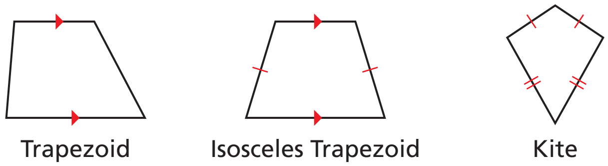
7.5 Properties of Trapezoids and Kites

Essential Question

What are some properties of trapezoids and kites?

Recall the types of quadrilaterals shown below.



EXPLORATION 1 Making a Conjecture about Trapezoids

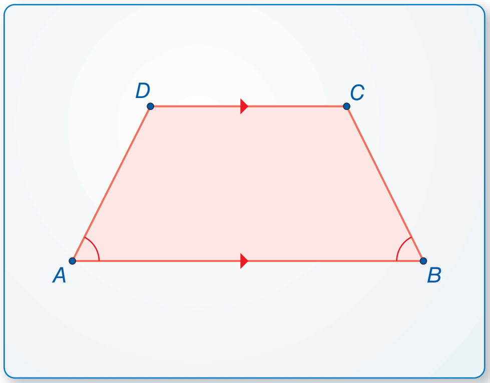
Work with a partner. Use dynamic geometry software.

a. Construct a trapezoid whose base angles are congruent. Explain your process.

b. Is the trapezoid isosceles? Justify your answer.

c. Repeat parts (a) and (b) for several other trapezoids. Write a conjecture based on your results.

Sample



PERSEVERE IN SOLVING PROBLEMS

To be proficient in math, you need to draw diagrams of important features and relationships, and search for regularity or trends.

EXPLORATION 2 Discovering a Property of Kites

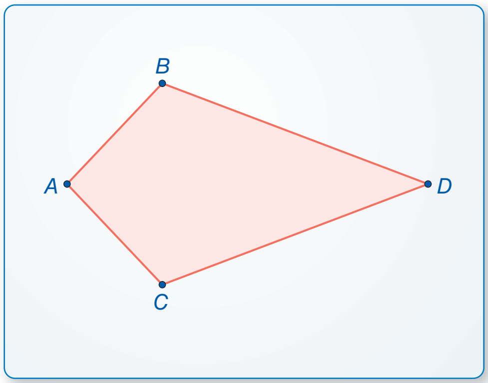
Work with a partner. Use dynamic geometry software.

a. Construct a kite. Explain your process.

b. Measure the angles of the kite. What do you observe?

c. Repeat parts (a) and (b) for several other kites. Write a conjecture based on your results.

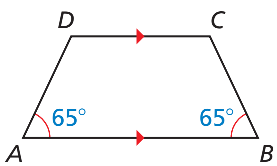
Sample



Communicate Your Answer

3. What are some properties of trapezoids and kites?

4. Is the trapezoid at the left isosceles? Explain.



5. A quadrilateral has angle measures of 70°, 70°, 110°, and 110°. Is the quadrilateral a kite? Explain.

7.5 Lesson

Core Vocabulary

trapezoid, *p. 442*
bases, *p. 442*
base angles, *p. 442*
legs, *p. 442*
isosceles trapezoid, *p. 442*
midsegment of a trapezoid, *p. 444*
kite, *p. 445*

Previous
diagonal
parallelogram

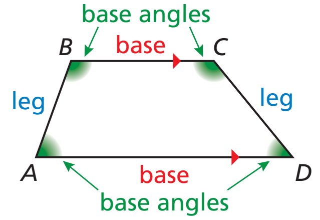
What You Will Learn

- Use properties of trapezoids.
- Use the Trapezoid Midsegment Theorem to find distances.
- Use properties of kites.
- Identify quadrilaterals.

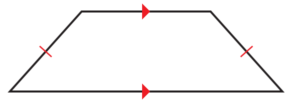
Using Properties of Trapezoids

A **trapezoid** is a quadrilateral with exactly one pair of parallel sides. The parallel sides are the **bases**.

Base angles of a trapezoid are two consecutive angles whose common side is a base. A trapezoid has two pairs of base angles. For example, in trapezoid $ABCD$, $\angle A$ and $\angle D$ are one pair of base angles, and $\angle B$ and $\angle C$ are the second pair. The nonparallel sides are the **legs** of the trapezoid.



