING AN ARGUMENT** A set of data pairs has a correlation coefficient $r = 0.3$. Your friend says that because the correlation coefficient is positive, it is logical to use the line of best fit to make predictions. Is your friend correct? Explain your reasoning.

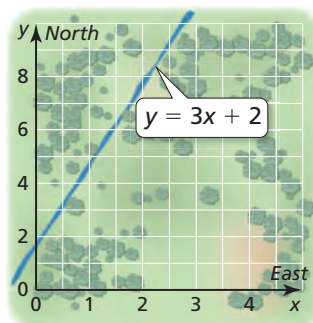
28. **THOUGHT PROVOKING** Points A and B lie on the line $y = -x + 4$. Choose coordinates for points A , B , and C where point C is the same distance from point A as it is from point B . Write equations for the lines connecting points A and C and points B and C .

29. **ABSTRACT REASONING** If x and y have a positive correlation, and y and z have a negative correlation, then what can you conclude about the correlation between x and z ? Explain.

30. **MATHEMATICAL CONNECTIONS** Which equation has a graph that is a line passing through the point $(8, -5)$ and is perpendicular to the graph of $y = -4x + 1$?

- (A) $y = \frac{1}{4}x - 5$ (B) $y = -4x + 27$
 (C) $y = -\frac{1}{4}x - 7$ (D) $y = \frac{1}{4}x - 7$

31. **PROBLEM SOLVING** You are participating in an orienteering competition. The diagram shows the position of a river that cuts through the woods. You are currently 2 miles east and 1 mile north of your starting point, the origin. What is the shortest distance you must travel to reach the river?



32. **ANALYZING RELATIONSHIPS** Data from North American countries show a positive correlation between the number of personal computers per capita and the average life expectancy in the country.

- Does a positive correlation make sense in this situation? Explain.
- Is it reasonable to conclude that giving residents of a country personal computers will lengthen their lives? Explain.



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the system of linear equations in two variables by elimination or substitution.

(Skills Review Handbook)

33. $3x + y = 7$
 $-2x - y = 9$

34. $4x + 3y = 2$
 $2x - 3y = 1$

35. $2x + 2y = 3$
 $x = 4y - 1$

36. $y = 1 + x$
 $2x + y = -2$

37. $\frac{1}{2}x + 4y = 4$
 $2x - y = 1$

38. $y = x - 4$
 $4x + y = 26$

2.4 Solving Linear Systems

Essential Question How can you determine the number of solutions of a linear system?

A linear system is *consistent* when it has at least one solution. A linear system is *inconsistent* when it has no solution.

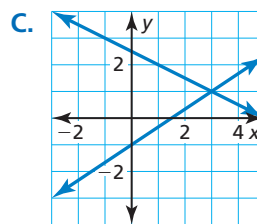
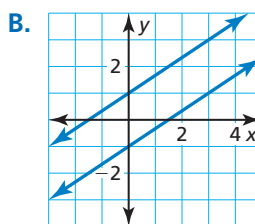
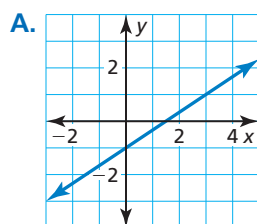
EXPLORATION 1 Recognizing Graphs of Linear Systems

Work with a partner. Match each linear system with its corresponding graph. Explain your reasoning. Then classify the system as *consistent* or *inconsistent*.

a. $2x - 3y = 3$
 $-4x + 6y = 6$

b. $2x - 3y = 3$
 $x + 2y = 5$

c. $2x - 3y = 3$
 $-4x + 6y = -6$



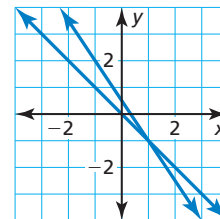
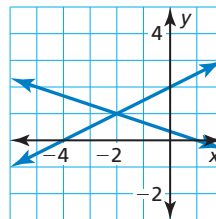
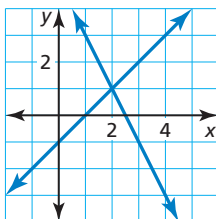
EXPLORATION 2 Solving Systems of Linear Equations

Work with a partner. Solve each linear system by substitution or elimination. Then use the graph of the system below to check your solution.

a. $2x + y = 5$
 $x - y = 1$

b. $x + 3y = 1$
 $-x + 2y = 4$

c. $x + y = 0$
 $3x + 2y = 1$



FINDING AN ENTRY POINT

To be proficient in math, you need to look for entry points to the solution of a problem.

Communicate Your Answer

- How can you determine the number of solutions of a linear system?
- Suppose you were given a system of *three* linear equations in *three* variables. Explain how you would approach solving such a system.
- Apply your strategy in Question 4 to solve the linear system.

$x + y + z = 1$ Equation 1

$x - y - z = 3$ Equation 2

$-x - y + z = -1$ Equation 3

2.4 Lesson

Core Vocabulary

linear equation in three variables, p. 64
system of three linear equations, p. 64
solution of a system of three linear equations, p. 64
ordered triple, p. 64

Previous

system of two linear equations

What You Will Learn

- ▶ Visualize solutions of systems of linear equations in three variables.
- ▶ Solve systems of linear equations in three variables algebraically.
- ▶ Solve real-life problems.

Visualizing Solutions of Systems

A **linear equation in three variables** x , y , and z is an equation of the form $ax + by + cz = d$, where a , b , and c are not all zero.

The following is an example of a **system of three linear equations** in three variables.

$$3x + 4y - 8z = -3 \quad \text{Equation 1}$$

$$x + y + 5z = -12 \quad \text{Equation 2}$$

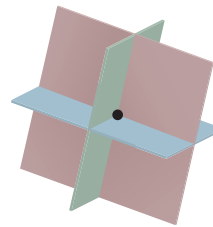
$$4x - 2y + z = 10 \quad \text{Equation 3}$$

A **solution** of such a system is an **ordered triple** (x, y, z) whose coordinates make each equation true.

The graph of a linear equation in three variables is a plane in three-dimensional space. The graphs of three such equations that form a system are three planes whose intersection determines the number of solutions of the system, as shown in the diagrams below.

Exactly One Solution

The planes intersect in a single point, which is the solution of the system.



Infinitely Many Solutions

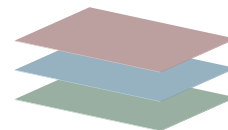
The planes intersect in a line. Every point on the line is a solution of the system.

The planes could also be the same plane. Every point in the plane is a solution of the system.



No Solution

There are no points in common with all three planes.



Solving Systems of Equations Algebraically

The algebraic methods you used to solve systems of linear equations in two variables can be extended to solve a system of linear equations in three variables.

Core Concept

Solving a Three-Variable System

Step 1 Rewrite the linear system in three variables as a linear system in two variables by using the substitution or elimination method.

Step 2 Solve the new linear system for both of its variables.

Step 3 Substitute the values found in Step 2 into one of the original equations and solve for the remaining variable.

When you obtain a false equation, such as $0 = 1$, in any of the steps, the system has no solution.

When you do not obtain a false equation, but obtain an identity such as $0 = 0$, the system has infinitely many solutions.

LOOKING FOR STRUCTURE

The coefficient of -1 in Equation 3 makes y a convenient variable to eliminate.

EXAMPLE 1

Solving a Three-Variable System (One Solution)

Solve the system.	$4x + 2y + 3z = 12$	Equation 1
	$2x - 3y + 5z = -7$	Equation 2
	$6x - y + 4z = -3$	Equation 3

SOLUTION

Step 1 Rewrite the system as a linear system in *two* variables.

$4x + 2y + 3z = 12$	Add 2 times Equation 3 to Equation 1 (to eliminate y).
$12x - 2y + 8z = -6$	
$16x \quad + 11z = 6$	New Equation 1
$2x - 3y + 5z = -7$	Add -3 times Equation 3 to Equation 2 (to eliminate y).
$-18x + 3y - 12z = 9$	
$-16x \quad - 7z = 2$	New Equation 2

Step 2 Solve the new linear system for both of its variables.

$16x + 11z = 6$	Add new Equation 1 and new Equation 2.
$-16x - 7z = 2$	
$4z = 8$	
$z = 2$	Solve for z .
$x = -1$	Substitute into new Equation 1 or 2 to find x .

Step 3 Substitute $x = -1$ and $z = 2$ into an original equation and solve for y .

$6x - y + 4z = -3$	Write original Equation 3.
$6(-1) - y + 4(2) = -3$	Substitute -1 for x and 2 for z .
$y = 5$	Solve for y .

► The solution is $x = -1$, $y = 5$, and $z = 2$, or the ordered triple $(-1, 5, 2)$. Check this solution in each of the original equations.

ANOTHER WAY

In Step 1, you could also eliminate x to get two equations in y and z , or you could eliminate z to get two equations in x and y .

EXAMPLE 2 Solving a Three-Variable System (No Solution)

Solve the system.

$$\begin{array}{rcl} x + y + z = 2 & & \text{Equation 1} \\ 5x + 5y + 5z = 3 & & \text{Equation 2} \\ 4x + y - 3z = -6 & & \text{Equation 3} \end{array}$$

SOLUTION

Step 1 Rewrite the system as a linear system in *two* variables.

$$\begin{array}{rcl} -5x - 5y - 5z = -10 & & \text{Add } -5 \text{ times Equation 1} \\ 5x + 5y + 5z = 3 & & \text{to Equation 2.} \\ \hline 0 = -7 \end{array}$$

▶ Because you obtain a false equation, the original system has no solution.

ANOTHER WAY

Subtracting Equation 2 from Equation 1 gives $z = 0$. After substituting 0 for z in each equation, you can see that each is equivalent to $y = x + 3$.

**EXAMPLE 3 Solving a Three-Variable System (Many Solutions)**

Solve the system.

$$\begin{array}{rcl} x - y + z = -3 & & \text{Equation 1} \\ x - y - z = -3 & & \text{Equation 2} \\ 5x - 5y + z = -15 & & \text{Equation 3} \end{array}$$

SOLUTION

Step 1 Rewrite the system as a linear system in *two* variables.

$$\begin{array}{rcl} x - y + z = -3 & & \text{Add Equation 1 to} \\ x - y - z = -3 & & \text{Equation 2 (to eliminate } z\text{).} \\ \hline 2x - 2y = -6 & & \text{New Equation 2} \\ x - y - z = -3 & & \text{Add Equation 2 to} \\ 5x - 5y + z = -15 & & \text{Equation 3 (to eliminate } z\text{).} \\ \hline 6x - 6y = -18 & & \text{New Equation 3} \end{array}$$

Step 2 Solve the new linear system for both of its variables.

$$\begin{array}{rcl} -6x + 6y = 18 & & \text{Add } -3 \text{ times new Equation 2} \\ 6x - 6y = -18 & & \text{to new Equation 3.} \\ \hline 0 = 0 \end{array}$$

Because you obtain the identity $0 = 0$, the system has infinitely many solutions.

Step 3 Describe the solutions of the system using an ordered triple. One way to do this is to solve new Equation 2 for y to obtain $y = x + 3$. Then substitute $x + 3$ for y in original Equation 1 to obtain $z = 0$.

▶ So, any ordered triple of the form $(x, x + 3, 0)$ is a solution of the system.

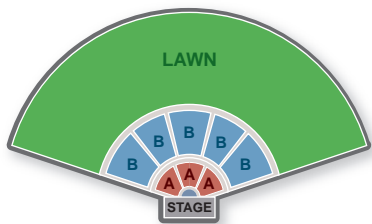
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Solve the system. Check your solution, if possible.

$$\begin{array}{lll} 1. \ x - 2y + z = -11 & 2. \ x + y - z = -1 & 3. \ x + y + z = 8 \\ 3x + 2y - z = 7 & 4x + 4y - 4z = -2 & x - y + z = 8 \\ -x + 2y + 4z = -9 & 3x + 2y + z = 0 & 2x + y + 2z = 16 \end{array}$$

4. In Example 3, describe the solutions of the system using an ordered triple in terms of y .

Solving Real-Life Problems



EXAMPLE 4 Solving a Multi-Step Problem

An amphitheater charges \$75 for each seat in Section A, \$55 for each seat in Section B, and \$30 for each lawn seat. There are three times as many seats in Section B as in Section A. The revenue from selling all 23,000 seats is \$870,000. How many seats are in each section of the amphitheater?

SOLUTION

Step 1 Write a verbal model for the situation.

$$\text{Number of seats in B, } y = 3 \cdot \text{Number of seats in A, } x$$

$$\text{Number of seats in A, } x + \text{Number of seats in B, } y + \text{Number of lawn seats, } z = \text{Total number of seats}$$

$$75 \cdot \text{Number of seats in A, } x + 55 \cdot \text{Number of seats in B, } y + 30 \cdot \text{Number of lawn seats, } z = \text{Total revenue}$$

Step 2 Write a system of equations.

$$y = 3x \quad \text{Equation 1}$$

$$x + y + z = 23,000 \quad \text{Equation 2}$$

$$75x + 55y + 30z = 870,000 \quad \text{Equation 3}$$

Step 3 Rewrite the system in Step 2 as a linear system in *two* variables by substituting $3x$ for y in Equations 2 and 3.

$$x + y + z = 23,000 \quad \text{Write Equation 2.}$$

$$x + 3x + z = 23,000 \quad \text{Substitute } 3x \text{ for } y.$$

$$4x + z = 23,000 \quad \text{New Equation 2}$$

$$75x + 55y + 30z = 870,000 \quad \text{Write Equation 3.}$$

$$75x + 55(3x) + 30z = 870,000 \quad \text{Substitute } 3x \text{ for } y.$$

$$240x + 30z = 870,000 \quad \text{New Equation 3}$$

Step 4 Solve the new linear system for both of its variables.

$$\begin{array}{r} -120x - 30z = -690,000 \\ \underline{240x + 30z = 870,000} \\ 120x = 180,000 \end{array} \quad \begin{array}{l} \text{Add } -30 \text{ times new Equation 2} \\ \text{to new Equation 3.} \end{array}$$

$$x = 1500 \quad \text{Solve for } x.$$

$$y = 4500 \quad \text{Substitute into Equation 1 to find } y.$$

$$z = 17,000 \quad \text{Substitute into Equation 2 to find } z.$$

▶ The solution is $x = 1500$, $y = 4500$, and $z = 17,000$, or $(1500, 4500, 17,000)$. So, there are 1500 seats in Section A, 4500 seats in Section B, and 17,000 lawn seats.

STUDY TIP

When substituting to find values of other variables, choose original or new equations that are easiest to use.



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5. **WHAT IF?** On the first day, 10,000 tickets sold, generating \$356,000 in revenue. The number of seats sold in Sections A and B are the same. How many lawn seats are still available?

2.4 Exercises

Vocabulary and Core Concept Check

- VOCABULARY** The solution of a system of three linear equations is expressed as a(n) _____.
- WRITING** Explain how you know when a linear system in three variables has infinitely many solutions.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, solve the system using the elimination method. (See Example 1.)


- | | |
|---|--|
| 3. $x + y - 2z = 5$
$-x + 2y + z = 2$
$2x + 3y - z = 9$ | 4. $x + 4y - 6z = -1$
$2x - y + 2z = -7$
$-x + 2y - 4z = 5$ |
| 5. $2x + y - z = 9$
$-x + 6y + 2z = -17$
$5x + 7y + z = 4$ | 6. $3x + 2y - z = 8$
$-3x + 4y + 5z = -14$
$x - 3y + 4z = -14$ |
| 7. $2x + 2y + 5z = -1$
$2x - y + z = 2$
$2x + 4y - 3z = 14$ | 8. $3x + 2y - 3z = -2$
$7x - 2y + 5z = -14$
$2x + 4y + z = 6$ |


- | | |
|---|---|
| 13. $x + 3y - z = 2$
$x + y - z = 0$
$3x + 2y - 3z = -1$ | 14. $x + 2y - z = 3$
$-2x - y + z = -1$
$6x - 3y - z = -7$ |
| 15. $x + 2y + 3z = 4$
$-3x + 2y - z = 12$
$-2x - 2y - 4z = -14$ | 16. $-2x - 3y + z = -6$
$x + y - z = 5$
$7x + 8y - 6z = 31$ |

17. **MODELING WITH MATHEMATICS** Three orders are placed at a pizza shop. Two small pizzas, a liter of soda, and a salad cost \$14; one small pizza, a liter of soda, and three salads cost \$15; and three small pizzas, a liter of soda, and two salads cost \$22. How much does each item cost?

ERROR ANALYSIS In Exercises 9 and 10, describe and correct the error in the first step of solving the system of linear equations.

