7.5 Properties of Trapezoids and Kites

**Essential Question**  What are some properties of trapezoids and kites?

Recall the types of quadrilaterals shown below.

![Trapezoid](image1.png)  ![Isosceles Trapezoid](image2.png)  ![Kite](image3.png)

**EXPLORATION 1** Making a Conjecture about Trapezoids

**Work with a partner.** Use dynamic geometry software.

a. Construct a trapezoid whose base angles are congruent. Explain your process.

b. Is the trapezoid isosceles? Justify your answer.

c. Repeat parts (a) and (b) for several other trapezoids. Write a conjecture based on your results.

**EXPLORATION 2** Discovering a Property of Kites

**Work with a partner.** Use dynamic geometry software.

a. Construct a kite. Explain your process.

b. Measure the angles of the kite. What do you observe?

c. Repeat parts (a) and (b) for several other kites. Write a conjecture based on your results.

**Communicate Your Answer**

3. What are some properties of trapezoids and kites?

4. Is the trapezoid at the left isosceles? Explain.

5. A quadrilateral has angle measures of $70^\circ$, $70^\circ$, $110^\circ$, and $110^\circ$. Is the quadrilateral a kite? Explain.
Chapter 7  Quadrilaterals and Other Polygons

7.5  Lesson

What You Will Learn

- Use properties of trapezoids.
- Use the Trapezoid Midsegment Theorem to find distances.
- Use properties of kites.
- Identify quadrilaterals.

Using Properties of Trapezoids

A trapezoid is a quadrilateral with exactly one pair of parallel sides. The parallel sides are the bases.

Base angles of a trapezoid are two consecutive angles whose common side is a base. A trapezoid has two pairs of base angles. For example, in trapezoid \(ABCD\), \(\angle A\) and \(\angle D\) are one pair of base angles, and \(\angle B\) and \(\angle C\) are the second pair. The nonparallel sides are the legs of the trapezoid.

If the legs of a trapezoid are congruent, then the trapezoid is an isosceles trapezoid.

EXAMPLE 1  Identifying a Trapezoid in the Coordinate Plane

Show that \(ORST\) is a trapezoid. Then decide whether it is isosceles.

**SOLUTION**

Step 1  Compare the slopes of opposite sides.

slope of \(RS\) = \(\frac{4 - 3}{2 - 0}\) = \(\frac{1}{2}\)

slope of \(OT\) = \(\frac{2 - 0}{4 - 0}\) = \(\frac{1}{2}\)

The slopes of \(RS\) and \(OT\) are the same, so \(RS \parallel OT\).

slope of \(ST\) = \(\frac{2 - 4}{4 - 2}\) = \(-\frac{2}{2}\) = \(-1\)

slope of \(RO\) = \(\frac{3 - 0}{0 - 0}\) = \(\frac{3}{0}\)  Undefined

The slopes of \(ST\) and \(RO\) are not the same, so \(ST\) is not parallel to \(OR\).

Because \(ORST\) has exactly one pair of parallel sides, it is a trapezoid.

Step 2  Compare the lengths of legs \(RO\) and \(ST\).

\(RO = |3 - 0| = 3\)  \(ST = \sqrt{(2 - 4)^2 + (4 - 2)^2} = \sqrt{8} = 2\sqrt{2}\)

Because \(RO \neq ST\), legs \(RO\) and \(ST\) are not congruent.

So, \(ORST\) is not an isosceles trapezoid.

Monitoring Progress

1. The points \(A(-5, 6), B(4, 9), C(4, 4),\) and \(D(-2, 2)\) form the vertices of a quadrilateral. Show that \(ABCD\) is a trapezoid. Then decide whether it is isosceles.
The stone above the arch in the diagram is an isosceles trapezoid. Find \(m \angle K\), \(m \angle M\), and \(m \angle J\).

**SOLUTION**

**Step 1** Find \(m \angle K\). The trapezoid \(JKLM\) is isosceles trapezoid. So, \(\angle K\) and \(\angle L\) are congruent base angles, and \(m \angle K = m \angle L = 85^\circ\).