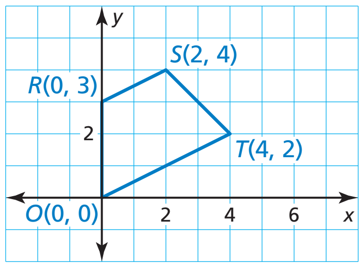
If the legs of a trapezoid are congruent, then the trapezoid is an **isosceles trapezoid**.



Isosceles trapezoid

EXAMPLE 1 Identifying a Trapezoid in the Coordinate Plane

Show that $ORST$ is a trapezoid. Then decide whether it is isosceles.



SOLUTION

Step 1 Compare the slopes of opposite sides.

$$\text{slope of } \overline{RS} = \frac{4-3}{2-0} = \frac{1}{2}$$

$$\text{slope of } \overline{OT} = \frac{2-0}{4-0} = \frac{2}{4} = \frac{1}{2}$$

The slopes of $\overline{RS}$ and $\overline{OT}$ are the same, so $\overline{RS} \parallel \overline{OT}$.

$$\text{slope of } \overline{ST} = \frac{2-4}{4-2} = \frac{-2}{2} = -1 \qquad \text{slope of } \overline{RO} = \frac{3-0}{0-0} = \frac{3}{0} \quad \text{Undefined}$$

The slopes of $\overline{ST}$ and $\overline{RO}$ are not the same, so $\overline{ST}$ is not parallel to $\overline{OR}$.

▶ Because $ORST$ has exactly one pair of parallel sides, it is a trapezoid.

Step 2 Compare the lengths of legs $\overline{RO}$ and $\overline{ST}$.

$$RO = |3-0| = 3 \qquad ST = \sqrt{(2-4)^2 + (4-2)^2} = \sqrt{8} = 2\sqrt{2}$$

Because $RO \neq ST$, legs $\overline{RO}$ and $\overline{ST}$ are *not* congruent.

▶ So, $ORST$ is not an isosceles trapezoid.

Monitoring Progress

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1. The points $A(-5, 6)$, $B(4, 9)$, $C(4, 4)$, and $D(-2, 2)$ form the vertices of a quadrilateral. Show that $ABCD$ is a trapezoid. Then decide whether it is isosceles.

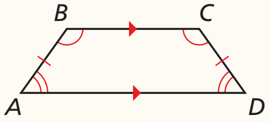
Theorems

Isosceles Trapezoid Base Angles Theorem

If a trapezoid is isosceles, then each pair of base angles is congruent.

If trapezoid *ABCD* is isosceles, then $\angle A \cong \angle D$ and $\angle B \cong \angle C$.

Proof Ex. 39, p. 449

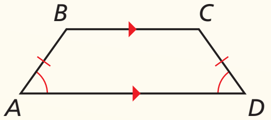


Isosceles Trapezoid Base Angles Converse

If a trapezoid has a pair of congruent base angles, then it is an isosceles trapezoid.

If $\angle A \cong \angle D$ (or if $\angle B \cong \angle C$), then trapezoid *ABCD* is isosceles.

Proof Ex. 40, p. 449

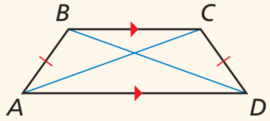


Isosceles Trapezoid Diagonals Theorem

A trapezoid is isosceles if and only if its diagonals are congruent.

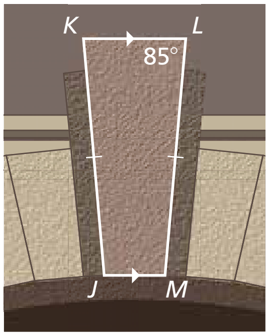
Trapezoid *ABCD* is isosceles if and only if $\overline{AC} \cong \overline{BD}$.

Proof Ex. 51, p. 450



EXAMPLE 2 Using Properties of Isosceles Trapezoids

The stone above the arch in the diagram is an isosceles trapezoid. Find $m\angle K$, $m\angle M$, and $m\angle J$.



SOLUTION

Step 1 Find $m\angle K$. *JKLM* is an isosceles trapezoid. So, $\angle K$ and $\angle L$ are congruent base angles, and $m\angle K = m\angle L = 85°$.

Step 2 Find $m\angle M$. Because $\angle L$ and $\angle M$ are consecutive interior angles formed by $\overleftrightarrow{LM}$ intersecting two parallel lines, they are supplementary. So, $m\angle M = 180° - 85° = 95°$.