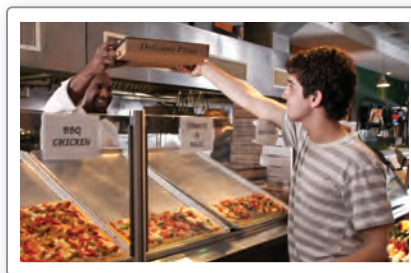
$$\begin{aligned} 4x - y + 2z &= -18 \\ -x + 2y + z &= 11 \\ 3x + 3y - 4z &= 44 \end{aligned}$$

9. 
$$\begin{aligned} 4x - y + 2z &= -18 \\ -4x + 2y + z &= 11 \\ \hline y + 3z &= -7 \end{aligned}$$

10. 
$$\begin{aligned} 12x - 3y + 6z &= -18 \\ 3x + 3y - 4z &= 44 \\ \hline 15x + 2z &= 26 \end{aligned}$$

In Exercises 11–16, solve the system using the elimination method. (See Examples 2 and 3.)

- | | |
|--|--|
| 11. $3x - y + 2z = 4$
$6x - 2y + 4z = -8$
$2x - y + 3z = 10$ | 12. $5x + y - z = 6$
$x + y + z = 2$
$12x + 4y = 10$ |
|--|--|



18. **MODELING WITH MATHEMATICS** Sam's Furniture Store places the following advertisement in the local newspaper. Write a system of equations for the three combinations of furniture. What is the price of each piece of furniture? Explain.

SALE

SAM'S Furniture Store

<p>\$1300 Sofa and love seat</p> <p>\$1400 Sofa and two chairs</p> <p>\$1600 Sofa, love seat, and one chair</p>	  
--	---

In Exercises 19–28, solve the system of linear equations using the substitution method. (See Example 4.)

19. $-2x + y + 6z = 1$
 $3x + 2y + 5z = 16$
 $7x + 3y - 4z = 11$
20. $x - 6y - 2z = -8$
 $-x + 5y + 3z = 2$
 $3x - 2y - 4z = 18$
21. $x + y + z = 4$
 $5x + 5y + 5z = 12$
 $x - 4y + z = 9$
22. $x + 2y = -1$
 $-x + 3y + 2z = -4$
 $-x + y - 4z = 10$
23. $2x - 3y + z = 10$
 $y + 2z = 13$
 $z = 5$
24. $x = 4$
 $x + y = -6$
 $4x - 3y + 2z = 26$
25. $x + y - z = 4$
 $3x + 2y + 4z = 17$
 $-x + 5y + z = 8$
26. $2x - y - z = 15$
 $4x + 5y + 2z = 10$
 $-x - 4y + 3z = -20$
27. $4x + y + 5z = 5$
 $8x + 2y + 10z = 10$
 $x - y - 2z = -2$
28. $x + 2y - z = 3$
 $2x + 4y - 2z = 6$
 $-x - 2y + z = -6$

29. **PROBLEM SOLVING** The number of left-handed people in the world is one-tenth the number of right-handed people. The percent of right-handed people is nine times the percent of left-handed people and ambidextrous people combined. What percent of people are ambidextrous?



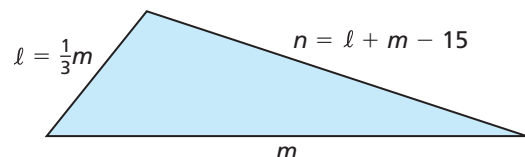
30. **MODELING WITH MATHEMATICS** Use a system of linear equations to model the data in the following newspaper article. Solve the system to find how many athletes finished in each place.

Lawrence High prevailed in Saturday's track meet with the help of 20 individual-event placers earning a combined 68 points. A first-place finish earns 5 points, a second-place finish earns 3 points, and a third-place finish earns 1 point. Lawrence had a strong second-place showing, with as many second place finishers as first- and third-place finishers combined.

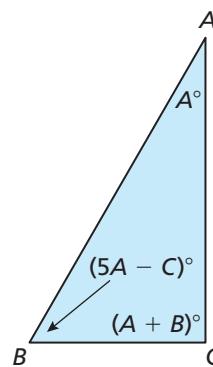
31. **WRITING** Explain when it might be more convenient to use the elimination method than the substitution method to solve a linear system. Give an example to support your claim.
32. **REPEATED REASONING** Using what you know about solving linear systems in two and three variables, plan a strategy for how you would solve a system that has four linear equations in four variables.

MATHEMATICAL CONNECTIONS In Exercises 33 and 34, write and use a linear system to answer the question.

33. The triangle has a perimeter of 65 feet. What are the lengths of sides ℓ , m , and n ?



34. What are the measures of angles A , B , and C ?



35. **OPEN-ENDED** Consider the system of linear equations below. Choose nonzero values for a , b , and c so the system satisfies the given condition. Explain your reasoning.

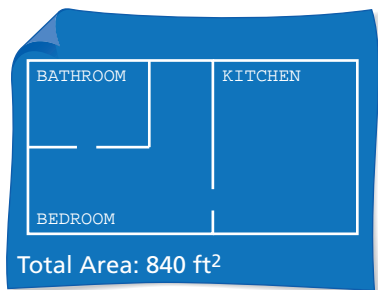
$$x + y + z = 2$$

$$ax + by + cz = 10$$

$$x - 2y + z = 4$$

- a. The system has no solution.
 b. The system has exactly one solution.
 c. The system has infinitely many solutions.
36. **MAKING AN ARGUMENT** A linear system in three variables has no solution. Your friend concludes that it is not possible for two of the three equations to have any points in common. Is your friend correct? Explain your reasoning.

- 37. PROBLEM SOLVING** A contractor is hired to build an apartment complex. Each 840-square-foot unit has a bedroom, kitchen, and bathroom. The bedroom will be the same size as the kitchen. The owner orders 980 square feet of tile to completely cover the floors of two kitchens and two bathrooms. Determine how many square feet of carpet is needed for each bedroom.



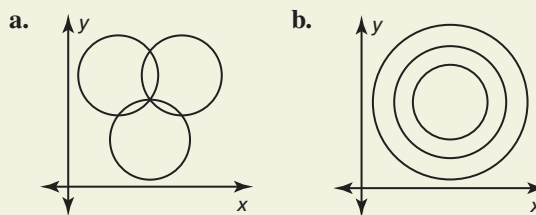
- 38. THOUGHT PROVOKING** Does the system of linear equations have more than one solution? Justify your answer.

$$\begin{aligned} 4x + y + z &= 0 \\ 2x + \frac{1}{2}y - 3z &= 0 \\ -x - \frac{1}{4}y - z &= 0 \end{aligned}$$

- 39. PROBLEM SOLVING** A florist must make 5 identical bridesmaid bouquets for a wedding. The budget is \$160, and each bouquet must have 12 flowers. Roses cost \$2.50 each, lilies cost \$4 each, and irises cost \$2 each. The florist wants twice as many roses as the other two types of flowers combined.

- Write a system of equations to represent this situation, assuming the florist plans to use the maximum budget.
- Solve the system to find how many of each type of flower should be in each bouquet.
- Suppose there is no limitation on the total cost of the bouquets. Does the problem still have exactly one solution? If so, find the solution. If not, give three possible solutions.

- 40. HOW DO YOU SEE IT?** Determine whether the system of equations that represents the circles has *no solution*, *one solution*, or *infinitely many solutions*. Explain your reasoning.



- 41. CRITICAL THINKING** Find the values of a , b , and c so that the linear system shown has $(-1, 2, -3)$ as its only solution. Explain your reasoning.

$$\begin{aligned} x + 2y - 3z &= a \\ -x - y + z &= b \\ 2x + 3y - 2z &= c \end{aligned}$$

- 42. ANALYZING RELATIONSHIPS** Determine which arrangement(s) of the integers -5 , 2 , and 3 produce a solution of the linear system that consist of only integers. Justify your answer.

$$\begin{aligned} x - 3y + 6z &= 21 \\ __x + __y + __z &= -30 \\ 2x - 5y + 2z &= -6 \end{aligned}$$

- 43. ABSTRACT REASONING** Write a linear system to represent the first three pictures below. Use the system to determine how many tangerines are required to balance the apple in the fourth picture. *Note:* The first picture shows that one tangerine and one apple balance one grapefruit.



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify. (*Skills Review Handbook*)

44. $(x - 2)^2$

45. $(3m + 1)^2$

46. $(2z - 5)^2$

47. $(4 - y)^2$

Write a function g described by the given transformation of $f(x) = |x| - 5$. (*Section 2.2*)

48. translation 2 units to the left

49. reflection in the x -axis

50. translation 4 units up

51. vertical stretch by a factor of 3

2.1–2.4 What Did You Learn?

Core Vocabulary

parent function, *p. 40*
transformation, *p. 41*
translation, *p. 41*
reflection, *p. 41*
vertical stretch, *p. 42*
vertical shrink, *p. 42*

line of fit, *p. 58*
line of best fit, *p. 59*
correlation coefficient, *p. 59*
linear equation in three variables,
p. 64

system of three linear equations,
p. 64
solution of a system of three linear
equations, *p. 64*
ordered triple, *p. 64*

Core Concepts

Section 2.1

Parent Functions, *p. 40*

Describing Transformations, *p. 41*

Section 2.2

Horizontal Translations, *p. 48*
Vertical Translations, *p. 48*
Reflections in the *x*-Axis, *p. 49*

Reflections in the *y*-Axis, *p. 49*
Horizontal Stretches and Shrinks, *p. 50*
Vertical Stretches and Shrinks, *p. 50*

Section 2.3

Writing an Equation of a Line, *p. 56*

Finding a Line of Fit, *p. 58*

Section 2.4

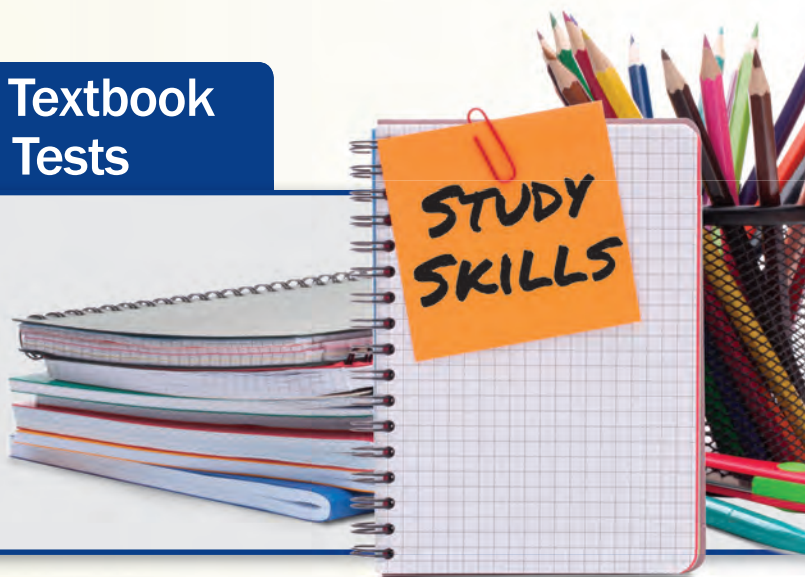
Solving a Three-Variable System, *p. 65*

Mathematical Practices

1. Explain how you would round your answer in Exercise 10 on page 52 if the extra expense is \$13,500.
2. Describe how you can write the equation of the line in Exercise 7 on page 60 using only one of the labeled points.

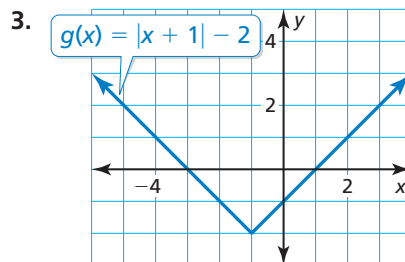
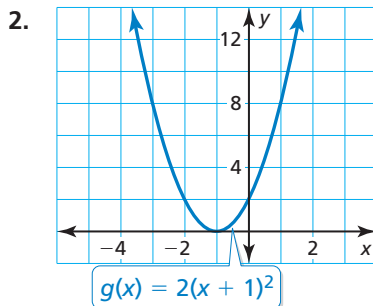
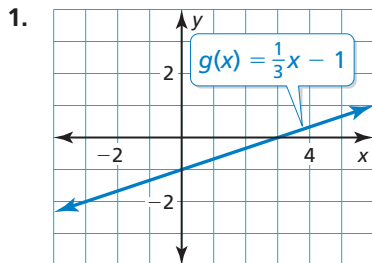
Using the Features of Your Textbook to Prepare for Quizzes and Tests

- Read and understand the core vocabulary and the contents of the Core Concept boxes.
- Review the Examples and the Monitoring Progress questions. Use the tutorials at BigIdeasMath.com for additional help.
- Review previously completed homework assignments.



2.1–2.4 Quiz

Identify the function family to which g belongs. Compare the graph of the function to the graph of its parent function. (Section 2.1)



Graph the function and its parent function. Then describe the transformation(s). (Section 2.1)

4. $f(x) = 3x$

5. $f(x) = -|x + 2| - 7$

6. $f(x) = \frac{1}{4}x^2 + 1$

Write a function g whose graph represents the indicated transformation(s) of the graph of f . (Section 2.2)

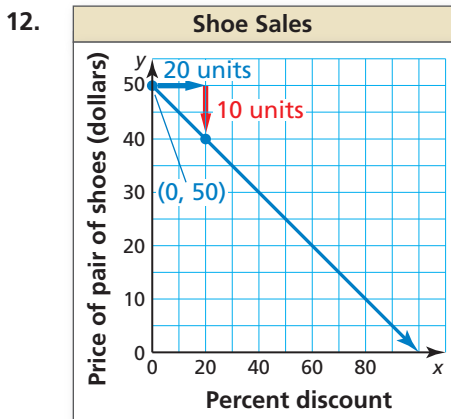
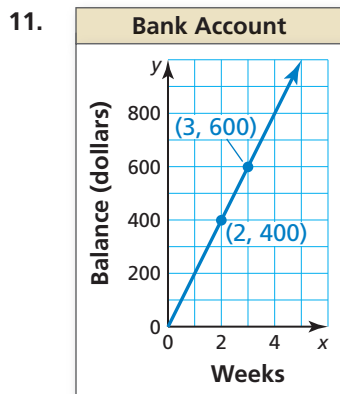
7. $f(x) = 2x + 1$; translation 3 units up

8. $f(x) = -3|x - 4|$; vertical shrink by a factor of $\frac{1}{2}$

9. $f(x) = |x|$; reflection in the x -axis and a vertical stretch by a factor of 4 followed by a translation 7 units down and 1 unit right

10. The total cost of an annual pass plus camping for x days in a National Park can be modeled by the function $f(x) = 20x + 80$. Senior citizens pay half of this price and receive an additional \$30 discount. Describe how to transform the graph of f to model the total cost for a senior citizen. What is the total cost for a senior citizen to go camping for three days? (Section 2.2)

Write an equation of the line and interpret the slope and y -intercept. (Section 2.3)



13. A bakery sells doughnuts, muffins, and bagels. The bakery makes three times as many doughnuts as bagels. The bakery earns a total of \$150 when all 130 baked items in stock are sold. How many of each item are in stock? Justify your answer. (Section 2.4)

Breakfast Specials

Doughnuts.....	\$1.00	
Muffins	\$1.50	
Bagels.....	\$1.20	

2.5 Transformations of Quadratic Functions

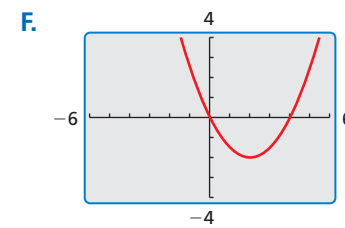
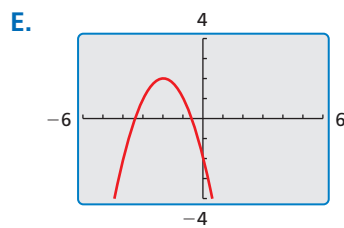
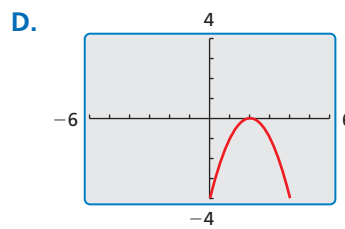
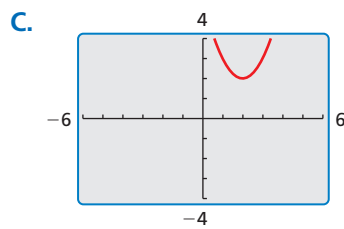
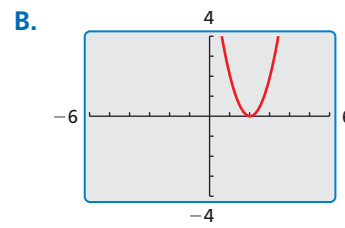
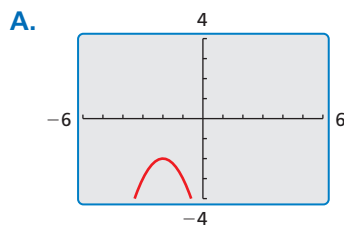
Essential Question How do the constants a , h , and k affect the graph of the quadratic function $g(x) = a(x - h)^2 + k$?

The parent function of the quadratic family is $f(x) = x^2$. A transformation of the graph of the parent function is represented by the function $g(x) = a(x - h)^2 + k$, where $a \neq 0$.

