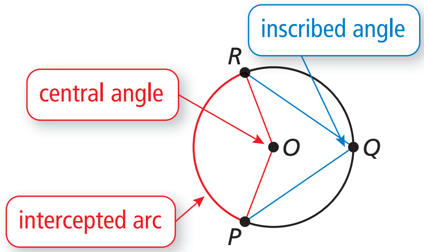
10.4 Inscribed Angles and Polygons

Essential Question

How are inscribed angles related to their intercepted arcs? How are the angles of an inscribed quadrilateral related to each other?

An **inscribed angle** is an angle whose vertex is on a circle and whose sides contain chords of the circle. An arc that lies between two lines, rays, or segments is called an **intercepted arc**. Recall that a polygon is an inscribed polygon when all its vertices lie on a circle.



EXPLORATION 1 Inscribed Angles and Central Angles

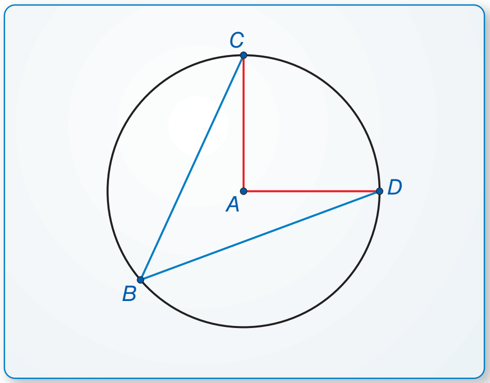
Work with a partner. Use dynamic geometry software.

a. Construct an inscribed angle in a circle. Then construct the corresponding central angle.

b. Measure both angles. How is the inscribed angle related to its intercepted arc?

c. Repeat parts (a) and (b) several times. Record your results in a table. Write a conjecture about how an inscribed angle is related to its intercepted arc.

Sample



ATTENDING TO PRECISION

To be proficient in math, you need to communicate precisely with others.

EXPLORATION 2 A Quadrilateral with Inscribed Angles

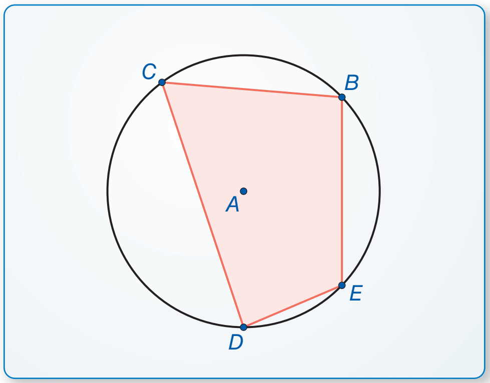
Work with a partner. Use dynamic geometry software.

a. Construct a quadrilateral with each vertex on a circle.

b. Measure all four angles. What relationships do you notice?

c. Repeat parts (a) and (b) several times. Record your results in a table. Then write a conjecture that summarizes the data.

Sample



Communicate Your Answer

3. How are inscribed angles related to their intercepted arcs? How are the angles of an inscribed quadrilateral related to each other?

4. Quadrilateral $EFGH$ is inscribed in $\odot C$, and $m\angle E = 80^\circ$. What is $m\angle G$? Explain.

10.4 Lesson

Core Vocabulary

inscribed angle, *p. 598*
intercepted arc, *p. 598*
subtend, *p. 598*

Previous
inscribed polygon
circumscribed circle

What You Will Learn

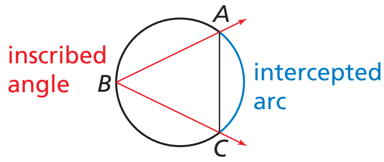
- Use inscribed angles.
- Use inscribed polygons.

Using Inscribed Angles

Core Concept

Inscribed Angle and Intercepted Arc

An **inscribed angle** is an angle whose vertex is on a circle and whose sides contain chords of the circle. An arc that lies between two lines, rays, or segments is called an **intercepted arc**. If the endpoints of a chord or arc lie on the sides of an inscribed angle, then the chord or arc is said to **subtend** the angle.