Step 3 Find $m\angle J$. Because $\angle J$ and $\angle M$ are a pair of base angles, they are congruent, and $m\angle J = m\angle M = 95°$.

▶ So, $m\angle K = 85°$, $m\angle M = 95°$, and $m\angle J = 95°$.

Monitoring Progress

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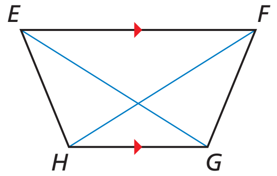
In Exercises 2 and 3, use trapezoid *EFGH*.

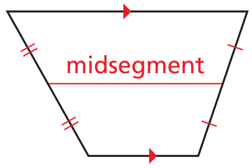


2. If $EG = FH$, is trapezoid *EFGH* isosceles? Explain.

3. If $m\angle HEF = 70°$ and $m\angle FGH = 110°$, is trapezoid *EFGH* isosceles? Explain.

Using the Trapezoid Midsegment Theorem

Recall that a midsegment of a triangle is a segment that connects the midpoints of two sides of the triangle. The **midsegment of a trapezoid** is the segment that connects the midpoints of its legs. The theorem below is similar to the Triangle Midsegment Theorem.



READING

The midsegment of a trapezoid is sometimes called the *median* of the trapezoid.

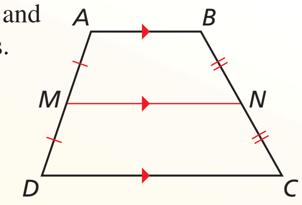
Theorem

Trapezoid Midsegment Theorem

The midsegment of a trapezoid is parallel to each base, and its length is one-half the sum of the lengths of the bases.

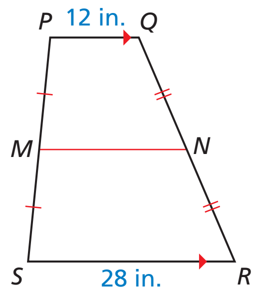
If $\overline{MN}$ is the midsegment of trapezoid *ABCD*, then $\overline{MN} \parallel \overline{AB}$, $\overline{MN} \parallel \overline{DC}$, and $MN = \frac{1}{2}(AB + CD)$.

Proof Ex. 49, p. 450



EXAMPLE 3 Using the Midsegment of a Trapezoid

In the diagram, $\overline{MN}$ is the midsegment of trapezoid *PQRS*. Find *MN*.



SOLUTION

$MN = \frac{1}{2}(PQ + SR)$ — Trapezoid Midsegment Theorem

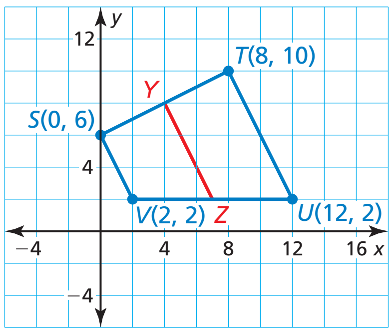
$= \frac{1}{2}(12 + 28)$ — Substitute 12 for *PQ* and 28 for *SR*.

$= 20$ — Simplify.

▶ The length of $\overline{MN}$ is 20 inches.

EXAMPLE 4 Using a Midsegment in the Coordinate Plane

Find the length of midsegment $\overline{YZ}$ in trapezoid *STUV*.



SOLUTION

Step 1 Find the lengths of $\overline{SV}$ and $\overline{TU}$.

$SV = \sqrt{(0 - 2)^2 + (6 - 2)^2} = \sqrt{20} = 2\sqrt{5}$

$TU = \sqrt{(8 - 12)^2 + (10 - 2)^2} = \sqrt{80} = 4\sqrt{5}$

Step 2 Multiply the sum of *SV* and *TU* by $\frac{1}{2}$.

$YZ = \frac{1}{2}\left(2\sqrt{5} + 4\sqrt{5}\right) = \frac{1}{2}\left(6\sqrt{5}\right) = 3\sqrt{5}$

▶ So, the length of $\overline{YZ}$ is $3\sqrt{5}$ units.

Monitoring Progress

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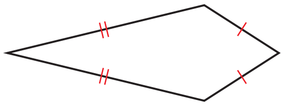
4. In trapezoid *JKLM*, ∠*J* and ∠*M* are right angles, and *JK* = 9 centimeters. The length of midsegment $\overline{NP}$ of trapezoid *JKLM* is 12 centimeters. Sketch trapezoid *JKLM* and its midsegment. Find *ML*. Explain your reasoning.
5. Explain another method you can use to find the length of $\overline{YZ}$ in Example 4.