EXPLORATION 1 Identifying Graphs of Quadratic Functions

Work with a partner. Match each quadratic function with its graph. Explain your reasoning. Then use a graphing calculator to verify that your answer is correct.

- a. $g(x) = -(x - 2)^2$ b. $g(x) = (x - 2)^2 + 2$ c. $g(x) = -(x + 2)^2 - 2$
 d. $g(x) = 0.5(x - 2)^2 - 2$ e. $g(x) = 2(x - 2)^2$ f. $g(x) = -(x + 2)^2 + 2$

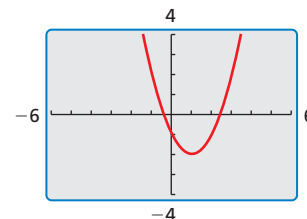


LOOKING FOR STRUCTURE

To be proficient in math, you need to look closely to discern a pattern or structure.

Communicate Your Answer

- How do the constants a , h , and k affect the graph of the quadratic function $g(x) = a(x - h)^2 + k$?
- Write the equation of the quadratic function whose graph is shown at the right. Explain your reasoning. Then use a graphing calculator to verify that your equation is correct.



2.5 Lesson

Core Vocabulary

quadratic function, p. 74
 parabola, p. 74
 vertex of a parabola, p. 76
 vertex form, p. 76

Previous
 transformations

What You Will Learn

- Describe transformations of quadratic functions.
- Write transformations of quadratic functions.

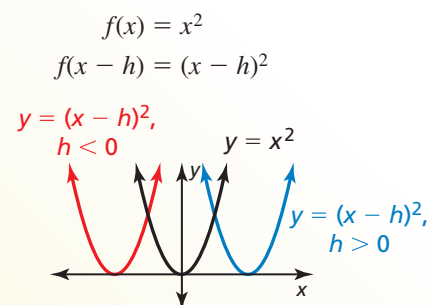
Describing Transformations of Quadratic Functions

A **quadratic function** is a function that can be written in the form $f(x) = a(x - h)^2 + k$, where $a \neq 0$. The U-shaped graph of a quadratic function is called a **parabola**.

In Section 2.1, you graphed quadratic functions using tables of values. You can also graph quadratic functions by applying transformations to the graph of the parent function $f(x) = x^2$.

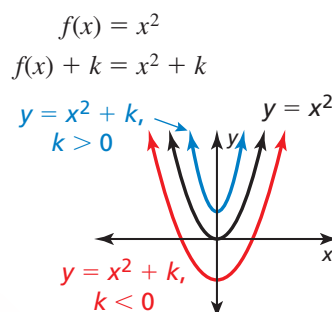
Core Concept

Horizontal Translations



- shifts left when $h < 0$
- shifts right when $h > 0$

Vertical Translations



- shifts down when $k < 0$
- shifts up when $k > 0$

EXAMPLE 1 Translations of a Quadratic Function

Describe the transformation of $f(x) = x^2$ represented by $g(x) = (x + 4)^2 - 1$. Then graph each function.

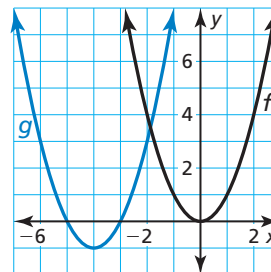
SOLUTION

Notice that the function is of the form $g(x) = (x - h)^2 + k$. Rewrite the function to identify h and k .

$$g(x) = (x - (-4))^2 + (-1)$$

↑ ↑
 h k

- Because $h = -4$ and $k = -1$, the graph of g is a translation 4 units left and 1 unit down of the graph of f .



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Describe the transformation of $f(x) = x^2$ represented by g . Then graph each function.

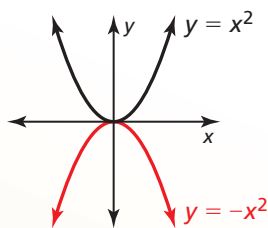
- $g(x) = (x - 3)^2$
- $g(x) = (x - 2)^2 - 2$
- $g(x) = (x + 5)^2 + 1$

Core Concept

Reflections in the x-Axis

$$f(x) = x^2$$

$$-f(x) = -(x^2) = -x^2$$

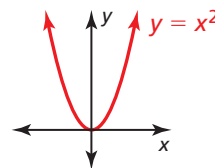


flips over the x -axis

Reflections in the y-Axis

$$f(x) = x^2$$

$$f(-x) = (-x)^2 = x^2$$

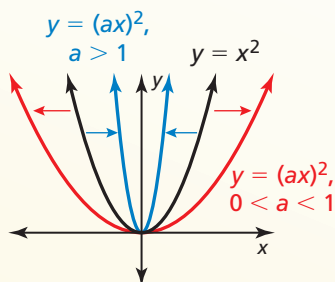


$y = x^2$ is its own reflection in the y -axis.

Horizontal Stretches and Shrinks

$$f(x) = x^2$$

$$f(ax) = (ax)^2$$

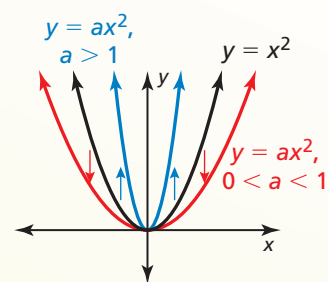


- horizontal stretch (away from y -axis) when $0 < a < 1$
- horizontal shrink (toward y -axis) when $a > 1$

Vertical Stretches and Shrinks

$$f(x) = x^2$$

$$a \cdot f(x) = ax^2$$



- vertical stretch (away from x -axis) when $a > 1$
- vertical shrink (toward x -axis) when $0 < a < 1$

EXAMPLE 2

Transformations of Quadratic Functions

Describe the transformation of $f(x) = x^2$ represented by g . Then graph each function.

a. $g(x) = -\frac{1}{2}x^2$

b. $g(x) = (2x)^2 + 1$

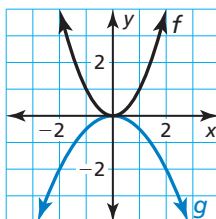
LOOKING FOR STRUCTURE

In Example 2b, notice that $g(x) = 4x^2 + 1$. So, you can also describe the graph of g as a vertical stretch by a factor of 4 followed by a translation 1 unit up of the graph of f .

SOLUTION

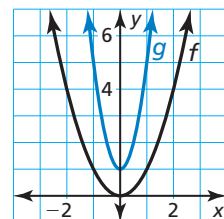
a. Notice that the function is of the form $g(x) = -ax^2$, where $a = \frac{1}{2}$.

- So, the graph of g is a reflection in the x -axis and a vertical shrink by a factor of $\frac{1}{2}$ of the graph of f .



b. Notice that the function is of the form $g(x) = (ax)^2 + k$, where $a = 2$ and $k = 1$.

- So, the graph of g is a horizontal shrink by a factor of $\frac{1}{2}$ followed by a translation 1 unit up of the graph of f .



Monitoring Progress



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Describe the transformation of $f(x) = x^2$ represented by g . Then graph each function.

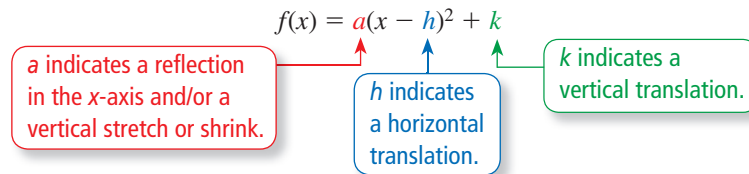
4. $g(x) = \left(\frac{1}{3}x\right)^2$

5. $g(x) = 3(x - 1)^2$

6. $g(x) = -(x + 3)^2 + 2$

Writing Transformations of Quadratic Functions

The lowest point on a parabola that opens up or the highest point on a parabola that opens down is the **vertex**. The **vertex form** of a quadratic function is $f(x) = a(x - h)^2 + k$, where $a \neq 0$ and the vertex is (h, k) .



EXAMPLE 3 Writing a Transformed Quadratic Function

Let the graph of g be a vertical stretch by a factor of 2 and a reflection in the x -axis, followed by a translation 3 units down of the graph of $f(x) = x^2$. Write a rule for g and identify the vertex.

SOLUTION

Method 1 Identify how the transformations affect the constants in vertex form.

$$\left. \begin{array}{l} \text{reflection in } x\text{-axis} \\ \text{vertical stretch by } 2 \\ \text{translation 3 units down} \end{array} \right\} \begin{array}{l} a = -2 \\ k = -3 \end{array}$$

Write the transformed function.

$$\begin{aligned} g(x) &= a(x - h)^2 + k && \text{Vertex form of a quadratic function} \\ &= -2(x - 0)^2 + (-3) && \text{Substitute } -2 \text{ for } a, 0 \text{ for } h, \text{ and } -3 \text{ for } k. \\ &= -2x^2 - 3 && \text{Simplify.} \end{aligned}$$

▶ The transformed function is $g(x) = -2x^2 - 3$. The vertex is $(0, -3)$.

Method 2 Begin with the parent function and apply the transformations one at a time in the stated order.

First write a function h that represents the reflection and vertical stretch of f .

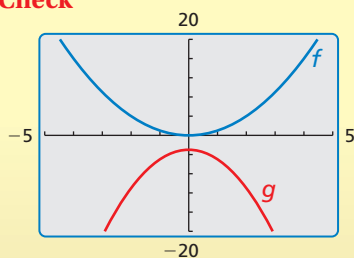
$$\begin{aligned} h(x) &= -2 \cdot f(x) && \text{Multiply the output by } -2. \\ &= -2x^2 && \text{Substitute } x^2 \text{ for } f(x). \end{aligned}$$

Then write a function g that represents the translation of h .

$$\begin{aligned} g(x) &= h(x) - 3 && \text{Subtract 3 from the output.} \\ &= -2x^2 - 3 && \text{Substitute } -2x^2 \text{ for } h(x). \end{aligned}$$

▶ The transformed function is $g(x) = -2x^2 - 3$. The vertex is $(0, -3)$.

Check



REMEMBER

To multiply two binomials, use the FOIL Method.

$$(x + 1)(x + 2) = x^2 + 2x + x + 2$$

First Inner
Outer Last

EXAMPLE 4 Writing a Transformed Quadratic Function

Let the graph of g be a translation 3 units right and 2 units up, followed by a reflection in the y -axis of the graph of $f(x) = x^2 - 5x$. Write a rule for g .

SOLUTION

Step 1 First write a function h that represents the translation of f .

$$\begin{aligned} h(x) &= f(x - 3) + 2 && \text{Subtract 3 from the input. Add 2 to the output.} \\ &= (x - 3)^2 - 5(x - 3) + 2 && \text{Replace } x \text{ with } x - 3 \text{ in } f(x). \\ &= x^2 - 11x + 26 && \text{Simplify.} \end{aligned}$$

Step 2 Then write a function g that represents the reflection of h .

$$\begin{aligned} g(x) &= h(-x) && \text{Multiply the input by } -1. \\ &= (-x)^2 - 11(-x) + 26 && \text{Replace } x \text{ with } -x \text{ in } h(x). \\ &= x^2 + 11x + 26 && \text{Simplify.} \end{aligned}$$



EXAMPLE 5 Modeling with Mathematics

The height h (in feet) of water spraying from a fire hose can be modeled by $h(x) = -0.03x^2 + x + 25$, where x is the horizontal distance (in feet) from the fire truck. The crew raises the ladder so that the water hits the ground 10 feet farther from the fire truck. Write a function that models the new path of the water.

SOLUTION

- 1. Understand the Problem** You are given a function that represents the path of water spraying from a fire hose. You are asked to write a function that represents the path of the water after the crew raises the ladder.
- 2. Make a Plan** Analyze the graph of the function to determine the translation of the ladder that causes water to travel 10 feet farther. Then write the function.
- 3. Solve the Problem** Use a graphing calculator to graph the original function.

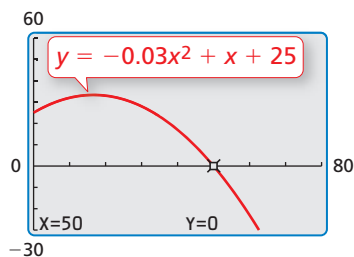
Because $h(50) = 0$, the water originally hits the ground 50 feet from the fire truck. The range of the function in this context does not include negative values. However, by observing that $h(60) = -23$, you can determine that a translation 23 units (feet) up causes the water to travel 10 feet farther from the fire truck.

$$\begin{aligned} g(x) &= h(x) + 23 && \text{Add 23 to the output.} \\ &= -0.03x^2 + x + 48 && \text{Substitute for } h(x) \text{ and simplify.} \end{aligned}$$

▶ The new path of the water can be modeled by $g(x) = -0.03x^2 + x + 48$.

- 4. Look Back** To check that your solution is correct, verify that $g(60) = 0$.

$$g(60) = -0.03(60)^2 + 60 + 48 = -108 + 60 + 48 = 0 \quad \checkmark$$



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- 7.** Let the graph of g be a vertical shrink by a factor of $\frac{1}{2}$ followed by a translation 2 units up of the graph of $f(x) = x^2$. Write a rule for g and identify the vertex.
- 8.** Let the graph of g be a translation 4 units left followed by a horizontal shrink by a factor of $\frac{1}{3}$ of the graph of $f(x) = x^2 + x$. Write a rule for g .
- 9. WHAT IF?** In Example 5, the water hits the ground 10 feet closer to the fire truck after lowering the ladder. Write a function that models the new path of the water.

Vocabulary and Core Concept Check

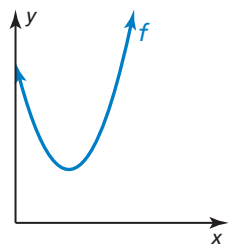
- COMPLETE THE SENTENCE** The graph of a quadratic function is called a(n) _____.
- VOCABULARY** Identify the vertex of the parabola given by $f(x) = (x + 2)^2 - 4$.

Monitoring Progress and Modeling with Mathematics

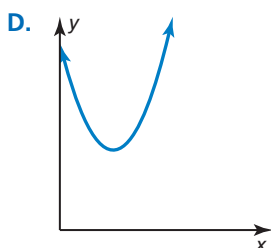
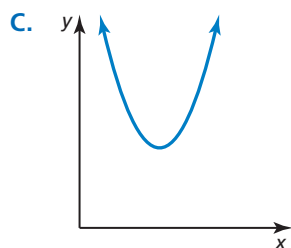
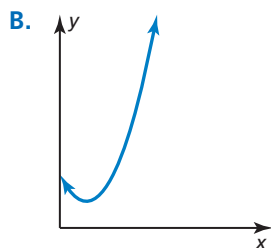
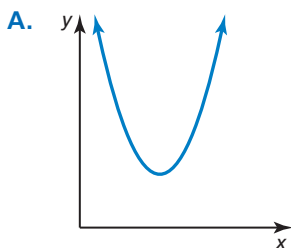
In Exercises 3–12, describe the transformation of $f(x) = x^2$ represented by g . Then graph each function. (See Example 1.)

- $g(x) = x^2 - 3$
- $g(x) = x^2 + 1$
- $g(x) = (x + 2)^2$
- $g(x) = (x - 4)^2$
- $g(x) = (x - 1)^2$
- $g(x) = (x + 3)^2$
- $g(x) = (x + 6)^2 - 2$
- $g(x) = (x - 9)^2 + 5$
- $g(x) = (x - 7)^2 + 1$
- $g(x) = (x + 10)^2 - 3$

ANALYZING RELATIONSHIPS
In Exercises 13–16, match the function with the correct transformation of the graph of f . Explain your reasoning.



- $y = f(x - 1)$
- $y = f(x) + 1$
- $y = f(x - 1) + 1$
- $y = f(x + 1) - 1$



In Exercises 17–24, describe the transformation of $f(x) = x^2$ represented by g . Then graph each function. (See Example 2.)

- $g(x) = -x^2$
- $g(x) = (-x)^2$
- $g(x) = 3x^2$
- $g(x) = \frac{1}{3}x^2$
- $g(x) = (2x)^2$
- $g(x) = -(2x)^2$
- $g(x) = \frac{1}{5}x^2 - 4$
- $g(x) = \frac{1}{2}(x - 1)^2$

ERROR ANALYSIS In Exercises 25 and 26, describe and correct the error in analyzing the graph of $f(x) = -6x^2 + 4$.

25. The graph is a reflection in the y -axis and a vertical stretch by a factor of 6, followed by a translation 4 units up of the graph of the parent quadratic function.

26. The graph is a translation 4 units up, followed by a vertical stretch by a factor of 6 and a reflection in the x -axis of the graph of the parent quadratic function.

USING STRUCTURE In Exercises 27–30, describe the transformation of the graph of the parent quadratic function. Then identify the vertex.