**Step 2** Find \(m \angle M\). Because \(\angle L\) and \(\angle M\) are consecutive interior angles formed by \(\overline{LM}\) intersecting two parallel lines, they are supplementary. So, 
\[m \angle M = 180^\circ - 85^\circ = 95^\circ\].

**Step 3** Find \(m \angle J\). Because \(\angle J\) and \(\angle M\) are a pair of base angles, they are congruent, and 
\[m \angle J = m \angle M = 95^\circ\].

So, \(m \angle K = 85^\circ, m \angle M = 95^\circ, \text{ and } m \angle J = 95^\circ\).

**Monitoring Progress**

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In Exercises 2 and 3, use trapezoid \(EFGH\).

2. If \(EG = FH\), is trapezoid \(EFGH\) isosceles? Explain.

3. If \(m \angle HEF = 70^\circ\) and \(m \angle FGH = 110^\circ\), is trapezoid \(EFGH\) isosceles? Explain.
Using the Trapezoid Midsegment Theorem

Recall that a midsegment of a triangle is a segment that connects the midpoints of two sides of the triangle. The midsegment of a trapezoid is the segment that connects the midpoints of its legs. The theorem below is similar to the Triangle Midsegment Theorem.

**Theorem**

**Trapezoid Midsegment Theorem**

The midsegment of a trapezoid is parallel to each base, and its length is one-half the sum of the lengths of the bases.

If \( MN \) is the midsegment of trapezoid \( ABCD \), then \( MN \parallel AB, MN \parallel DC \), and \( MN = \frac{1}{2}(AB + CD) \).

**Proof** Ex. 49, p. 450

Using the Midsegment of a Trapezoid

In the diagram, \( MN \) is the midsegment of trapezoid \( PQRS \). Find \( MN \).

**SOLUTION**

\[
MN = \frac{1}{2}(PQ + SR) \quad \text{Trapezoid Midsegment Theorem}
\]

\[
= \frac{1}{2}(12 + 28) \quad \text{Substitute 12 for } PQ \text{ and 28 for } SR.
\]

\[
= 20 \quad \text{Simplify.}
\]

\( MN \) is 20 inches.

Using a Midsegment in the Coordinate Plane

Find the length of midsegment \( YZ \) in trapezoid \( STUV \).

**SOLUTION**

Step 1 Find the lengths of \( SV \) and \( TU \).

\[
SV = \sqrt{(0 - 2)^2 + (6 - 2)^2} = \sqrt{20} = 2\sqrt{5}
\]

\[
TU = \sqrt{(8 - 12)^2 + (10 - 2)^2} = \sqrt{80} = 4\sqrt{5}
\]

Step 2 Multiply the sum of \( SV \) and \( TU \) by \( \frac{1}{2} \).

\[
YZ = \frac{1}{2}(2\sqrt{5} + 4\sqrt{5}) = \frac{1}{2}(6\sqrt{5}) = 3\sqrt{5}
\]

\( YZ \) is 3\( \sqrt{5} \) units.

Monitoring Progress

4. In trapezoid \( JKL \), \( \angle J \) and \( \angle M \) are right angles, and \( JK = 9 \) centimeters. The length of midsegment \( NP \) of trapezoid \( JKL \) is 12 centimeters. Sketch trapezoid \( JKL \) and its midsegment. Find \( ML \). Explain your reasoning.

5. Explain another method you can use to find the length of \( YZ \) in Example 4.
Using Properties of Kites
A kite is a quadrilateral that has two pairs of consecutive congruent sides, but opposite sides are not congruent.

Kite Diagonals Theorem
If a quadrilateral is a kite, then its diagonals are perpendicular.
If quadrilateral $ABCD$ is a kite, then $AC \perp BD$.