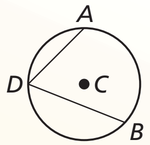


$\angle B$ intercepts $\overparen{AC}$.
$\overparen{AC}$ subtends $\angle B$.
$\overline{AC}$ subtends $\angle B$.

Theorem

Measure of an Inscribed Angle Theorem

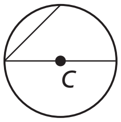
The measure of an inscribed angle is one-half the measure of its intercepted arc.



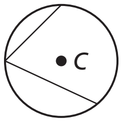
$m\angle ADB = \frac{1}{2}m\overparen{AB}$

Proof Ex. 35, p. 604

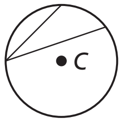
The proof of the Measure of an Inscribed Angle Theorem involves three cases.



Case 1 Center C is on a side of the inscribed angle.



Case 2 Center C is inside the inscribed angle.



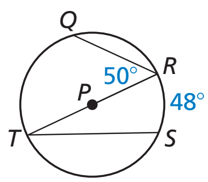
Case 3 Center C is outside the inscribed angle.

EXAMPLE 1 Using Inscribed Angles

Find the indicated measure.

a. $m\angle T$

b. $m\overparen{QR}$



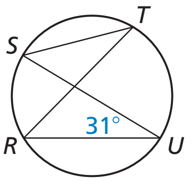
SOLUTION

a. $m\angle T = \frac{1}{2}m\overparen{RS} = \frac{1}{2}(48°) = 24°$

b. $m\overparen{TQ} = 2m\angle R = 2 \cdot 50° = 100°$
Because $\overparen{TQR}$ is a semicircle, $m\overparen{QR} = 180° - m\overparen{TQ} = 180° - 100° = 80°$.

EXAMPLE 2 Finding the Measure of an Intercepted Arc

Find $m\widehat{RS}$ and $m\angle STR$. What do you notice about $\angle STR$ and $\angle RUS$?



SOLUTION

From the Measure of an Inscribed Angle Theorem, you know that $m\widehat{RS} = 2m\angle RUS = 2(31°) = 62°$.

Also, $m\angle STR = \frac{1}{2}m\widehat{RS} = \frac{1}{2}(62°) = 31°$.

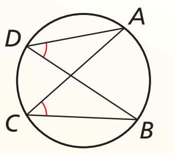
▶ So, $\angle STR \cong \angle RUS$.

Example 2 suggests the Inscribed Angles of a Circle Theorem.

Theorem

Inscribed Angles of a Circle Theorem

If two inscribed angles of a circle intercept the same arc, then the angles are congruent.

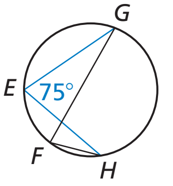


Proof Ex. 36, p. 604

$\angle ADB \cong \angle ACB$

EXAMPLE 3 Finding the Measure of an Angle

Given $m\angle E = 75°$, find $m\angle F$.



SOLUTION

Both $\angle E$ and $\angle F$ intercept $\widehat{GH}$. So, $\angle E \cong \angle F$ by the Inscribed Angles of a Circle Theorem.

▶ So, $m\angle F = m\angle E = 75°$.

Monitoring Progress

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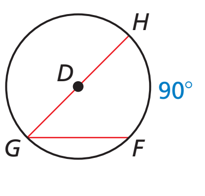
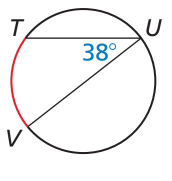
Find the measure of the red arc or angle.

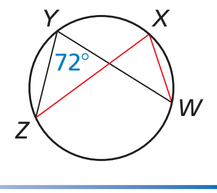
1.



2.