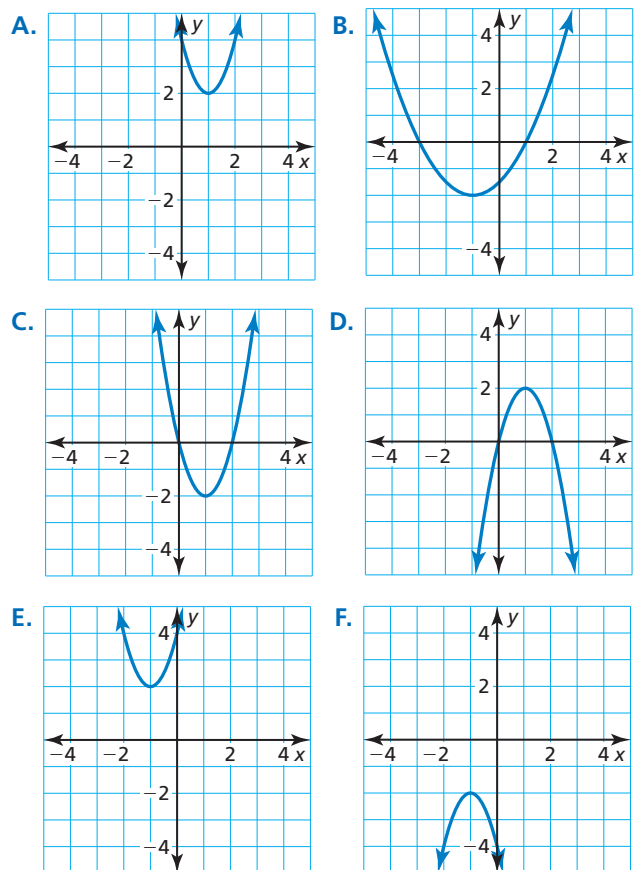
- $f(x) = 3(x + 2)^2 + 1$
- $f(x) = -4(x + 1)^2 - 5$
- $f(x) = -2x^2 + 5$
- $f(x) = \frac{1}{2}(x - 1)^2$

In Exercises 31–34, write a rule for g described by the transformations of the graph of f . Then identify the vertex. (See Examples 3 and 4.)

31. $f(x) = x^2$; vertical stretch by a factor of 4 and a reflection in the x -axis, followed by a translation 2 units up
32. $f(x) = x^2$; vertical shrink by a factor of $\frac{1}{3}$ and a reflection in the y -axis, followed by a translation 3 units right
33. $f(x) = 8x^2 - 6$; horizontal stretch by a factor of 2 and a translation 2 units up, followed by a reflection in the y -axis
34. $f(x) = (x + 6)^2 + 3$; horizontal shrink by a factor of $\frac{1}{2}$ and a translation 1 unit down, followed by a reflection in the x -axis

USING TOOLS In Exercises 35–40, match the function with its graph. Explain your reasoning.

35. $g(x) = 2(x - 1)^2 - 2$ 36. $g(x) = \frac{1}{2}(x + 1)^2 - 2$
37. $g(x) = -2(x - 1)^2 + 2$
38. $g(x) = 2(x + 1)^2 + 2$ 39. $g(x) = -2(x + 1)^2 - 2$
40. $g(x) = 2(x - 1)^2 + 2$



JUSTIFYING STEPS In Exercises 41 and 42, justify each step in writing a function g based on the transformations of $f(x) = 2x^2 + 6x$.

41. translation 6 units down followed by a reflection in the x -axis
- $$h(x) = f(x) - 6$$
- $$= 2x^2 + 6x - 6$$
- $$g(x) = -h(x)$$
- $$= -(2x^2 + 6x - 6)$$
- $$= -2x^2 - 6x + 6$$
42. reflection in the y -axis followed by a translation 4 units right
- $$h(x) = f(-x)$$
- $$= 2(-x)^2 + 6(-x)$$
- $$= 2x^2 - 6x$$
- $$g(x) = h(x - 4)$$
- $$= 2(x - 4)^2 - 6(x - 4)$$
- $$= 2x^2 - 22x + 56$$

43. **MODELING WITH MATHEMATICS** The function $h(x) = -0.03(x - 14)^2 + 6$ models the jump of a red kangaroo, where x is the horizontal distance traveled (in feet) and $h(x)$ is the height (in feet). When the kangaroo jumps from a higher location, it lands 5 feet farther away. Write a function that models the second jump. (See Example 5.)



44. **MODELING WITH MATHEMATICS** The function $f(t) = -16t^2 + 10$ models the height (in feet) of an object t seconds after it is dropped from a height of 10 feet on Earth. The same object dropped from the same height on the moon is modeled by $g(t) = -\frac{8}{3}t^2 + 10$. Describe the transformation of the graph of f to obtain g . From what height must the object be dropped on the moon so it hits the ground at the same time as on Earth?

45. MODELING WITH MATHEMATICS Flying fish use their pectoral fins like airplane wings to glide through the air.

- Write an equation of the form $y = a(x - h)^2 + k$ with vertex $(33, 5)$ that models the flight path, assuming the fish leaves the water at $(0, 0)$.
- What are the domain and range of the function? What do they represent in this situation?
- Does the value of a change when the flight path has vertex $(30, 4)$? Justify your answer.

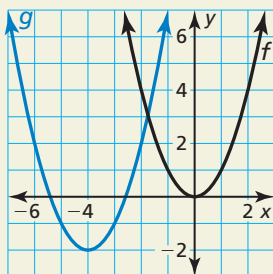


47. COMPARING METHODS Let the graph of g be a translation 3 units up and 1 unit right followed by a vertical stretch by a factor of 2 of the graph of $f(x) = x^2$.

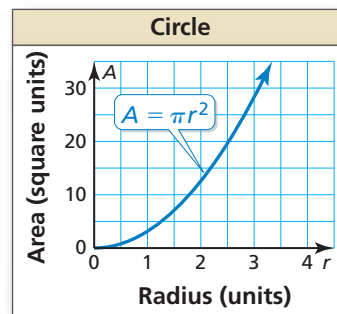
- Identify the values of a , h , and k and use vertex form to write the transformed function.
- Use function notation to write the transformed function. Compare this function with your function in part (a).
- Suppose the vertical stretch was performed first, followed by the translations. Repeat parts (a) and (b).
- Which method do you prefer when writing a transformed function? Explain.

48. THOUGHT PROVOKING A jump on a pogo stick with a conventional spring can be modeled by $f(x) = -0.5(x - 6)^2 + 18$, where x is the horizontal distance (in inches) and $f(x)$ is the vertical distance (in inches). Write at least one transformation of the function and provide a possible reason for your transformation.

46. HOW DO YOU SEE IT? Describe the graph of g as a transformation of the graph of $f(x) = x^2$.



49. MATHEMATICAL CONNECTIONS The area of a circle depends on the radius, as shown in the graph. A circular earring with a radius of r millimeters has a circular hole with a radius of $\frac{3r}{4}$ millimeters. Describe a transformation of the graph below that models the area of the blue portion of the earring.



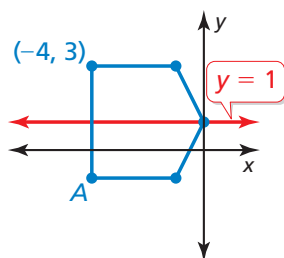
Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

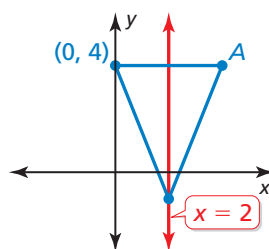
A line of symmetry for the figure is shown in red. Find the coordinates of point A .

(Skills Review Handbook)

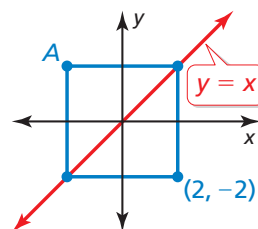
50.



51.



52.



2.6 Characteristics of Quadratic Functions

Essential Question What type of symmetry does the graph of $f(x) = a(x - h)^2 + k$ have and how can you describe this symmetry?

EXPLORATION 1 Parabolas and Symmetry

Work with a partner.

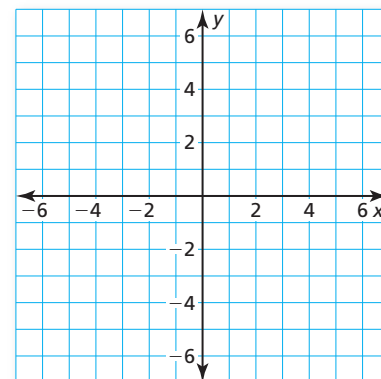
- a. Complete the table. Then use the values in the table to sketch the graph of the function

$$f(x) = \frac{1}{2}x^2 - 2x - 2$$

on graph paper.

x	-2	-1	0	1	2
f(x)					

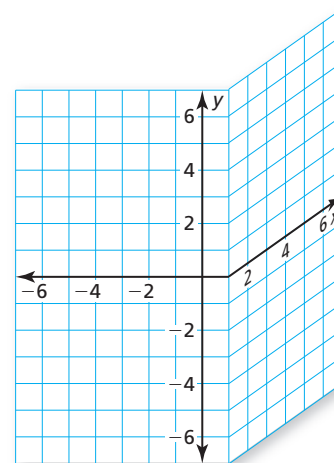
x	3	4	5	6
f(x)				



- b. Use the results in part (a) to identify the vertex of the parabola.
- c. Find a vertical line on your graph paper so that when you fold the paper, the left portion of the graph coincides with the right portion of the graph. What is the equation of this line? How does it relate to the vertex?
- d. Show that the vertex form

$$f(x) = \frac{1}{2}(x - 2)^2 - 4$$

is equivalent to the function given in part (a).



EXPLORATION 2 Parabolas and Symmetry

Work with a partner. Repeat Exploration 1 for the function given by

$$f(x) = -\frac{1}{3}x^2 + 2x + 3 = -\frac{1}{3}(x - 3)^2 + 6.$$

Communicate Your Answer

3. What type of symmetry does the graph of $f(x) = a(x - h)^2 + k$ have and how can you describe this symmetry?
4. Describe the symmetry of each graph. Then use a graphing calculator to verify your answer.
- a. $f(x) = -(x - 1)^2 + 4$ b. $f(x) = (x + 1)^2 - 2$ c. $f(x) = 2(x - 3)^2 + 1$
 d. $f(x) = \frac{1}{2}(x + 2)^2$ e. $f(x) = -2x^2 + 3$ f. $f(x) = 3(x - 5)^2 + 2$

ATTENDING TO PRECISION

To be proficient in math, you need to use clear definitions in your reasoning and discussions with others.



2.6 Lesson

Core Vocabulary

axis of symmetry, p. 82
 standard form, p. 82
 minimum value, p. 84
 maximum value, p. 84
 intercept form, p. 85

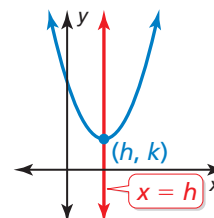
Previous
 x-intercept

What You Will Learn

- ▶ Explore properties of parabolas.
- ▶ Find maximum and minimum values of quadratic functions.
- ▶ Graph quadratic functions using x-intercepts.
- ▶ Rewrite equations.

Exploring Properties of Parabolas

An **axis of symmetry** is a line that divides a parabola into mirror images and passes through the vertex. Because the vertex of $f(x) = a(x - h)^2 + k$ is (h, k) , the axis of symmetry is the vertical line $x = h$.



Previously, you used transformations to graph quadratic functions in vertex form. You can also use the axis of symmetry and the vertex to graph quadratic functions written in vertex form.

EXAMPLE 1 Using Symmetry to Graph Quadratic Functions

Graph $f(x) = -2(x + 3)^2 + 4$. Label the vertex and axis of symmetry.

SOLUTION

Step 1 Identify the constants $a = -2$, $h = -3$, and $k = 4$.

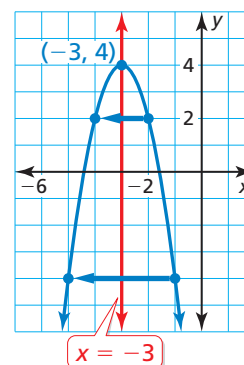
Step 2 Plot the vertex $(h, k) = (-3, 4)$ and draw the axis of symmetry $x = -3$.

Step 3 Evaluate the function for two values of x .

$$\begin{aligned} x = -2: f(-2) &= -2(-2 + 3)^2 + 4 = 2 \\ x = -1: f(-1) &= -2(-1 + 3)^2 + 4 = -4 \end{aligned}$$

Plot the points $(-2, 2)$, $(-1, -4)$, and their reflections in the axis of symmetry.

Step 4 Draw a parabola through the plotted points.



Quadratic functions can also be written in **standard form**, $f(x) = ax^2 + bx + c$, where $a \neq 0$. You can derive standard form by expanding vertex form.

$$\begin{aligned} f(x) &= a(x - h)^2 + k && \text{Vertex form} \\ f(x) &= a(x^2 - 2hx + h^2) + k && \text{Expand } (x - h)^2. \\ f(x) &= ax^2 - 2ahx + ah^2 + k && \text{Distributive Property} \\ f(x) &= ax^2 + (-2ah)x + (ah^2 + k) && \text{Group like terms.} \\ f(x) &= ax^2 + bx + c && \text{Let } b = -2ah \text{ and let } c = ah^2 + k. \end{aligned}$$

This allows you to make the following observations.

$a = a$: So, a has the same meaning in vertex form and standard form.

$b = -2ah$: Solve for h to obtain $h = -\frac{b}{2a}$. So, the axis of symmetry is $x = -\frac{b}{2a}$.

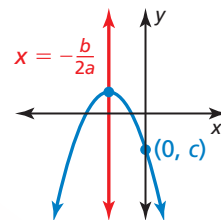
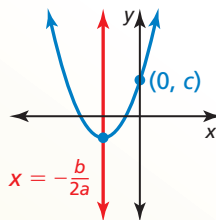
$c = ah^2 + k$: In vertex form $f(x) = a(x - h)^2 + k$, notice that $f(0) = ah^2 + k$. So, c is the y-intercept.

Core Concept

Properties of the Graph of $f(x) = ax^2 + bx + c$

$$y = ax^2 + bx + c, a > 0$$

$$y = ax^2 + bx + c, a < 0$$



- The parabola opens up when $a > 0$ and opens down when $a < 0$.
- The graph is narrower than the graph of $f(x) = x^2$ when $|a| > 1$ and wider when $|a| < 1$.
- The axis of symmetry is $x = -\frac{b}{2a}$ and the vertex is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.
- The y-intercept is c . So, the point $(0, c)$ is on the parabola.

EXAMPLE 2 Graphing a Quadratic Function in Standard Form

Graph $f(x) = 3x^2 - 6x + 1$. Label the vertex and axis of symmetry.

COMMON ERROR

Be sure to include the negative sign when writing the expression for the x -coordinate of the vertex.

SOLUTION

Step 1 Identify the coefficients $a = 3$, $b = -6$, and $c = 1$. Because $a > 0$, the parabola opens up.

Step 2 Find the vertex. First calculate the x -coordinate.

$$x = -\frac{b}{2a} = -\frac{-6}{2(3)} = 1$$

Then find the y -coordinate of the vertex.

$$f(1) = 3(1)^2 - 6(1) + 1 = -2$$

So, the vertex is $(1, -2)$. Plot this point.

Step 3 Draw the axis of symmetry $x = 1$.

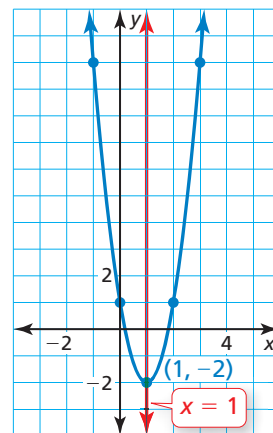
Step 4 Identify the y -intercept c , which is 1. Plot the point $(0, 1)$ and its reflection in the axis of symmetry, $(2, 1)$.

Step 5 Evaluate the function for another value of x , such as $x = 3$.

$$f(3) = 3(3)^2 - 6(3) + 1 = 10$$

Plot the point $(3, 10)$ and its reflection in the axis of symmetry, $(-1, 10)$.

Step 6 Draw a parabola through the plotted points.



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Graph the function. Label the vertex and axis of symmetry.

1. $f(x) = -3(x + 1)^2$

2. $g(x) = 2(x - 2)^2 + 5$

3. $h(x) = x^2 + 2x - 1$

4. $p(x) = -2x^2 - 8x + 1$

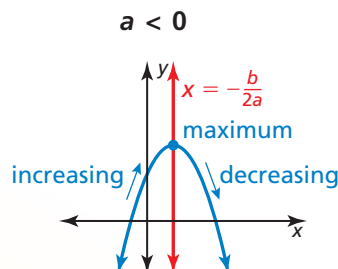
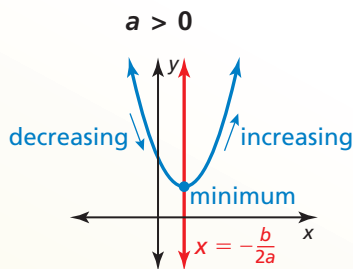
Finding Maximum and Minimum Values

Because the vertex is the highest or lowest point on a parabola, its y -coordinate is the *maximum value* or *minimum value* of the function. The vertex lies on the axis of symmetry, so the function is *increasing* on one side of the axis of symmetry and *decreasing* on the other side.