**Proof** p. 445

Kite Opposite Angles Theorem
If a quadrilateral is a kite, then exactly one pair of opposite angles are congruent.
If quadrilateral $ABCD$ is a kite and $BC \equiv BA$, then $\angle A \equiv \angle C$ and $\angle B \equiv \angle D$.

**Proof** Ex. 47, p. 450

**PROOF** Kite Diagonals Theorem

Given $ABCD$ is a kite, $BC \equiv BA$ and $DC \equiv DA$.

Prove $AC \perp BD$

<table>
<thead>
<tr>
<th>STATEMENTS</th>
<th>REASONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $ABCD$ is a kite with $BC \equiv BA$ and $DC \equiv DA$.</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. $B$ and $D$ lie on the $\perp$ bisector of $AC$.</td>
<td>2. Converse of the $\perp$ Bisector Theorem</td>
</tr>
<tr>
<td>3. $BD$ is the $\perp$ bisector of $AC$.</td>
<td>3. Through any two points, there exists exactly one line.</td>
</tr>
<tr>
<td>4. $AC \perp BD$</td>
<td>4. Definition of $\perp$ bisector</td>
</tr>
</tbody>
</table>

**EXAMPLE 5** Finding Angle Measures in a Kite

Find $m\angle D$ in the kite shown.

**SOLUTION**
By the Kite Opposite Angles Theorem, $DEFG$ has exactly one pair of congruent opposite angles. Because $\angle E \not\equiv \angle G$, $\angle D$ and $\angle F$ must be congruent. So, $m\angle D = m\angle F$. Write and solve an equation to find $m\angle D$.

$$m\angle D + m\angle F + 115^\circ + 73^\circ = 360^\circ$$

Corollary to the Polygon Interior Angles Theorem

$$m\angle D + m\angle D + 115^\circ + 73^\circ = 360^\circ$$

Substitute $m\angle D$ for $m\angle F$. Combine like terms.

$$2m\angle D + 188^\circ = 360^\circ$$

Solve for $m\angle D$.

$$m\angle D = 86^\circ$$
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6. In a kite, the measures of the angles are $3x^\circ$, $75^\circ$, $90^\circ$, and $120^\circ$. Find the value of $x$. What are the measures of the angles that are congruent?

Identifying Special Quadrilaterals

The diagram shows relationships among the special quadrilaterals you have studied in this chapter. Each shape in the diagram has the properties of the shapes linked above it. For example, a rhombus has the properties of a parallelogram and a quadrilateral.

### EXAMPLE 6  Identifying a Quadrilateral

What is the most specific name for quadrilateral $ABCD$?

**SOLUTION**

The diagram shows $AE \cong CE$ and $BE \cong DE$. So, the diagonals bisect each other. By the Parallelogram Diagonals Converse, $ABCD$ is a parallelogram.

Rectangles, rhombuses, and squares are also parallelograms. However, there is no information given about the side lengths or angle measures of $ABCD$. So, you cannot determine whether it is a rectangle, a rhombus, or a square.

So, the most specific name for $ABCD$ is a parallelogram.

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7. Quadrilateral $DEFG$ has at least one pair of opposite sides congruent. What types of quadrilaterals meet this condition?

Give the most specific name for the quadrilateral. Explain your reasoning.

8. $\quad$ 9. $\quad$ 10.
7.5 Exercises

Vocabulary and Core Concept Check

1. **WRITING** Describe the differences between a trapezoid and a kite.

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find “both” answers.

   - Is there enough information to prove that trapezoid \(ABCD\) is isosceles?
   - Is there enough information to prove that \(AB \cong DC\)?
   - Is there enough information to prove that the non-parallel sides of trapezoid \(ABCD\) are congruent?
   - Is there enough information to prove that the legs of trapezoid \(ABCD\) are congruent?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, show that the quadrilateral with the given vertices is a trapezoid. Then decide whether it is isosceles. (See Example 1.)