Using Properties of Kites

A **kite** is a quadrilateral that has two pairs of consecutive congruent sides, but opposite sides are not congruent.



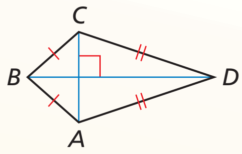
Theorems

Kite Diagonals Theorem

If a quadrilateral is a kite, then its diagonals are perpendicular.

If quadrilateral $ABCD$ is a kite, then $\overline{AC} \perp \overline{BD}$.

Proof p. 445

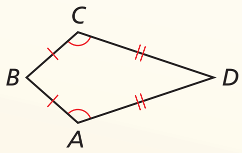


Kite Opposite Angles Theorem

If a quadrilateral is a kite, then exactly one pair of opposite angles are congruent.

If quadrilateral $ABCD$ is a kite and $\overline{BC} \cong \overline{BA}$, then $\angle A \cong \angle C$ and $\angle B \ncong \angle D$.

Proof Ex. 47, p. 450



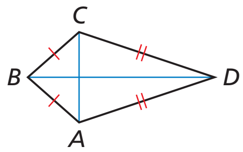
STUDY TIP

The congruent angles of a kite are formed by the noncongruent adjacent sides.

PROOF Kite Diagonals Theorem

Given $ABCD$ is a kite, $\overline{BC} \cong \overline{BA}$, and $\overline{DC} \cong \overline{DA}$.

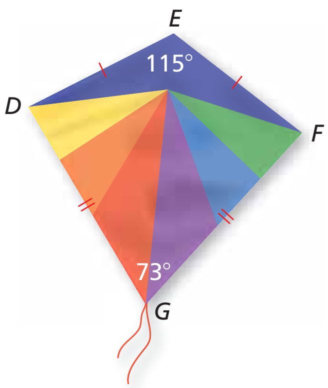
Prove $\overline{AC} \perp \overline{BD}$



STATEMENTS	REASONS
1. $ABCD$ is a kite with $\overline{BC} \cong \overline{BA}$ and $\overline{DC} \cong \overline{DA}$.	**1.** Given
2. B and D lie on the $\perp$ bisector of $\overline{AC}$.	**2.** Converse of the $\perp$ Bisector Theorem
3. $\overline{BD}$ is the $\perp$ bisector of $\overline{AC}$.	**3.** Through any two points, there exists exactly one line.
4. $\overline{AC} \perp \overline{BD}$	**4.** Definition of $\perp$ bisector

EXAMPLE 5 Finding Angle Measures in a Kite

Find $m\angle D$ in the kite shown.



SOLUTION

By the Kite Opposite Angles Theorem, $DEFG$ has exactly one pair of congruent opposite angles. Because $\angle E \ncong \angle G$, $\angle D$ and $\angle F$ must be congruent. So, $m\angle D = m\angle F$. Write and solve an equation to find $m\angle D$.

$m\angle D + m\angle F + 115° + 73° = 360°$ Corollary to the Polygon Interior Angles Theorem

$m\angle D + m\angle D + 115° + 73° = 360°$ Substitute $m\angle D$ for $m\angle F$.

$2m\angle D + 188° = 360°$ Combine like terms.

$m\angle D = 86°$ Solve for $m\angle D$.

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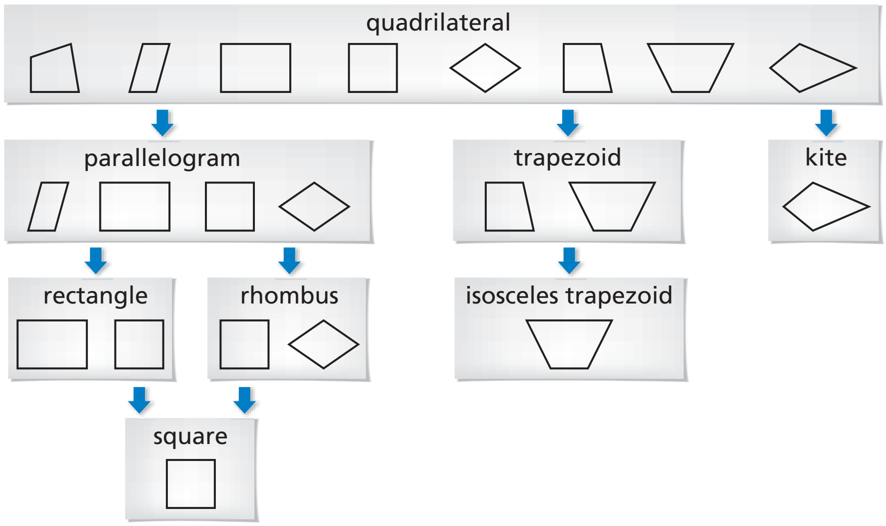
6. In a kite, the measures of the angles are $3x°$, $75°$, $90°$, and $120°$. Find the value of x. What are the measures of the angles that are congruent?