Core Concept

Minimum and Maximum Values

For the quadratic function $f(x) = ax^2 + bx + c$, the y -coordinate of the vertex is the **minimum value** of the function when $a > 0$ and the **maximum value** when $a < 0$.



STUDY TIP

When a function f is written in vertex form, you can use $h = -\frac{b}{2a}$ and $k = f\left(-\frac{b}{2a}\right)$ to state the properties shown.

- | | |
|--|--|
| <ul style="list-style-type: none"> • Minimum value: $f\left(-\frac{b}{2a}\right)$ • Domain: All real numbers • Range: $y \geq f\left(-\frac{b}{2a}\right)$ • Decreasing to the left of $x = -\frac{b}{2a}$ • Increasing to the right of $x = -\frac{b}{2a}$ | <ul style="list-style-type: none"> • Maximum value: $f\left(-\frac{b}{2a}\right)$ • Domain: All real numbers • Range: $y \leq f\left(-\frac{b}{2a}\right)$ • Increasing to the left of $x = -\frac{b}{2a}$ • Decreasing to the right of $x = -\frac{b}{2a}$ |
|--|--|

EXAMPLE 3 Finding a Minimum or a Maximum Value

Find the minimum value or maximum value of $f(x) = \frac{1}{2}x^2 - 2x - 1$. Describe the domain and range of the function, and where the function is increasing and decreasing.

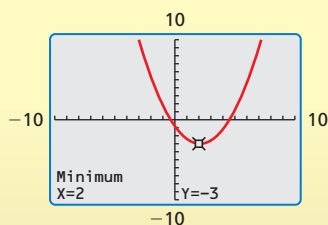
SOLUTION

Identify the coefficients $a = \frac{1}{2}$, $b = -2$, and $c = -1$. Because $a > 0$, the parabola opens up and the function has a minimum value. To find the minimum value, calculate the coordinates of the vertex.

$$x = -\frac{b}{2a} = -\frac{-2}{2\left(\frac{1}{2}\right)} = 2 \quad \Rightarrow \quad f(2) = \frac{1}{2}(2)^2 - 2(2) - 1 = -3$$

- ▶ The minimum value is -3 . So, the domain is all real numbers and the range is $y \geq -3$. The function is decreasing to the left of $x = 2$ and increasing to the right of $x = 2$.

Check



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5. Find the minimum value or maximum value of (a) $f(x) = 4x^2 + 16x - 3$ and (b) $h(x) = -x^2 + 5x + 9$. Describe the domain and range of each function, and where each function is increasing and decreasing.

Graphing Quadratic Functions Using x-Intercepts

When the graph of a quadratic function has at least one x -intercept, the function can be written in **intercept form**, $f(x) = a(x - p)(x - q)$, where $a \neq 0$.

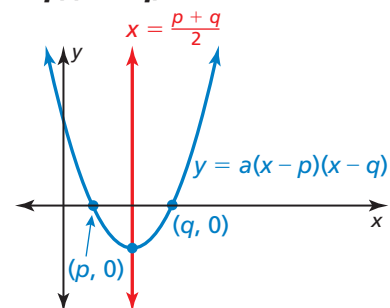
REMEMBER

An x -intercept of a graph is the x -coordinate of a point where the graph intersects the x -axis. It occurs where $f(x) = 0$.

Core Concept

Properties of the Graph of $f(x) = a(x - p)(x - q)$

- Because $f(p) = 0$ and $f(q) = 0$, p and q are the x -intercepts of the graph of the function.
- The axis of symmetry is halfway between $(p, 0)$ and $(q, 0)$. So, the axis of symmetry is $x = \frac{p + q}{2}$.
- The parabola opens up when $a > 0$ and opens down when $a < 0$.



COMMON ERROR

Remember that the x -intercepts of the graph of $f(x) = a(x - p)(x - q)$ are p and q , not $-p$ and $-q$.

EXAMPLE 4 Graphing a Quadratic Function in Intercept Form

Graph $f(x) = -2(x + 3)(x - 1)$. Label the x -intercepts, vertex, and axis of symmetry.

SOLUTION

Step 1 Identify the x -intercepts. The x -intercepts are $p = -3$ and $q = 1$, so the parabola passes through the points $(-3, 0)$ and $(1, 0)$.

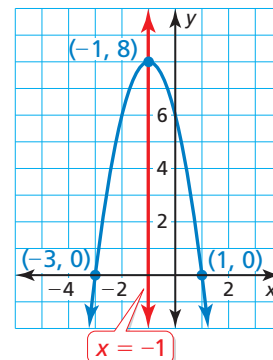
Step 2 Find the coordinates of the vertex.

$$x = \frac{p + q}{2} = \frac{-3 + 1}{2} = -1$$

$$f(-1) = -2(-1 + 3)(-1 - 1) = 8$$

So, the axis of symmetry is $x = -1$ and the vertex is $(-1, 8)$.

Step 3 Draw a parabola through the vertex and the points where the x -intercepts occur.



Check You can check your answer by generating a table of values for f on a graphing calculator.

X	Y1
-4	-10
-3	0
-2	6
-1	8
0	6
1	0
2	-10

X=-1

x -intercept

x -intercept

The values show symmetry about $x = -1$. So, the vertex is $(-1, 8)$.

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Graph the function. Label the x -intercepts, vertex, and axis of symmetry.

6. $f(x) = -(x + 1)(x + 5)$

7. $g(x) = \frac{1}{4}(x - 6)(x - 2)$

Rewriting Equations

You can use completing the square to rewrite equations of the form

$$ax^2 + by^2 + cx + dy + e = 0$$

into forms that make it easier to identify characteristics of their graphs.

EXAMPLE 5 Rewriting Equations

Use completing the square to find the vertex of the parabola or the center and radius of the circle. Then graph the equation.

a. $-2x^2 + 8x - y + 7 = 0$

b. $x^2 + y^2 - 2x + 6y - 6 = 0$

SOLUTION

a. The equation has no y^2 -term. So, isolate the x -terms and complete the square.

$$-2x^2 + 8x = y - 7$$

$$-2(x^2 - 4x) = y - 7$$

$$-2(x^2 - 4x + 4) = y - 7 - 8$$

$$-2(x - 2)^2 + 15 = y$$

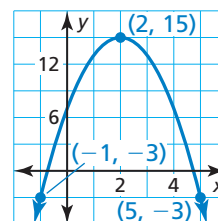
Isolate the x -terms.

Factor out -2 on the left side.

Complete the square for $x^2 - 4x$.

Write in vertex form.

► The graph of the equation is a parabola with a vertex of $(2, 15)$.



b. The equation has an x^2 -term and a y^2 -term. So, complete the square twice.

$$x^2 + y^2 - 2x + 6y = 6$$

$$(x^2 - 2x) + (y^2 + 6y) = 6$$

$$(x^2 - 2x + 1) + (y^2 + 6y + 9) = 6 + 1 + 9$$

$$(x - 1)^2 + (y + 3)^2 = 16$$

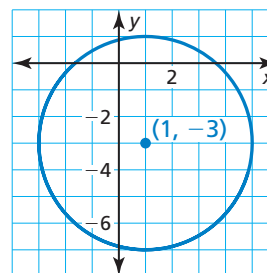
Add 6 to each side.

Group x -terms and y -terms.

Complete the square for $x^2 - 2x$ and $y^2 + 6y$.

Factor left side. Simplify right side.

► The graph of the equation is a circle with a radius of 4 and center $(1, -3)$.



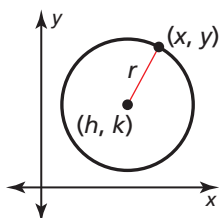
REMEMBER

To complete the square for an expression of the form $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$ to the expression. In this example, you must also add $-2(4) = -8$ to the right side of the equation to preserve equality.

REMEMBER

Recall that the standard equation of a circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2.$$



Monitoring Progress



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Use completing the square to find the vertex of the parabola or the center and radius of the circle. Then graph the equation.

8. $x^2 - 8x - y + 15 = 0$

9. $x^2 + y^2 + 2x - 8y - 8 = 0$

2.6 Exercises

Vocabulary and Core Concept Check

- WRITING** Explain how to determine whether a quadratic function will have a minimum value or a maximum value.
- WHICH ONE DOESN'T BELONG?** The graph of which function does *not* belong with the other three? Explain.

$$f(x) = 3x^2 + 6x - 24$$

$$f(x) = 3x^2 + 24x - 6$$

$$f(x) = 3(x - 2)(x + 4)$$

$$f(x) = 3(x + 1)^2 - 27$$

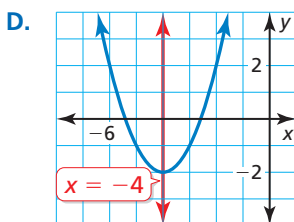
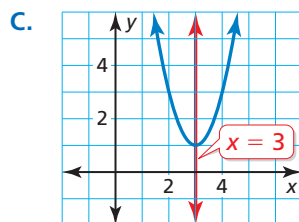
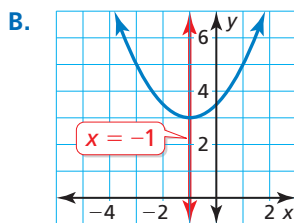
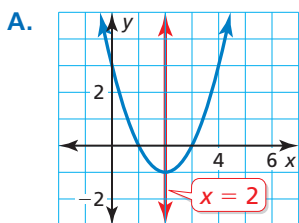
Monitoring Progress and Modeling with Mathematics

In Exercises 3–14, graph the function. Label the vertex and axis of symmetry. (See Example 1.)

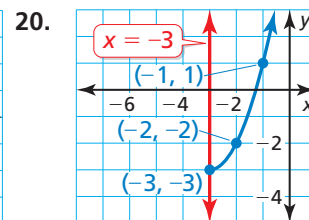
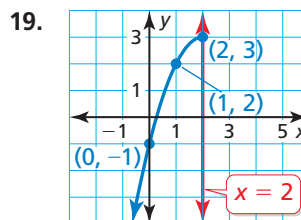
- | | |
|-------------------------------------|------------------------------------|
| 3. $f(x) = (x - 3)^2$ | 4. $h(x) = (x + 4)^2$ |
| 5. $g(x) = (x + 3)^2 + 5$ | 6. $y = (x - 7)^2 - 1$ |
| 7. $y = -4(x - 2)^2 + 4$ | 8. $g(x) = 2(x + 1)^2 - 3$ |
| 9. $f(x) = -2(x - 1)^2 - 5$ | 10. $h(x) = 4(x + 4)^2 + 6$ |
| 11. $y = -\frac{1}{4}(x + 2)^2 + 1$ | 12. $y = \frac{1}{2}(x - 3)^2 + 2$ |
| 13. $f(x) = 0.4(x - 1)^2$ | 14. $g(x) = 0.75x^2 - 5$ |

ANALYZING RELATIONSHIPS In Exercises 15–18, use the axis of symmetry to match the equation with its graph.

- | | |
|------------------------------------|-------------------------|
| 15. $y = 2(x - 3)^2 + 1$ | 16. $y = (x + 4)^2 - 2$ |
| 17. $y = \frac{1}{2}(x + 1)^2 + 3$ | 18. $y = (x - 2)^2 - 1$ |



REASONING In Exercises 19 and 20, use the axis of symmetry to plot the reflection of each point and complete the parabola.





In Exercises 21–30, graph the function. Label the vertex and axis of symmetry. (See Example 2.)

- | | |
|-----------------------------------|------------------------------------|
| 21. $y = x^2 + 2x + 1$ | 22. $y = 3x^2 - 6x + 4$ |
| 23. $y = -4x^2 + 8x + 2$ | 24. $f(x) = -x^2 - 6x + 3$ |
| 25. $g(x) = -x^2 - 1$ | 26. $f(x) = 6x^2 - 5$ |
| 27. $g(x) = -1.5x^2 + 3x + 2$ | |
| 28. $f(x) = 0.5x^2 + x - 3$ | |
| 29. $y = \frac{3}{2}x^2 - 3x + 6$ | 30. $y = -\frac{5}{2}x^2 - 4x - 1$ |

- WRITING** Two quadratic functions have graphs with vertices (2, 4) and (2, -3). Explain why you can not use the axes of symmetry to distinguish between the two functions.
- WRITING** A quadratic function is increasing to the left of $x = 2$ and decreasing to the right of $x = 2$. Will the vertex be the highest or lowest point on the graph of the parabola? Explain.

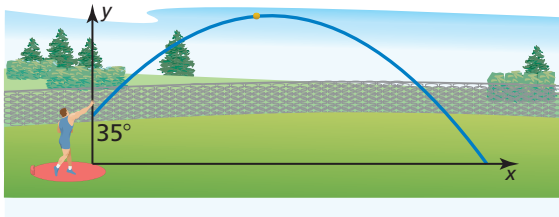
ERROR ANALYSIS In Exercises 33 and 34, describe and correct the error in analyzing the graph of $y = 4x^2 + 24x - 7$.

33.  The x -coordinate of the vertex is $x = \frac{b}{2a} = \frac{24}{2(4)} = 3$.

34.  The y -intercept of the graph is the value of c , which is 7.

MODELING WITH MATHEMATICS In Exercises 35 and 36, x is the horizontal distance (in feet) and y is the vertical distance (in feet). Find and interpret the coordinates of the vertex.

35. The path of a basketball thrown at an angle of 45° can be modeled by $y = -0.02x^2 + x + 6$.
36. The path of a shot put released at an angle of 35° can be modeled by $y = -0.01x^2 + 0.7x + 6$.

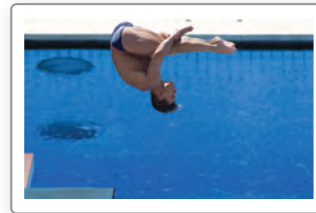


37. **ANALYZING EQUATIONS** The graph of which function has the same axis of symmetry as the graph of $y = x^2 + 2x + 2$?
- (A) $y = 2x^2 + 2x + 2$
- (B) $y = -3x^2 - 6x + 2$
- (C) $y = x^2 - 2x + 2$
- (D) $y = -5x^2 + 10x + 2$
38. **USING STRUCTURE** Which function represents the widest parabola? Explain your reasoning.
- (A) $y = 2(x + 3)^2$
- (B) $y = x^2 - 5$
- (C) $y = 0.5(x - 1)^2 + 1$
- (D) $y = -x^2 + 6$

In Exercises 39–48, find the minimum or maximum value of the function. Describe the domain and range of the function, and where the function is increasing and decreasing. (See Example 3.)

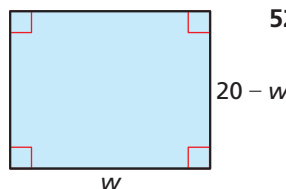
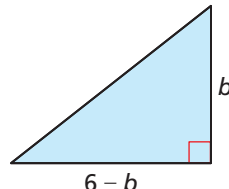
39. $y = 6x^2 - 1$ 40. $y = 9x^2 + 7$
41. $y = -x^2 - 4x - 2$ 42. $g(x) = -3x^2 - 6x + 5$
43. $f(x) = -2x^2 + 8x + 7$
44. $g(x) = 3x^2 + 18x - 5$
45. $h(x) = 2x^2 - 12x$ 46. $h(x) = x^2 - 4x$
47. $y = \frac{1}{4}x^2 - 3x + 2$ 48. $f(x) = \frac{3}{2}x^2 + 6x + 4$
49. **PROBLEM SOLVING** The path of a diver is modeled by the function $f(x) = -9x^2 + 9x + 1$, where $f(x)$ is the height of the diver (in meters) above the water and x is the horizontal distance (in meters) from the end of the diving board.

- a. What is the height of the diving board?
- b. What is the maximum height of the diver?
- c. Describe where the diver is ascending and where the diver is descending.



50. **PROBLEM SOLVING** The engine torque y (in foot-pounds) of one model of car is given by $y = -3.75x^2 + 23.2x + 38.8$, where x is the speed (in thousands of revolutions per minute) of the engine.
- a. Find the engine speed that maximizes torque. What is the maximum torque?
- b. Explain what happens to the engine torque as the speed of the engine increases.

MATHEMATICAL CONNECTIONS In Exercises 51 and 52, write an equation for the area of the figure. Then determine the maximum possible area of the figure.

51.  52. 

In Exercises 53–60, graph the function. Label the x -intercept(s), vertex, and axis of symmetry. (See Example 4.)

53. $y = (x + 3)(x - 3)$ 54. $y = (x + 1)(x - 3)$

55. $y = 3(x + 2)(x + 6)$ 56. $f(x) = 2(x - 5)(x - 1)$

57. $g(x) = -x(x + 6)$ 58. $y = -4x(x + 7)$

59. $f(x) = -2(x - 3)^2$ 60. $y = 4(x - 7)^2$

USING TOOLS In Exercises 61–64, identify the x -intercepts of the function and describe where the graph is increasing and decreasing. Use a graphing calculator to verify your answer.