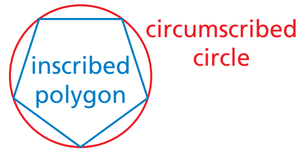


3.



Using Inscribed Polygons

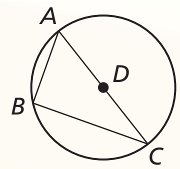
Recall that a polygon is an inscribed polygon when all its vertices lie on a circle. The circle that contains the vertices is a circumscribed circle.



Theorems

Inscribed Right Triangle Theorem

If a right triangle is inscribed in a circle, then the hypotenuse is a diameter of the circle. Conversely, if one side of an inscribed triangle is a diameter of the circle, then the triangle is a right triangle and the angle opposite the diameter is the right angle.

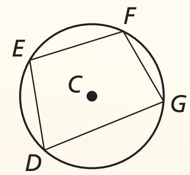


$m\angle ABC = 90°$ if and only if $\overline{AC}$ is a diameter of the circle.

Proof Ex. 37, p. 604

Inscribed Quadrilateral Theorem

A quadrilateral can be inscribed in a circle if and only if its opposite angles are supplementary.



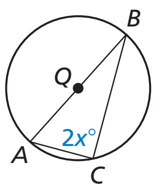
D, E, F, and G lie on $\odot C$ if and only if $m\angle D + m\angle F = m\angle E + m\angle G = 180°$.

Proof Ex. 38, p. 604; *BigIdeasMath.com*

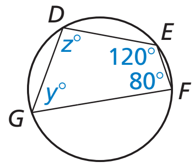
EXAMPLE 4 Using Inscribed Polygons

Find the value of each variable.

a.



b.



SOLUTION

a. $\overline{AB}$ is a diameter. So, $\angle C$ is a right angle, and $m\angle C = 90°$ by the Inscribed Right Triangle Theorem.

$$2x° = 90°$$

$$x = 45$$

▶ The value of x is 45.

b. *DEFG* is inscribed in a circle, so opposite angles are supplementary by the Inscribed Quadrilateral Theorem.

$$m\angle D + m\angle F = 180° \qquad m\angle E + m\angle G = 180°$$

$$z + 80 = 180 \qquad 120 + y = 180$$

$$z = 100 \qquad y = 60$$

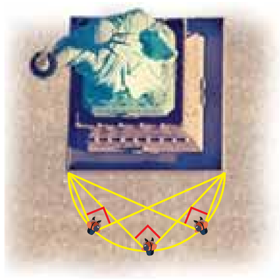
▶ The value of z is 100 and the value of y is 60.

EXAMPLE 5 Using a Circumscribed Circle

Your camera has a 90° field of vision, and you want to photograph the front of a statue. You stand at a location in which the front of the statue is all that appears in your camera's field of vision, as shown. You want to change your location. Where else can you stand so that the front of the statue is all that appears in your camera's field of vision?

SOLUTION

From the Inscribed Right Triangle Theorem, you know that if a right triangle is inscribed in a circle, then the hypotenuse of the triangle is a diameter of the circle. So, draw the circle that has the front of the statue as a diameter.



▶ The statue fits perfectly within your camera's 90° field of vision from any point on the semicircle in front of the statue.

Monitoring Progress

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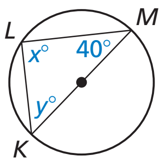
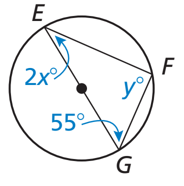
Find the value of each variable.

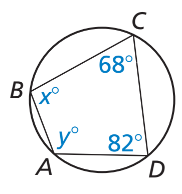
4.



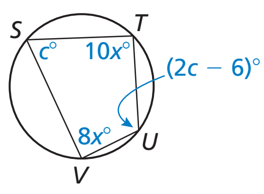
5.



6.



7.



8. In Example 5, explain how to find locations where the left side of the statue is all that appears in your camera's field of vision.