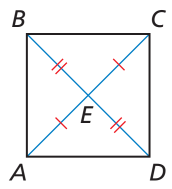
Identifying Special Quadrilaterals

The diagram shows relationships among the special quadrilaterals you have studied in this chapter. Each shape in the diagram has the properties of the shapes linked above it. For example, a rhombus has the properties of a parallelogram and a quadrilateral.



EXAMPLE 6 Identifying a Quadrilateral

What is the most specific name for quadrilateral *ABCD*?



SOLUTION

The diagram shows $\overline{AE} \cong \overline{CE}$ and $\overline{BE} \cong \overline{DE}$. So, the diagonals bisect each other. By the Parallelogram Diagonals Converse, *ABCD* is a parallelogram.

READING DIAGRAMS

In Example 6, *ABCD* looks like a square. But you must rely only on marked information when you interpret a diagram.

Rectangles, rhombuses, and squares are also parallelograms. However, there is no information given about the side lengths or angle measures of *ABCD*. So, you cannot determine whether it is a rectangle, a rhombus, or a square.

▶ So, the most specific name for *ABCD* is a parallelogram.

Monitoring Progress

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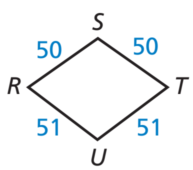
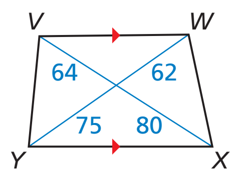
7. Quadrilateral *DEFG* has at least one pair of opposite sides congruent. What types of quadrilaterals meet this condition?

Give the most specific name for the quadrilateral. Explain your reasoning.

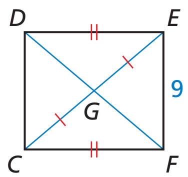
8.



9.



10.



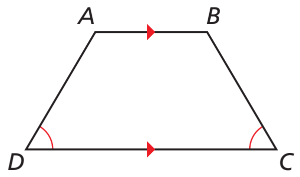
7.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Describe the differences between a trapezoid and a kite.

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.



Is there enough information to prove that trapezoid *ABCD* is isosceles?

Is there enough information to prove that $\overline{AB} \cong \overline{DC}$?

Is there enough information to prove that the non-parallel sides of trapezoid *ABCD* are congruent?

Is there enough information to prove that the legs of trapezoid *ABCD* are congruent?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, show that the quadrilateral with the given vertices is a trapezoid. Then decide whether it is isosceles. *(See Example 1.)*

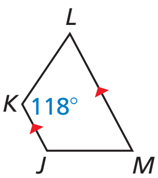
3. $W(1, 4)$, $X(1, 8)$, $Y(-3, 9)$, $Z(-3, 3)$

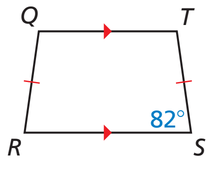
4. $D(-3, 3)$, $E(-1, 1)$, $F(1, -4)$, $G(-3, 0)$

5. $M(-2, 0)$, $N(0, 4)$, $P(5, 4)$, $Q(8, 0)$

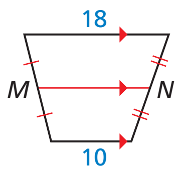
6. $H(1, 9)$, $J(4, 2)$, $K(5, 2)$, $L(8, 9)$

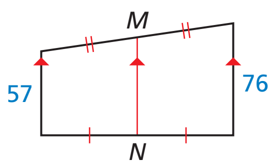
In Exercises 7 and 8, find the measure of each angle in the isosceles trapezoid. *(See Example 2.)*

7. 

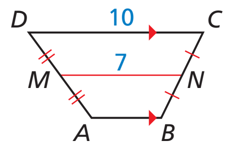
8. 