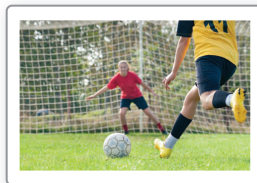
61. $f(x) = \frac{1}{2}(x - 2)(x + 6)$

62. $y = \frac{3}{4}(x + 1)(x - 3)$

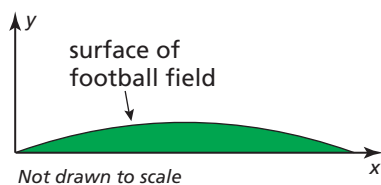
63. $g(x) = -4(x - 4)(x - 2)$

64. $h(x) = -5(x + 5)(x + 1)$

65. **MODELING WITH MATHEMATICS** A soccer player kicks a ball downfield. The height of the ball increases until it reaches a maximum height of 8 yards, 20 yards away from the player. A second kick is modeled by $y = x(0.4 - 0.008x)$. Which kick travels farther before hitting the ground? Which kick travels higher?



66. **MODELING WITH MATHEMATICS** Although a football field appears to be flat, some are actually shaped like a parabola so that rain runs off to both sides. The cross section of a field can be modeled by $y = -0.000234x(x - 160)$, where x and y are measured in feet. What is the width of the field? What is the maximum height of the surface of the field?



67. **OPEN-ENDED** Write two different quadratic functions in intercept form whose graphs have the axis of symmetry $x = 3$.
68. **USING STRUCTURE** Write the quadratic function $f(x) = x^2 + x - 12$ in intercept form. Graph the function. Label the x -intercepts, y -intercept, vertex, and axis of symmetry.

In Exercises 69–72, use completing the square to find the vertex of the parabola or the center and radius of the circle. Then graph the equation. (See Example 5.)

69. $3x^2 + 6x - y - 2 = 0$

70. $-2x^2 + 6x - 2y - 1 = 0$

71. $x^2 + y^2 + 12y - 13 = 0$

72. $4x^2 + 4y^2 - 24x - 24y - 9 = 0$

73. **USING STRUCTURE** Recall that the standard equation of a parabola that opens right or left with vertex at (h, k) is $x = \frac{1}{4p}(y - k)^2 + h$. Use completing the square to find the focus, directrix, and vertex of $y^2 - 4x - 8y + 20 = 0$. Then graph the equation.

74. **REASONING** Consider an equation of the form $ax^2 + by^2 + cx + dy + e = 0$. What must be true about the coefficients a and b for the graph of the equation to be a parabola? a circle? Explain your reasoning.

75. **PROBLEM SOLVING** An online music store sells about 4000 songs each day when it charges \$1 per song. For each \$0.05 increase in price, about 80 fewer songs per day are sold. Use the verbal model and quadratic function to determine how much the store should charge per song to maximize daily revenue.

Revenue (dollars)	=	Price (dollars/song)	•	Sales (songs)
----------------------	---	-------------------------	---	------------------

$$R(x) = (1 + 0.05x) \cdot (4000 - 80x)$$

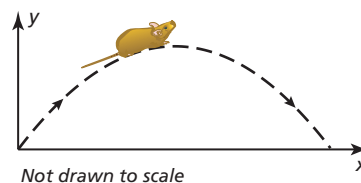
76. **DRAWING CONCLUSIONS** Compare the graphs of the three quadratic functions. What do you notice? Rewrite the functions f and g in standard form to justify your answer.

$$f(x) = (x + 3)(x + 1)$$

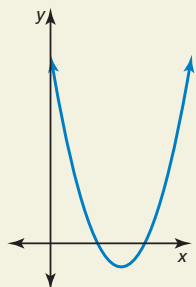
$$g(x) = (x + 2)^2 - 1$$

$$h(x) = x^2 + 4x + 3$$

77. **PROBLEM SOLVING** A woodland jumping mouse hops along a parabolic path given by $y = -0.2x^2 + 1.3x$, where x is the mouse's horizontal distance traveled (in feet) and y is the corresponding height (in feet). Can the mouse jump over a fence that is 3 feet high? Justify your answer.



78. **HOW DO YOU SEE IT?** Consider the graph of the function $f(x) = a(x - p)(x - q)$.



- What does $f\left(\frac{p+q}{2}\right)$ represent in the graph?
- If $a < 0$, how does your answer in part (a) change? Explain.

79. **MODELING WITH MATHEMATICS** The Gateshead Millennium Bridge spans the River Tyne. The arch of the bridge can be modeled by a parabola. The arch reaches a maximum height of 50 meters at a point roughly 63 meters across the river. Graph the curve of the arch. What are the domain and range? What do they represent in this situation?



80. **THOUGHT PROVOKING** You have 100 feet of fencing to enclose a rectangular garden. Draw three possible designs for the garden. Of these, which has the greatest area? Make a conjecture about the dimensions of the rectangular garden with the greatest possible area. Explain your reasoning.

81. **MAKING AN ARGUMENT** The point $(1, 5)$ lies on the graph of a quadratic function with axis of symmetry $x = -1$. Your friend says the vertex could be the point $(0, 5)$. Is your friend correct? Explain.

82. **CRITICAL THINKING** Find the y-intercept in terms of a , p , and q for the quadratic function $f(x) = a(x - p)(x - q)$.

83. **MODELING WITH MATHEMATICS** A kernel of popcorn contains water that expands when the kernel is heated, causing it to pop. The equations below represent the “popping volume” y (in cubic centimeters per gram) of popcorn with moisture content x (as a percent of the popcorn’s weight).

Hot-air popping: $y = -0.761(x - 5.52)(x - 22.6)$

Hot-oil popping: $y = -0.652(x - 5.35)(x - 21.8)$



- For hot-air popping, what moisture content maximizes popping volume? What is the maximum volume?
 - For hot-oil popping, what moisture content maximizes popping volume? What is the maximum volume?
 - Use a graphing calculator to graph both functions in the same coordinate plane. What are the domain and range of each function in this situation? Explain.
84. **ABSTRACT REASONING** A function is written in intercept form with $a > 0$. What happens to the vertex of the graph as a increases? as a approaches 0?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Sketch the solid of revolution. Then identify and describe the solid. (Section 1.4)

- a right triangle with legs of length 3 and 5 rotated around its shorter leg
- a semicircle with radius 4 rotated around its diameter

Use a graphing calculator to find an equation for the line of best fit. (Section 2.3)

87.

x	0	3	6	7	11
y	4	9	24	29	46

88.

x	0	5	10	12	16
y	18	15	9	7	2

2.7 Modeling with Quadratic Functions

Essential Question How can you use a quadratic function to model a real-life situation?

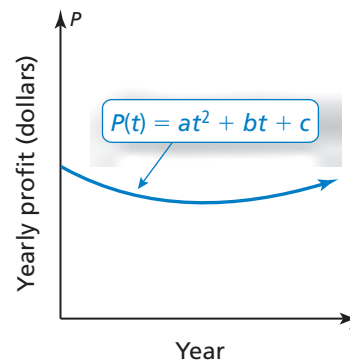
EXPLORATION 1 Modeling with a Quadratic Function

Work with a partner. The graph shows a quadratic function of the form

$$P(t) = at^2 + bt + c$$

which approximates the yearly profits for a company, where $P(t)$ is the profit in year t .

- Is the value of a positive, negative, or zero? Explain.
- Write an expression in terms of a and b that represents the year t when the company made the least profit.
- The company made the same yearly profits in 2004 and 2012. Estimate the year in which the company made the least profit.
- Assume that the model is still valid today. Are the yearly profits currently increasing, decreasing, or constant? Explain.

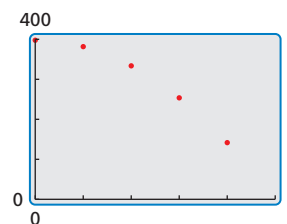


EXPLORATION 2 Modeling with a Graphing Calculator

Work with a partner. The table shows the heights h (in feet) of a wrench t seconds after it has been dropped from a building under construction.

Time, t	0	1	2	3	4
Height, h	400	384	336	256	144

- Use a graphing calculator to create a scatter plot of the data, as shown at the right. Explain why the data appear to fit a quadratic model.
- Use the *quadratic regression* feature to find a quadratic model for the data.
- Graph the quadratic function on the same screen as the scatter plot to verify that it fits the data.
- When does the wrench hit the ground? Explain.



MODELING WITH MATHEMATICS

To be proficient in math, you need to routinely interpret your results in the context of the situation.

Communicate Your Answer

- How can you use a quadratic function to model a real-life situation?
- Use the Internet or some other reference to find examples of real-life situations that can be modeled by quadratic functions.

2.7 Lesson

Core Vocabulary

Previous

average rate of change
first differences
second differences
system of three linear equations
Quadratic Formula

What You Will Learn

- ▶ Write equations of quadratic functions using vertices, points, and x-intercepts.
- ▶ Write quadratic equations to model data sets.

Writing Quadratic Equations

Core Concept

Writing Quadratic Equations

Given a point and the vertex (h, k)

Use vertex form:

$$y = a(x - h)^2 + k$$

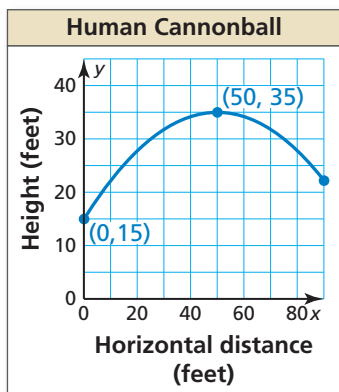
Given a point and x-intercepts p and q

Use intercept form:

$$y = a(x - p)(x - q)$$

Given three points

Write and solve a system of three equations in three variables.



EXAMPLE 1 Writing an Equation Using a Vertex and a Point

The graph shows the parabolic path of a performer who is shot out of a cannon, where y is the height (in feet) and x is the horizontal distance traveled (in feet). Write an equation of the parabola. The performer lands in a net 90 feet from the cannon. What is the height of the net?

SOLUTION

From the graph, you can see that the vertex (h, k) is $(50, 35)$ and the parabola passes through the point $(0, 15)$. Use the vertex and the point to solve for a in vertex form.

$$y = a(x - h)^2 + k$$

Vertex form

$$15 = a(0 - 50)^2 + 35$$

Substitute for $h, k, x,$ and y .

$$-20 = 2500a$$

Simplify.

$$-0.008 = a$$

Divide each side by 2500.

Because $a = -0.008$, $h = 50$, and $k = 35$, the path can be modeled by the equation $y = -0.008(x - 50)^2 + 35$, where $0 \leq x \leq 90$. Find the height when $x = 90$.

$$y = -0.008(90 - 50)^2 + 35$$

Substitute 90 for x .

$$= -0.008(1600) + 35$$

Simplify.

$$= 22.2$$

Simplify.

- ▶ So, the height of the net is about 22 feet.

Monitoring Progress

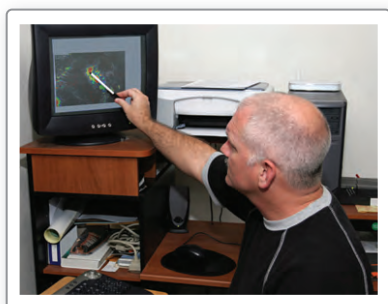
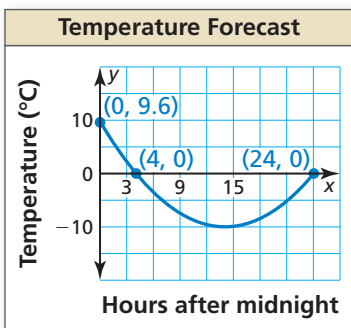


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1. **WHAT IF?** The vertex of the parabola is $(50, 37.5)$. What is the height of the net?
2. Write an equation of the parabola that passes through the point $(-1, 2)$ and has vertex $(4, -9)$.

EXAMPLE 2

Writing an Equation Using a Point and x -Intercepts



A meteorologist creates a parabola to predict the temperature tomorrow, where x is the number of hours after midnight and y is the temperature (in degrees Celsius).

- Write a function f that models the temperature over time. What is the coldest temperature?
- What is the average rate of change in temperature over the interval in which the temperature is decreasing? increasing? Compare the average rates of change.

SOLUTION

- The x -intercepts are 4 and 24 and the parabola passes through $(0, 9.6)$. Use the x -intercepts and the point to solve for a in intercept form.

$$y = a(x - p)(x - q) \quad \text{Intercept form}$$

$$9.6 = a(0 - 4)(0 - 24) \quad \text{Substitute for } p, q, x, \text{ and } y.$$

$$9.6 = 96a \quad \text{Simplify.}$$

$$0.1 = a \quad \text{Divide each side by 96.}$$

Because $a = 0.1$, $p = 4$, and $q = 24$, the temperature over time can be modeled by $f(x) = 0.1(x - 4)(x - 24)$, where $0 \leq x \leq 24$. The coldest temperature is the minimum value. So, find $f(x)$ when $x = \frac{4 + 24}{2} = 14$.

$$f(14) = 0.1(14 - 4)(14 - 24) \quad \text{Substitute 14 for } x.$$

$$= -10 \quad \text{Simplify.}$$

- So, the coldest temperature is -10°C at 14 hours after midnight, or 2 P.M.

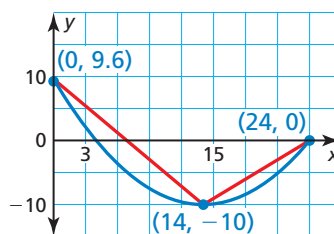
- The parabola opens up and the axis of symmetry is $x = 14$. So, the function is decreasing over the interval $0 < x < 14$ and increasing over the interval $14 < x < 24$.

Average rate of change
over $0 < x < 14$:

$$\frac{f(14) - f(0)}{14 - 0} = \frac{-10 - 9.6}{14} = -1.4$$

Average rate of change
over $14 < x < 24$:

$$\frac{f(24) - f(14)}{24 - 14} = \frac{0 - (-10)}{10} = 1$$



- Because $|-1.4| > |1|$, the average rate at which the temperature decreases from midnight to 2 P.M. is greater than the average rate at which it increases from 2 P.M. to midnight.

REMEMBER

The average rate of change of a function f from x_1 to x_2 is the slope of the line connecting $(x_1, f(x_1))$ and $(x_2, f(x_2))$:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



Monitoring Progress



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- WHAT IF?** The y -intercept is 4.8. How does this change your answers in parts (a) and (b)?
- Write an equation of the parabola that passes through the point $(2, 5)$ and has x -intercepts -2 and 4 .

Writing Equations to Model Data

When data have equally-spaced inputs, you can analyze patterns in the differences of the outputs to determine what type of function can be used to model the data. Linear data have constant *first differences*. Quadratic data have constant *second differences*.

EXAMPLE 3 Writing a Quadratic Equation Using Three Points

Time, t	Height, h
10	26,900
15	29,025
20	30,600
25	31,625
30	32,100
35	32,025
40	31,400

NASA can create a weightless environment by flying a plane in parabolic paths. The table shows heights h (in feet) of a plane t seconds after starting the flight path. Passengers experience a weightless environment above 30,800 feet. Write and solve an equation to approximate the period of weightlessness.

SOLUTION

Step 1 The input values are equally spaced. So, analyze the differences in the outputs to determine what type of function you can use to model the data.

	$h(10)$	$h(15)$	$h(20)$	$h(25)$	$h(30)$	$h(35)$	$h(40)$
	26,900	29,025	30,600	31,625	32,100	32,025	31,400
first differences:	2125	1575	1025	475	-75	-625	
second differences:		-550	-550	-550	-550	-550	

Because the second differences are constant, you can model the data with a quadratic function.

Step 2 Write a quadratic function of the form $h(t) = at^2 + bt + c$ that models the data. Use any three points (t, h) from the table to write a system of equations.

Use (10, 26,900): $100a + 10b + c = 26,900$ Equation 1

Use (20, 30,600): $400a + 20b + c = 30,600$ Equation 2

Use (30, 32,100): $900a + 30b + c = 32,100$ Equation 3

Use the elimination method to solve the system.

Subtract Equation 1 from Equation 2.

$$\rightarrow 300a + 10b = 3700$$

New Equation 1

Subtract Equation 1 from Equation 3.

$$\rightarrow 800a + 20b = 5200$$

New Equation 2

$$200a = -2200$$

Subtract 2 times new Equation 1 from new Equation 2.

$$a = -11$$

Solve for a .

$$b = 700$$

Substitute into new Equation 1 to find b .

$$c = 21,000$$

Substitute into Equation 1 to find c .

The data can be modeled by the function $h(t) = -11t^2 + 700t + 21,000$.

Step 3 To find the period of weightlessness, find the t -values for which $h(t) = 30,800$.

$$30,800 = -11t^2 + 700t + 21,000$$

Substitute 30,800 for $h(t)$.

$$0 = -11t^2 + 700t - 9800$$

Write in standard form.

$$t = \frac{-700 \pm \sqrt{700^2 - 4(-11)(-9800)}}{2(-11)}$$

Substitute -11 for a , 700 for b , and -9800 for c in the Quadratic Formula.

$$t = \frac{-700 \pm \sqrt{58,800}}{-22}$$

Simplify.

$$t \approx 20.8, \text{ or } t \approx 42.8$$

Use a calculator.