10.4 Exercises

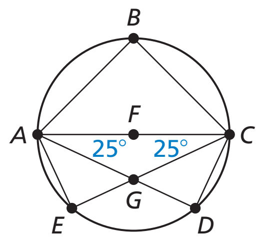
Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** An arc that lies between two lines, rays, or segments is called a(n) ________________.

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

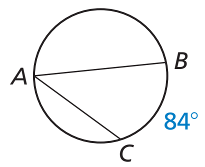
Find $m\angle ABC$.	Find $m\angle AGC$.
Find $m\angle AEC$.	Find $m\angle ADC$.



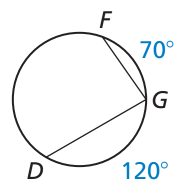
Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, find the indicated measure. *(See Examples 1 and 2.)*

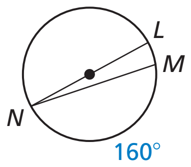
3. $m\angle A$



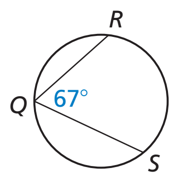
4. $m\angle G$



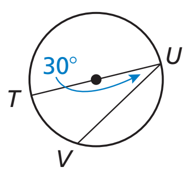
5. $m\angle N$



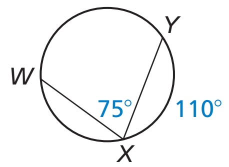
6. $m\overparen{RS}$



7. $m\overparen{VU}$

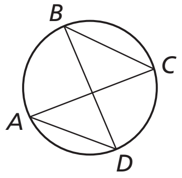


8. $m\overparen{WX}$

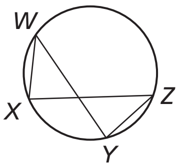


In Exercises 9 and 10, name two pairs of congruent angles.

9.

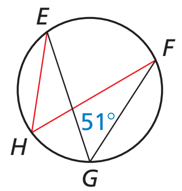


10.

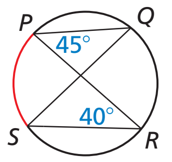


In Exercises 11 and 12, find the measure of the red arc or angle. *(See Example 3.)*

11.

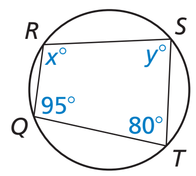


12.



In Exercises 13–16, find the value of each variable. *(See Example 4.)*

13.



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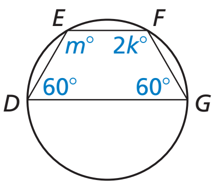
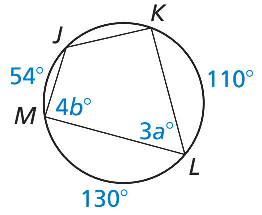
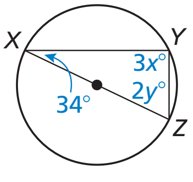


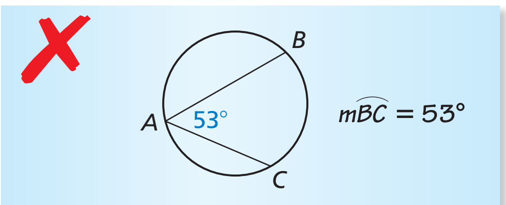
15.



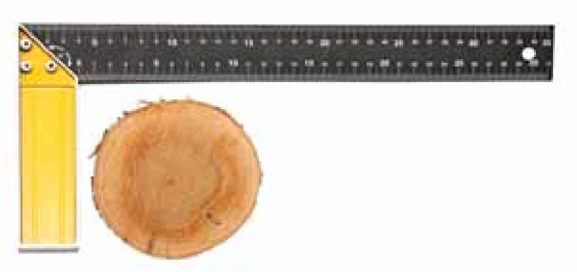
16.



17. **ERROR ANALYSIS** Describe and correct the error in finding $m\overparen{BC}$.