3. \(W(1, 4), X(1, 8), Y(-3, 9), Z(-3, 3)\)

4. \(D(-3, 3), E(-1, 1), F(1, -4), G(-3, 0)\)

5. \(M(-2, 0), N(0, 4), P(5, 4), Q(8, 0)\)

6. \(H(1, 9), J(4, 2), K(5, 2), L(8, 9)\)

In Exercises 7 and 8, find the measure of each angle in the isosceles trapezoid. (See Example 2.)

7. \(\angle KLM = 118^\circ\)

8. \(\angle QTS = 82^\circ\)

In Exercises 9 and 10, find the length of the midsegment of the trapezoid. (See Example 3.)

9. \(MN = 18\)

10. \(MN = 57\)

In Exercises 11 and 12, find \(AB\).

11. \(D(7, 10), C(11, 2)\)

12. \(D(15.5, 18.7), C(22.5, 8.7)\)

In Exercises 13 and 14, find the length of the midsegment of the trapezoid with the given vertices. (See Example 4.)

13. \(A(2, 0), B(8, -4), C(12, 2), D(0, 10)\)

14. \(S(-2, 4), T(-2, -4), U(3, -2), V(13, 10)\)

In Exercises 15–18, find \(m \angle G\). (See Example 5.)

15. \(\angle EFG = 100^\circ, \angle FGH = 40^\circ\)

16. \(\angle EFG = 150^\circ\)

17. \(\angle EFG = 60^\circ, \angle FGH = 110^\circ\)

18. \(\angle EFG = 110^\circ\)
19. ERROR ANALYSIS Describe and correct the error in finding $DC$.

\begin{align*}
\text{Corrected:} \quad DC &= AB - MN \\
&= 14 - 8 \\
&= 6
\end{align*}

20. ERROR ANALYSIS Describe and correct the error in finding $m\angle A$.

\begin{align*}
\text{Corrected:} \quad m\angle A &= 120^\circ - 50^\circ \\
&= 70^\circ
\end{align*}

In Exercises 21–24, give the most specific name for the quadrilateral. Explain your reasoning. (See Example 6.)

21. \[J \quad K \quad \text{square}\]
22. \[P \quad Q \quad \text{isosceles trapezoid}\]
23. \[A \quad B \quad \text{parallelogram}\]
24. \[Z \quad W \quad \text{rhombus}\]

REASONING In Exercises 25 and 26, tell whether enough information is given in the diagram to classify the quadrilateral by the indicated name. Explain.

25. \[A \quad B \quad \text{square}\]
26. \[J \quad K \quad \text{isosceles trapezoid}\]

MATHEMATICAL CONNECTIONS In Exercises 27 and 28, find the value of $x$.

27. \[3x + 1 \quad 12.5\]
28. \[3x + 2 \quad 15\]

29. MODELING WITH MATHEMATICS In the diagram, $NP = 8$ inches, and $LR = 20$ inches. What is the diameter of the bottom layer of the cake?

30. PROBLEM SOLVING You and a friend are building a kite. You need a stick to place from $X$ to $W$ and a stick to place from $W$ to $Z$ to finish constructing the frame. You want the kite to have the geometric shape of a kite. How long does each stick need to be? Explain your reasoning.

31. isosceles trapezoid
32. kite
33. parallelogram
34. square
35. PROOF Write a proof.
Given $JL \cong LN, KM$ is a midsegment of $\triangle JLN$.
Prove Quadrilateral $JKMN$ is an isosceles trapezoid.

36. PROOF Write a proof.
Given $ABCD$ is a kite.
$AB \cong CB, AD \cong CD$
Prove $CE \cong AE$

37. ABSTRACT REASONING Point $U$ lies on the perpendicular bisector of $RT$. Describe the set of points $S$ for which $RSTU$ is a kite.

38. REASONING Determine whether the points $A(4, 5)$, $B(3, 3), C(-6, -13)$, and $D(6, -2)$ are the vertices of a kite. Explain your reasoning.

PROVING A THEOREM In Exercises 39 and 40, use the diagram to prove the given theorem. In the diagram, $EC$ is drawn parallel to $AB$.