REMEMBER

The Quadratic Formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

gives the solutions of $ax^2 + bx + c = 0$.

▶ The plane rises above 30,800 feet after 20.8 seconds and falls below 30,800 feet after 42.8 seconds. So, the period of weightlessness is $42.8 - 20.8 \approx 22$ seconds.

Real-life data that show a quadratic relationship usually do not have constant second differences because the data are not *exactly* quadratic. Relationships that are *approximately* quadratic have second differences that are relatively “close” in value. Many technology tools have a *quadratic regression* feature that you can use to find a quadratic function that best models a set of data.

EXAMPLE 4 Using Quadratic Regression

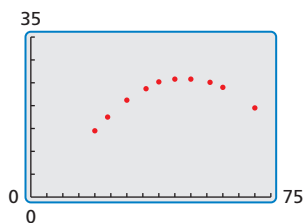
Miles per hour, x	Miles per gallon, y
20	14.5
24	17.5
30	21.2
36	23.7
40	25.2
45	25.8
50	25.8
56	25.1
60	24.0
70	19.5

The table shows fuel efficiencies of a vehicle at different speeds. Write a function that models the data. Use the model to approximate the optimal driving speed.

SOLUTION

Because the x -values are not equally spaced, you cannot analyze the differences in the outputs. Use a graphing calculator to find a function that models the data.

Step 1 Enter the data in a graphing calculator using two lists and create a scatter plot. The data show a quadratic relationship.



Step 2 Use the *quadratic regression* feature. A quadratic model that represents the data is $y = -0.014x^2 + 1.37x - 7.1$.

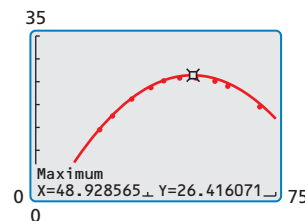
```
QuadReg
y=ax^2+bx+c
a=-.014097349
b=1.366218867
c=-7.144052413
R^2=.9992475882
```

STUDY TIP

The *coefficient of determination* R^2 shows how well an equation fits a set of data. The closer R^2 is to 1, the better the fit.

Step 3 Graph the regression equation with the scatter plot.

In this context, the “optimal” driving speed is the speed at which the mileage per gallon is maximized. Using the *maximum* feature, you can see that the maximum mileage per gallon is about 26.4 miles per gallon when driving about 48.9 miles per hour.



► So, the optimal driving speed is about 49 miles per hour.

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- Write an equation of the parabola that passes through the points $(-1, 4)$, $(0, 1)$, and $(2, 7)$.
- The table shows the estimated profits y (in dollars) for a concert when the charge is x dollars per ticket. Write and solve an equation to determine what ticket prices result in profits above \$8000. What ticket price maximizes profit?

Ticket price, x	2	5	8	11	14	17
Profit, y	2600	6500	8600	8900	7400	4100

- The table shows the results of an experiment testing the maximum weights y (in tons) supported by ice x inches thick. Write a function that models the data. How much weight can be supported by ice that is 22 inches thick?

Ice thickness, x	12	14	15	18	20	24	27
Maximum weight, y	3.4	7.6	10.0	18.3	25.0	40.6	54.3

Vocabulary and Core Concept Check

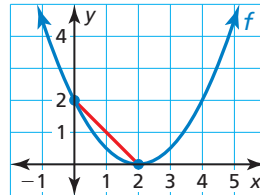
- WRITING** Explain when it is appropriate to use a quadratic model for a set of data.
- DIFFERENT WORDS, SAME QUESTION** Which is different? Find “both” answers.

What is the average rate of change over $0 \leq x \leq 2$?

What is the distance from $f(0)$ to $f(2)$?

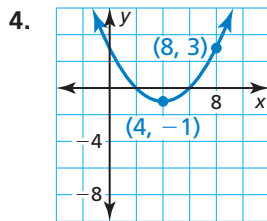
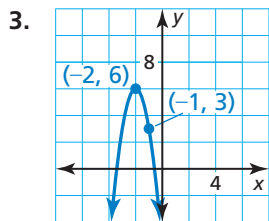
What is the slope of the line segment?

What is $\frac{f(2) - f(0)}{2 - 0}$?



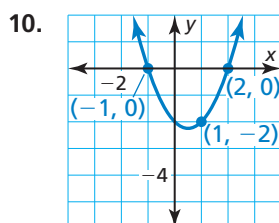
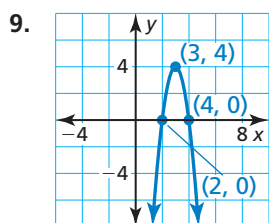
Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, write an equation of the parabola in vertex form. (See Example 1.)



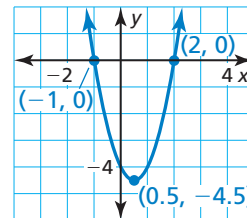
- passes through (13, 8) and has vertex (3, 2)
- passes through (-7, -15) and has vertex (-5, 9)
- passes through (0, -24) and has vertex (-6, -12)
- passes through (6, 35) and has vertex (-1, 14)

In Exercises 9–14, write an equation of the parabola in intercept form. (See Example 2.)



- x-intercepts of 12 and -6; passes through (14, 4)
- x-intercepts of 9 and 1; passes through (0, -18)
- x-intercepts of -16 and -2; passes through (-18, 72)
- x-intercepts of -7 and -3; passes through (-2, 0.05)

- WRITING** Explain when to use intercept form and when to use vertex form when writing an equation of a parabola.
- ANALYZING EQUATIONS** Which of the following equations represent the parabola?

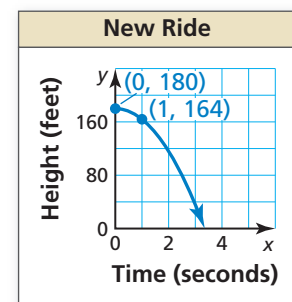
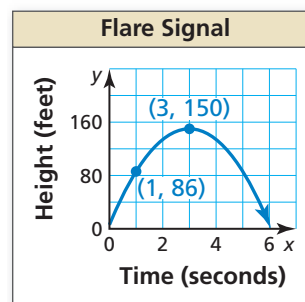


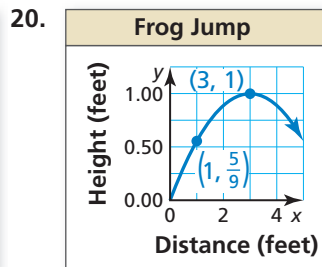
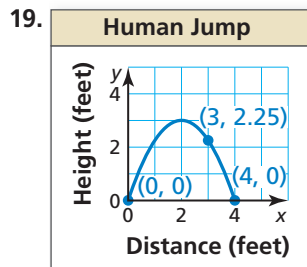
- $y = 2(x - 2)(x + 1)$
- $y = 2(x + 0.5)^2 - 4.5$
- $y = 2(x - 0.5)^2 - 4.5$
- $y = 2(x + 2)(x - 1)$

In Exercises 17–20, write an equation of the parabola in vertex form or intercept form.

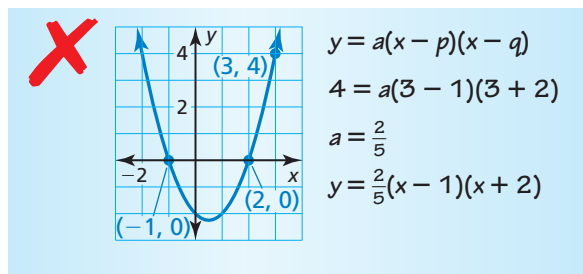
17.

18.

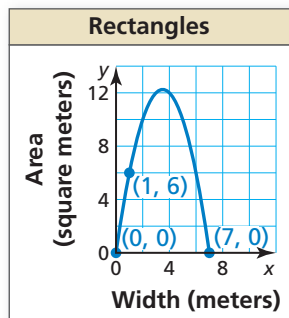




21. **ERROR ANALYSIS** Describe and correct the error in writing an equation of the parabola.



22. **MATHEMATICAL CONNECTIONS** The area of a rectangle is modeled by the graph where y is the area (in square meters) and x is the width (in meters). Write an equation of the parabola. Find the dimensions and corresponding area of one possible rectangle. What dimensions result in the maximum area?



23. **MODELING WITH MATHEMATICS** Every rope has a safe working load. A rope should not be used to lift a weight greater than its safe working load. The table shows the safe working loads S (in pounds) for ropes with circumference C (in inches). Write an equation for the safe working load for a rope. Find the safe working load for a rope that has a circumference of 10 inches. (See Example 3.)

Circumference, C	0	1	2	3
Safe working load, S	0	180	720	1620

24. **MODELING WITH MATHEMATICS** A baseball is thrown up in the air. The table shows the heights y (in feet) of the baseball after x seconds. Write and solve an equation to determine how long the ball is above 10 feet. How long is the ball in the air?

Time, x	0	2	4	6
Baseball height, y	6	22	22	6

25. **COMPARING METHODS** You use a system with three variables to find the equation of a parabola that passes through the points $(-8, 0)$, $(2, -20)$, and $(1, 0)$. Your friend uses intercept form to find the equation. Whose method is easier? Justify your answer.
26. **MODELING WITH MATHEMATICS** The table shows the distances y a motorcyclist is from home after x hours.

Time (hours), x	0	1	2	3
Distance (miles), y	0	45	90	135

- a. Determine what type of function you can use to model the data. Explain your reasoning.
- b. Write and evaluate a function to determine the distance the motorcyclist is from home after 6 hours.
27. **USING TOOLS** The table shows the heights h (in feet) of a sponge t seconds after it was dropped by a window cleaner on top of a skyscraper. (See Example 4.)
- | Time, t | 0 | 1 | 1.5 | 2.5 | 3 |
|-------------|-----|-----|-----|-----|-----|
| Height, h | 280 | 264 | 244 | 180 | 136 |
- a. Use a graphing calculator to create a scatter plot. Which better represents the data, a line or a parabola? Explain.
- b. Use the *regression* feature of your calculator to find the model that best fits the data.
- c. Use the model in part (b) to predict when the sponge will hit the ground.
- d. Identify and interpret the domain and range in this situation.
28. **MAKING AN ARGUMENT** Your friend states that quadratic functions with the same x -intercepts have the same equations, vertex, and axis of symmetry. Is your friend correct? Explain your reasoning.

In Exercises 29–32, analyze the differences in the outputs to determine whether the data are *linear*, *quadratic*, or *neither*. Explain. If linear or quadratic, write an equation that fits the data.

29.

Price decrease (dollars), x	0	5	10	15	20
Revenue (\$1000s), y	470	630	690	650	510

30.

Time (hours), x	0	1	2	3	4
Height (feet), y	40	42	44	46	48

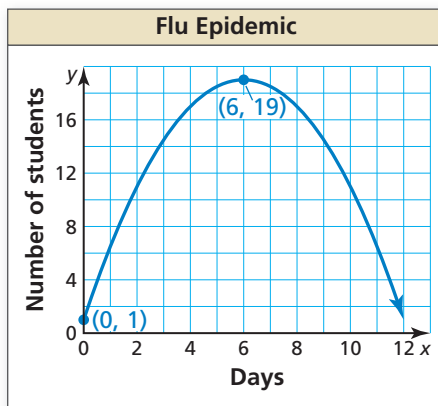
31.

Time (hours), x	1	2	3	4	5
Population (hundreds), y	2	4	8	16	32

32.

Time (days), x	0	1	2	3	4
Height (feet), y	320	303	254	173	60

33. **PROBLEM SOLVING** The graph shows the number y of students absent from school due to the flu each day x .



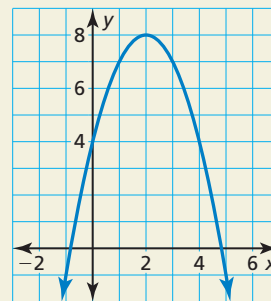
- Interpret the meaning of the vertex in this situation.
- Write an equation for the parabola to predict the number of students absent on day 10.
- Compare the average rates of change in the students with the flu from 0 to 6 days and 6 to 11 days.

34. **THOUGHT PROVOKING** Describe a real-life situation that can be modeled by a quadratic equation. Justify your answer.

35. **PROBLEM SOLVING** The table shows the heights y of a competitive water-skier x seconds after jumping off a ramp. Write a function that models the height of the water-skier over time. When is the water-skier 5 feet above the water? How long is the skier in the air?

Time (seconds), x	0	0.25	0.75	1	1.1
Height (feet), y	22	22.5	17.5	12	9.24

36. **HOW DO YOU SEE IT?** Use the graph to determine whether the average rate of change over each interval is *positive*, *negative*, or *zero*.



- $0 \leq x \leq 2$
- $2 \leq x \leq 5$
- $2 \leq x \leq 4$
- $0 \leq x \leq 4$

37. **REPEATED REASONING** The table shows the number of tiles in each figure. Verify that the data show a quadratic relationship. Predict the number of tiles in the 12th figure.

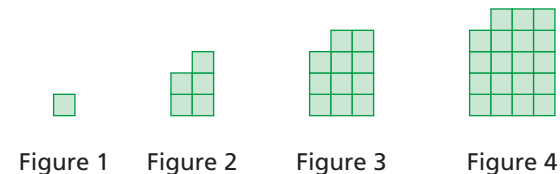


Figure	1	2	3	4
Number of Tiles	1	5	11	19

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Factor the trinomial. (*Skills Review Handbook*)

38. $x^2 + 4x + 3$

39. $x^2 - 3x + 2$

40. $3x^2 - 15x + 12$

41. $5x^2 + 5x - 30$

2.5–2.7 What Did You Learn?

Core Vocabulary

quadratic function, *p.* 74
parabola, *p.* 74
vertex of a parabola, *p.* 76

vertex form, *p.* 76
axis of symmetry, *p.* 82
standard form, *p.* 82

minimum value, *p.* 84
maximum value, *p.* 84
intercept form, *p.* 85

Core Concepts

Section 2.5

Horizontal Translations, *p.* 74
Vertical Translations, *p.* 74
Reflections in the *x*-Axis, *p.* 75

Reflections in the *y*-Axis, *p.* 75
Horizontal Stretches and Shrinks, *p.* 75
Vertical Stretches and Shrinks, *p.* 75

Section 2.6

Properties of the Graph of $f(x) = ax^2 + bx + c$,
p. 83
Minimum and Maximum Values, *p.* 84

Properties of the Graph of $f(x) = a(x - p)(x - q)$,
p. 85

Section 2.7

Writing Quadratic Equations, *p.* 92

Writing Quadratic Equations to Model Data, *p.* 94

Mathematical Practices

1. Why does the height you found in Exercise 44 on page 79 make sense in the context of the situation?
2. How can you use technology to deepen your understanding of the concepts in Exercise 83 on page 90?
3. Describe how you were able to construct a viable argument in Exercise 28 on page 97.

Performance Task:

Changing the Course

Designers of motocross races use mathematics to create ramps and jumps for their courses. How could you modify their models so that riders will catch more air on your track?

To explore the answer to this question and more, check out the Performance Task and Real-Life STEM video at BigIdeasMath.com.

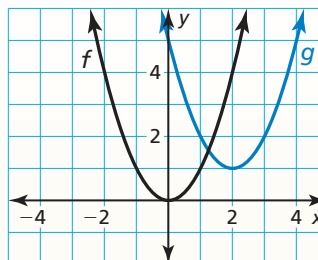


2.1 Parent Functions and Transformations (pp. 39–46)

Graph $g(x) = (x - 2)^2 + 1$ and its parent function. Then describe the transformation.

The function g is a quadratic function.

- The graph of g is a translation 2 units right and 1 unit up of the graph of the parent quadratic function.



Graph the function and its parent function. Then describe the transformation.

- | | | |
|-------------------|----------------------|----------------------------|
| 1. $f(x) = x + 3$ | 2. $g(x) = x - 1$ | 3. $h(x) = \frac{1}{2}x^2$ |
| 4. $h(x) = 4$ | 5. $f(x) = - x - 3$ | 6. $g(x) = -3(x + 3)^2$ |

2.2 Transformations of Linear and Absolute Value Functions (pp. 47–54)

Let the graph of g be a translation 2 units to the right followed by a reflection in the y -axis of the graph of $f(x) = |x|$. Write a rule for g .

Step 1 First write a function h that represents the translation of f .

$$\begin{aligned} h(x) &= f(x - 2) && \text{Subtract 2 from the input.} \\ &= |x - 2| && \text{Replace } x \text{ with } x - 2 \text{ in } f(x). \end{aligned}$$

Step 2 Then write a function g that represents the reflection of h .

$$\begin{aligned} g(x) &= h(-x) && \text{Multiply the input by } -1. \\ &= |-x - 2| && \text{Replace } x \text{ with } -x \text{ in } h(x). \\ &= |-(x + 2)| && \text{Factor out } -1. \\ &= |-1| \cdot |x + 2| && \text{Product Property of Absolute Value} \\ &= |x + 2| && \text{Simplify.} \end{aligned}$$

- The transformed function is $g(x) = |x + 2|$.