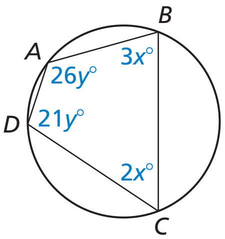


18. **MODELING WITH MATHEMATICS** A *carpenter's square* is an L-shaped tool used to draw right angles. You need to cut a circular piece of wood into two semicircles. How can you use the carpenter's square to draw a diameter on the circular piece of wood? *(See Example 5.)*

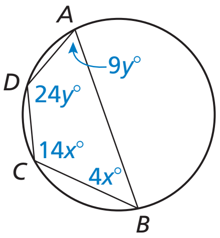


MATHEMATICAL CONNECTIONS **In Exercises 19–21, find the values of x and y. Then find the measures of the interior angles of the polygon.**

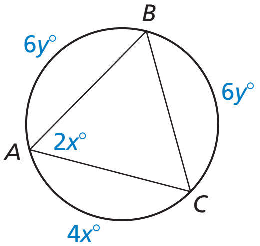
19.



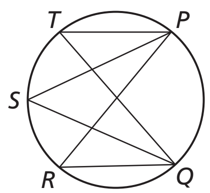
20.



21.



22. **MAKING AN ARGUMENT** Your friend claims that $\angle PTQ \cong \angle PSQ \cong \angle PRQ$. Is your friend correct? Explain your reasoning.



REASONING **In Exercises 23–28, determine whether a quadrilateral of the given type can always be inscribed inside a circle. Explain your reasoning.**

23. square

24. rectangle

25. parallelogram

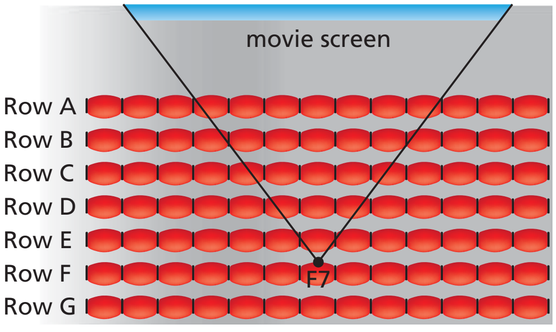
26. kite

27. rhombus

28. isosceles trapezoid

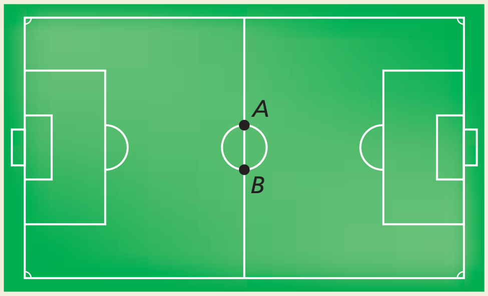
29. **MODELING WITH MATHEMATICS** Three moons, A, B, and C, are in the same circular orbit 100,000 kilometers above the surface of a planet. The planet is 20,000 kilometers in diameter and $m\angle ABC = 90^\circ$. Draw a diagram of the situation. How far is moon A from moon C?

30. **MODELING WITH MATHEMATICS** At the movie theater, you want to choose a seat that has the best *viewing angle*, so that you can be close to the screen and still see the whole screen without moving your eyes. You previously decided that seat F7 has the best viewing angle, but this time someone else is already sitting there. Where else can you sit so that your seat has the same viewing angle as seat F7? Explain.



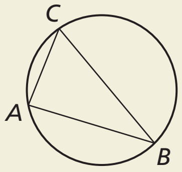
31. **WRITING** A right triangle is inscribed in a circle, and the radius of the circle is given. Explain how to find the length of the hypotenuse.

32. **HOW DO YOU SEE IT?** Let point Y represent your location on the soccer field below. What type of angle is $\angle AYB$ if you stand anywhere on the circle except at point A or point B?



33. **WRITING** Explain why the diagonals of a rectangle inscribed in a circle are diameters of the circle.