39. Isosceles Trapezoid Base Angles Theorem
Given $ABCD$ is an isosceles trapezoid.
Prove $\angle A \cong \angle D, \angle B \cong \angle C$

40. Isosceles Trapezoid Base Angles Converse
Given $ABCD$ is a trapezoid.
$\angle A \cong \angle D, BC \parallel AD$
Prove $ABCD$ is an isosceles trapezoid.

41. MAKING AN ARGUMENT Your cousin claims there is enough information to prove that $JKLM$ is an isosceles trapezoid. Is your cousin correct? Explain.

42. MATHEMATICAL CONNECTIONS The bases of a trapezoid lie on the lines $y = 2x + 7$ and $y = 2x - 5$. Write the equation of the line that contains the midsegment of the trapezoid.

43. CONSTRUCTION $AC$ and $BD$ bisect each other.
   a. Construct quadrilateral $ABCD$ so that $AC$ and $BD$ are congruent, but not perpendicular. Classify the quadrilateral. Justify your answer.
   b. Construct quadrilateral $ABCD$ so that $AC$ and $BD$ are perpendicular, but not congruent. Classify the quadrilateral. Justify your answer.

44. PROOF Write a proof.
Given $QRST$ is an isosceles trapezoid.
Prove $\angle TQS \cong \angle SRT$

45. MODELING WITH MATHEMATICS A plastic spiderweb is made in the shape of a regular dodecagon (12-sided polygon). $AB \parallel PQ$, and $X$ is equidistant from the vertices of the dodecagon.
   a. Are you given enough information to prove that $ABPQ$ is an isosceles trapezoid?
   b. What is the measure of each interior angle of $ABPQ$?

46. ATTENDING TO PRECISION In trapezoid $PQRS, \overline{PQ} \parallel RS$ and $MN$ is the midsegment of $PQRS$. If $RS = 5 \cdot PQ$, what is the ratio of $MN$ to $RS$?
   A) $\frac{3}{5}$  B) $\frac{5}{3}$  C) $\frac{1}{2}$  D) $\frac{3}{1}$

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47. **PROVING A THEOREM** Use the plan for proof below to write a paragraph proof of the Kite Opposite Angles Theorem.

**Given** \( EFGH \) is a kite.

\[
\overline{EF} \cong \overline{FG}, \overline{EH} \cong \overline{GH}
\]

**Prove** \( \angle E \cong \angle G, \angle F \neq \angle H \)

---

**Plan for Proof** First show that \( \angle E \cong \angle G \). Then use an indirect argument to show that \( \angle F \neq \angle H \).

48. **HOW DO YOU SEE IT?** One of the earliest shapes used for cut diamonds is called the **table cut**, as shown in the figure. Each face of a cut gem is called a **facet**.

a. \( \overline{BC} \parallel \overline{AD} \), and \( \overline{AB} \) and \( \overline{DC} \) are not parallel. What shape is the facet labeled \( \overline{ABCD} \)?

b. \( \overline{DE} \parallel \overline{GF} \), and \( \overline{DG} \) and \( \overline{EF} \) are congruent but not parallel. What shape is the facet labeled \( \overline{DEFG} \)?

49. **PROVING A THEOREM** In the diagram below, \( \overline{BG} \) is the midsegment of \( \triangle ACD \), and \( \overline{GE} \) is the midsegment of \( \triangle ADF \). Use the diagram to prove the Trapezoid Midsegment Theorem.

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**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

Graph \( \triangle PQR \) with vertices \( P(-3, 2), Q(2, 3), \) and \( R(4, -2) \) and its image after the translation. *(Skills Review Handbook)*

53. \((x, y) \rightarrow (x + 5, y + 8)\)

54. \((x, y) \rightarrow (x + 6, y - 3)\)

55. \((x, y) \rightarrow (x - 4, y - 7)\)

Tell whether the two figures are similar. Explain your reasoning. *(Skills Review Handbook)*

56.

57.

58.