Write a function g whose graph represents the indicated transformations of the graph of f . Use a graphing calculator to check your answer.

- $f(x) = |x|$; reflection in the x -axis followed by a translation 4 units to the left
- $f(x) = |x|$; vertical shrink by a factor of $\frac{1}{2}$ followed by a translation 2 units up
- $f(x) = x$; translation 3 units down followed by a reflection in the y -axis

2.3 Modeling with Linear Functions (pp. 55–62)

The table shows the numbers of ice cream cones sold for different outside temperatures (in degrees Fahrenheit). Do the data show a linear relationship? If so, write an equation of a line of fit and use it to estimate how many ice cream cones are sold when the temperature is 60°F.

Temperature, x	53	62	70	82	90
Number of cones, y	90	105	117	131	147

Step 1 Create a scatter plot of the data. The data show a linear relationship.

Step 2 Sketch the line that appears to most closely fit the data. One possibility is shown.

Step 3 Choose two points on the line. For the line shown, you might choose (70, 117) and (90, 147).

Step 4 Write an equation of the line. First, find the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{147 - 117}{90 - 70} = \frac{30}{20} = 1.5$$

Use point-slope form to write an equation.

Use $(x_1, y_1) = (70, 117)$.

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$

$$y - 117 = 1.5(x - 70) \quad \text{Substitute for } m, x_1, \text{ and } y_1.$$

$$y - 117 = 1.5x - 105 \quad \text{Distributive Property}$$

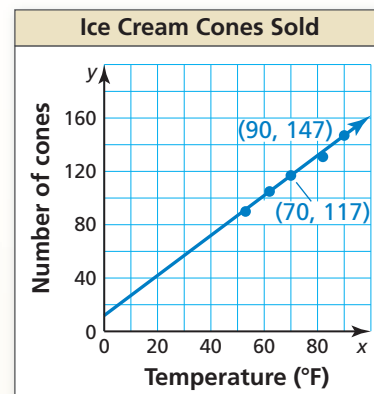
$$y = 1.5x + 12 \quad \text{Add 117 to each side.}$$

Use the equation to estimate the number of ice cream cones sold.

$$y = 1.5(60) + 12 \quad \text{Substitute 60 for } x.$$

$$= 102 \quad \text{Simplify.}$$

▶ Approximately 102 ice cream cones are sold when the temperature is 60°F.



Write an equation of the line.

10. The table shows the total number y (in billions) of U.S. movie admissions each year for x years. Use a graphing calculator to find an equation of the line of best fit for the data.

Year, x	0	2	4	6	8	10
Admissions, y	1.24	1.26	1.39	1.47	1.49	1.57

11. You ride your bike and measure how far you travel. After 10 minutes, you travel 3.5 miles. After 30 minutes, you travel 10.5 miles. Write an equation to model your distance. How far can you ride your bike in 45 minutes?

2.4 Solving Linear Systems (pp. 63–70)

Solve the system.

$$x - y + z = -3 \quad \text{Equation 1}$$

$$2x - y + 5z = 4 \quad \text{Equation 2}$$

$$4x + 2y - z = 2 \quad \text{Equation 3}$$

Step 1 Rewrite the system as a linear system in two variables.

$$\begin{array}{r} x - y + z = -3 \\ 4x + 2y - z = 2 \\ \hline 5x + y = -1 \end{array} \quad \begin{array}{l} \text{Add Equation 1 to} \\ \text{Equation 3 (to eliminate } z\text{).} \end{array}$$

$$5x + y = -1 \quad \text{New Equation 3}$$

$$\begin{array}{r} -5x + 5y - 5z = 15 \\ 2x - y + 5z = 4 \\ \hline -3x + 4y = 19 \end{array} \quad \begin{array}{l} \text{Add } -5 \text{ times Equation 1 to} \\ \text{Equation 2 (to eliminate } z\text{).} \end{array}$$

$$-3x + 4y = 19 \quad \text{New Equation 2}$$

Step 2 Solve the new linear system for both of its variables.

$$\begin{array}{r} -20x - 4y = 4 \\ -3x + 4y = 19 \\ \hline -23x = 23 \end{array} \quad \begin{array}{l} \text{Add } -4 \text{ times new Equation 3} \\ \text{to new Equation 2.} \end{array}$$

$$-23x = 23$$

$$x = -1 \quad \text{Solve for } x.$$

$$y = 4 \quad \text{Substitute into new Equation 2 or 3 to find } y.$$

Step 3 Substitute $x = -1$ and $y = 4$ into an original equation and solve for z .

$$x - y + z = -3 \quad \text{Write original Equation 1.}$$

$$(-1) - 4 + z = -3 \quad \text{Substitute } -1 \text{ for } x \text{ and } 4 \text{ for } y.$$

$$z = 2 \quad \text{Solve for } z.$$

▶ The solution is $x = -1$, $y = 4$, and $z = 2$, or the ordered triple $(-1, 4, 2)$.

Solve the system. Check your solution, if possible.

12. $x + y + z = 3$

$$-x + 3y + 2z = -8$$

$$x = 4z$$

13. $2x - 5y - z = 17$

$$x + y + 3z = 19$$

$$-4x + 6y + z = -20$$

14. $x + y + z = 2$

$$2x - 3y + z = 11$$

$$-3x + 2y - 2z = -13$$

15. $x + 4y - 2z = 3$

$$x + 3y + 7z = 1$$

$$2x + 9y - 13z = 2$$

16. $x - y + 3z = 6$

$$x - 2y = 5$$

$$2x - 2y + 5z = 9$$

17. $x + 2z = 4$

$$x + y + z = 6$$

$$3x + 3y + 4z = 28$$

18. A school band performs a spring concert for a crowd of 600 people. The revenue for the concert is \$3150. There are 150 more adults at the concert than students. How many of each type of ticket are sold?



2.5 Transformations of Quadratic Functions (pp. 73–80)

Let the graph of g be a translation 1 unit left and 2 units up of the graph of $f(x) = x^2 + 1$. Write a rule for g .

$$\begin{aligned} g(x) &= f(x - (-1)) + 2 && \text{Subtract } -1 \text{ from the input. Add 2 to the output.} \\ &= (x + 1)^2 + 1 + 2 && \text{Replace } x \text{ with } x + 1 \text{ in } f(x). \\ &= x^2 + 2x + 4 && \text{Simplify.} \end{aligned}$$

▶ The transformed function is $g(x) = x^2 + 2x + 4$.

Describe the transformation of $f(x) = x^2$ represented by g . Then graph each function.

19. $g(x) = (x + 4)^2$ 20. $g(x) = (x - 7)^2 + 2$ 21. $g(x) = -3(x + 2)^2 - 1$

Write a rule for g .

22. Let the graph of g be a horizontal shrink by a factor of $\frac{2}{3}$, followed by a translation 5 units left and 2 units down of the graph of $f(x) = x^2$.
23. Let the graph of g be a translation 2 units left and 3 units up, followed by a reflection in the y -axis of the graph of $f(x) = x^2 - 2x$.

2.6 Characteristics of Quadratic Functions (pp. 81–90)

Graph $f(x) = 2x^2 - 8x + 1$. Label the vertex and axis of symmetry.

Step 1 Identify the coefficients $a = 2$, $b = -8$, and $c = 1$.
Because $a > 0$, the parabola opens up.

Step 2 Find the vertex. First calculate the x -coordinate.

$$x = -\frac{b}{2a} = -\frac{-8}{2(2)} = 2$$

Then find the y -coordinate of the vertex.

$$f(2) = 2(2)^2 - 8(2) + 1 = -7$$

So, the vertex is $(2, -7)$. Plot this point.

Step 3 Draw the axis of symmetry $x = 2$.

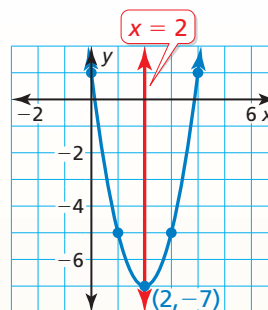
Step 4 Identify the y -intercept c , which is 1. Plot the point $(0, 1)$ and its reflection in the axis of symmetry, $(4, 1)$.

Step 5 Evaluate the function for another value of x , such as $x = 1$.

$$f(1) = 2(1)^2 - 8(1) + 1 = -5$$

Plot the point $(1, -5)$ and its reflection in the axis of symmetry, $(3, -5)$.

Step 6 Draw a parabola through the plotted points.

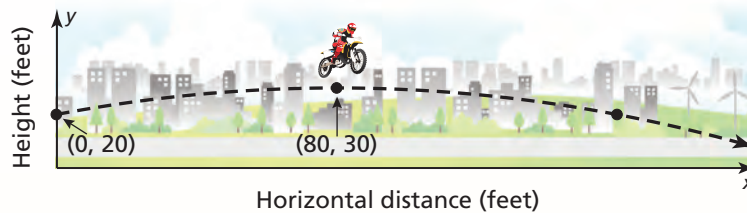


Graph the function. Label the vertex and axis of symmetry. Find the minimum or maximum value of f . Describe where the function is increasing and decreasing.

24. $f(x) = 3(x - 1)^2 - 4$ 25. $g(x) = -2x^2 + 16x + 3$ 26. $h(x) = (x - 3)(x + 7)$

2.7 Modeling with Quadratic Functions (pp. 91–98)

The graph shows the parabolic path of a stunt motorcyclist jumping off a ramp, where y is the height (in feet) and x is the horizontal distance traveled (in feet). Write an equation of the parabola. The motorcyclist lands on another ramp 160 feet from the first ramp. What is the height of the second ramp?



Step 1 First write an equation of the parabola.

From the graph, you can see that the vertex (h, k) is $(80, 30)$ and the parabola passes through the point $(0, 20)$. Use the vertex and the point to solve for a in vertex form.

$$\begin{aligned}
 y &= a(x - h)^2 + k && \text{Vertex form} \\
 20 &= a(0 - 80)^2 + 30 && \text{Substitute for } h, k, x, \text{ and } y. \\
 -10 &= 6400a && \text{Simplify.} \\
 -\frac{1}{640} &= a && \text{Divide each side by 6400.}
 \end{aligned}$$

Because $a = -\frac{1}{640}$, $h = 80$, and $k = 30$, the path can be modeled by

$$y = -\frac{1}{640}(x - 80)^2 + 30, \text{ where } 0 \leq x \leq 160.$$

Step 2 Then find the height of the second ramp.

$$\begin{aligned}
 y &= -\frac{1}{640}(160 - 80)^2 + 30 && \text{Substitute 160 for } x. \\
 &= 20 && \text{Simplify.}
 \end{aligned}$$

► So, the height of the second ramp is 20 feet.

Write an equation of the parabola with the given characteristics.

27. passes through $(1, 12)$ and has vertex $(10, -4)$
28. passes through $(4, 3)$ and has x -intercepts of -1 and 5
29. passes through $(-2, 7)$, $(1, 10)$, and $(2, 27)$
30. The table shows the heights y of a dropped object after x seconds. Verify that the data show a quadratic relationship. Write a function that models the data. How long is the object in the air?

Time (seconds), x	0	0.5	1	1.5	2	2.5
Height (feet), y	150	146	134	114	86	50

2 Chapter Test

Graph the function and its parent function. Then describe the transformation.

1. $f(x) = |x - 1|$

2. $f(x) = (3x)^2$

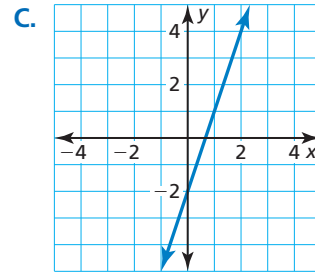
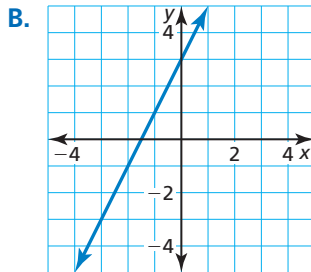
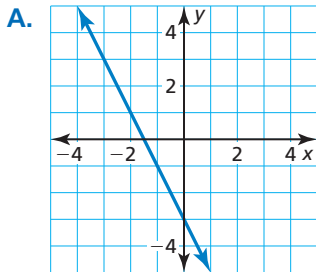
3. $f(x) = -\frac{1}{2}x - 4$

Match the transformation of $f(x) = x$ with its graph. Then write a rule for g .

4. $g(x) = 2f(x) + 3$

5. $g(x) = 3f(x) - 2$

6. $g(x) = -2f(x) - 3$



7. Graph $f(x) = 8x^2 - 4x + 3$. Label the vertex and axis of symmetry. Describe where the function is increasing and decreasing.

8. Let the graph of g be a translation 2 units left and 1 unit down, followed by a reflection in the y -axis of the graph of $f(x) = (2x + 1)^2 - 4$. Write a rule for g .

Write a linear function or a quadratic function that models the data.

9.

x	-2	-1	0	1	2
f(x)	-1	3	7	11	15

10.

x	2	4	6	8	10
f(x)	0	-13	-34	-63	-100

Solve the system. Check your solution, if possible.

11. $-2x + y + 4z = 5$

12. $y = \frac{1}{2}z$

13. $x - y + 5z = 3$

$x + 3y - z = 2$

$x + 2y + 5z = 2$

$2x + 3y - z = 2$

$4x + y - 6z = 11$

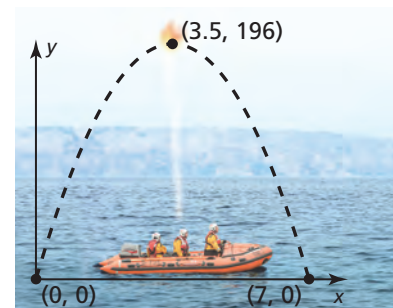
$3x + 6y - 3z = 9$

$-4x - y - 9z = -8$

14. The graph of a quadratic function f has an axis of symmetry $x = 3$ and passes through the point $(0, 6)$. Find another point that lies on the parabola. Then write an equation of the parabola when the minimum value of f is -4 .

15. A passenger on a stranded lifeboat shoots a distress flare into the air. The height (in feet) of the flare above the water is given by $f(t) = -16t(t - 8)$, where t is time (in seconds) since the flare was shot. The passenger shoots a second flare, whose path is modeled in the graph. Which flare travels higher? Which remains in the air longer? Justify your answer.

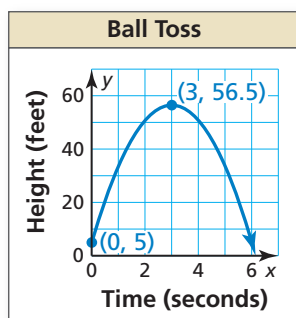
16. A surfboard shop sells 40 surfboards per month when it charges \$500 per surfboard. Each time the shop decreases the price by \$10, it sells 1 additional surfboard per month. How much should the shop charge per surfboard to maximize the amount of money earned? What is the maximum amount the shop can earn per month? Explain.



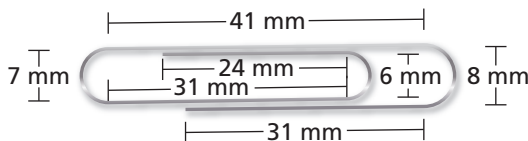
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Cumulative Assessment

- The function $g(x) = \frac{1}{2}|x - 4| + 4$ is a combination of transformations of $f(x) = |x|$. Which combinations describe the transformation from the graph of f to the graph of g ?
 - translation 4 units right and vertical shrink by a factor of $\frac{1}{2}$, followed by a translation 4 units up
 - translation 4 units right and 4 units up, followed by a vertical shrink by a factor of $\frac{1}{2}$
 - vertical shrink by a factor of $\frac{1}{2}$, followed by a translation 4 units up and 4 units right
 - translation 4 units right and 8 units up, followed by a vertical shrink by a factor of $\frac{1}{2}$
- Two balls are thrown in the air. The path of the first ball is represented in the graph. The second ball is released 1.5 feet higher than the first ball and after 3 seconds reaches its maximum height 5 feet lower than the first ball.



- Write an equation for the path of the second ball.
 - Do the balls hit the ground at the same time? If so, how long are the balls in the air? If not, which ball hits the ground first? Explain your reasoning.
- The paper clip is made from cylindrical metal wire with a diameter of 1 millimeter. The density of the metal is about 7.8 grams per cubic centimeter. Approximate the mass of the paper clip to the nearest gram. Explain your procedure.



- Gym A charges \$10 per month plus an initiation fee of \$100. Gym B charges \$30 per month, but due to a special promotion, is not currently charging an initiation fee.
 - Write an equation for each gym modeling the total cost y for a membership lasting x months.
 - When is it more economical for a person to choose Gym A over Gym B?
 - Gym A lowers its initiation fee to \$25. Describe the transformation this change represents and how it affects your decision in part (b).