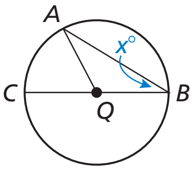
34. **THOUGHT PROVOKING** The figure shows a circle that is circumscribed about $\triangle ABC$. Is it possible to circumscribe a circle about any triangle? Justify your answer.



35. **PROVING A THEOREM** If an angle is inscribed in $\odot Q$, the center Q can be on a side of the inscribed angle, inside the inscribed angle, or outside the inscribed angle. Prove each case of the Measure of an Inscribed Angle Theorem.

a. Case 1

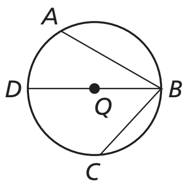
Given $\angle ABC$ is inscribed in $\odot Q$.
Let $m\angle B = x°$.
Center Q lies on $\overline{BC}$.



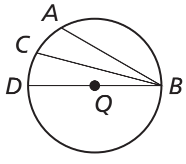
Prove $m\angle ABC = \frac{1}{2} m\widehat{AC}$

(*Hint*: Show that $\triangle AQB$ is isosceles. Then write $m\widehat{AC}$ in terms of x.)

b. Case 2 Use the diagram and auxiliary line to write **Given** and **Prove** statements for Case 2. Then write a proof.



c. Case 3 Use the diagram and auxiliary line to write **Given** and **Prove** statements for Case 3. Then write a proof.



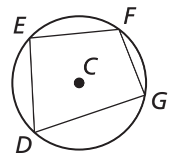
36. **PROVING A THEOREM** Write a paragraph proof of the Inscribed Angles of a Circle Theorem. First, draw a diagram and write **Given** and **Prove** statements.

37. **PROVING A THEOREM** The Inscribed Right Triangle Theorem is written as a conditional statement and its converse. Write a plan for proof for each statement.

38. **PROVING A THEOREM** Copy and complete the paragraph proof for one part of the Inscribed Quadrilateral Theorem.

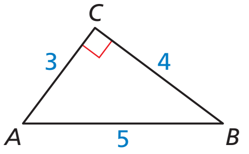
Given $\odot C$ with inscribed quadrilateral $DEFG$

Prove $m\angle D + m\angle F = 180°$,
$m\angle E + m\angle G = 180°$

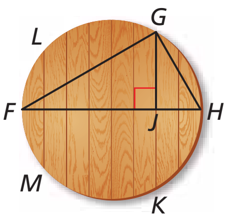


By the Arc Addition Postulate, $m\widehat{EFG}$ + ____ = 360° and $m\widehat{FGD} + m\widehat{DEF} = 360°$. Using the ________ Theorem, $m\widehat{EDG} = 2m\angle F$, $m\widehat{EFG} = 2m\angle D$, $m\widehat{DEF} = 2m\angle G$, and $m\widehat{FGD} = 2m\angle E$. By the Substitution Property of Equality, $2m\angle D$ + ___ = 360°, so ___. Similarly, ___.

39. **CRITICAL THINKING** In the diagram, $\angle C$ is a right angle. If you draw the smallest possible circle through C tangent to $\overline{AB}$, the circle will intersect $\overline{AC}$ at J and $\overline{BC}$ at K. Find the exact length of $\overline{JK}$.



40. **CRITICAL THINKING** You are making a circular cutting board. To begin, you glue eight 1-inch boards together, as shown. Then you draw and cut a circle with an 8-inch diameter from the boards.



a. $\overline{FH}$ is a diameter of the circular cutting board. Write a proportion relating GJ and JH. State a theorem to justify your answer.

b. Find FJ, JH, and GJ. What is the length of the cutting board seam labeled $\overline{GK}$?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution. *(Skills Review Handbook)*

41. $3x = 145$

42. $\frac{1}{2}x = 63$

43. $240 = 2x$

44. $75 = \frac{1}{2}(x - 30)$