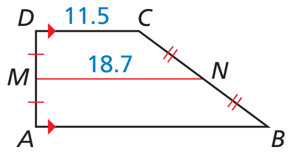
In Exercises 9 and 10, find the length of the midsegment of the trapezoid. *(See Example 3.)*

9. 

10. 

In Exercises 11 and 12, find *AB*.

11. 

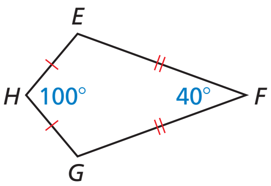
12. 

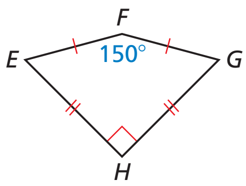
In Exercises 13 and 14, find the length of the midsegment of the trapezoid with the given vertices. *(See Example 4.)*

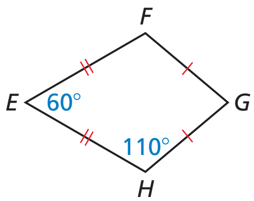
13. $A(2, 0)$, $B(8, -4)$, $C(12, 2)$, $D(0, 10)$

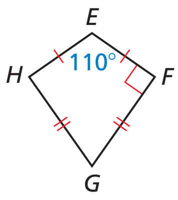
14. $S(-2, 4)$, $T(-2, -4)$, $U(3, -2)$, $V(13, 10)$

In Exercises 15–18, find $m\angle G$. *(See Example 5.)*

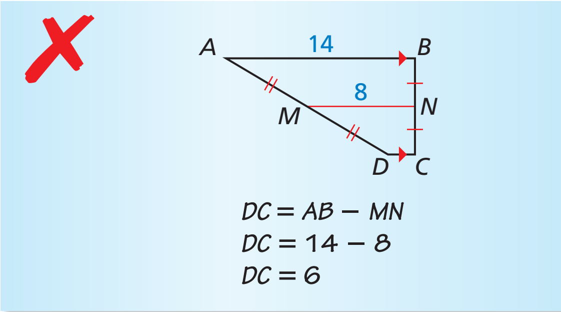
15. 

16. 

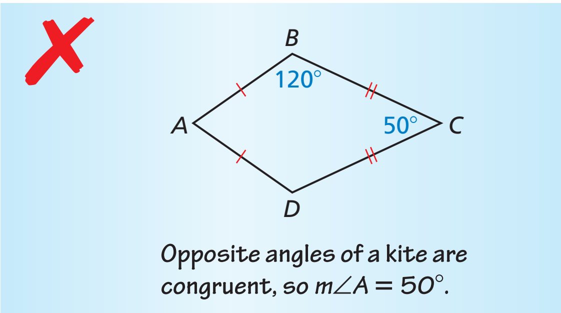
17. 

18. 

19. **ERROR ANALYSIS** Describe and correct the error in finding *DC*.

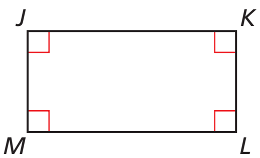


20. **ERROR ANALYSIS** Describe and correct the error in finding $m\angle A$.

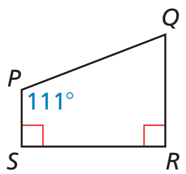


In Exercises 21–24, give the most specific name for the quadrilateral. Explain your reasoning. *(See Example 6.)*

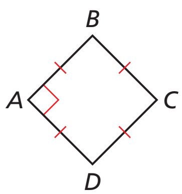
21.



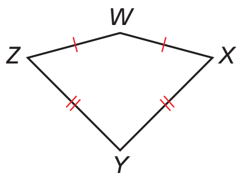
22.



23.

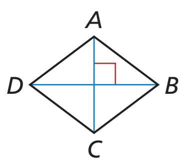


24.

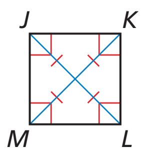


REASONING **In Exercises 25 and 26, tell whether enough information is given in the diagram to classify the quadrilateral by the indicated name. Explain.**

25. rhombus

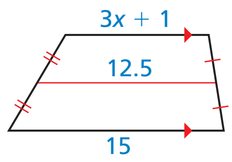


26. square

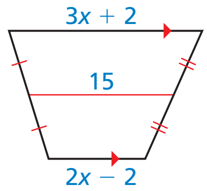


MATHEMATICAL CONNECTIONS **In Exercises 27 and 28, find the value of *x*.**

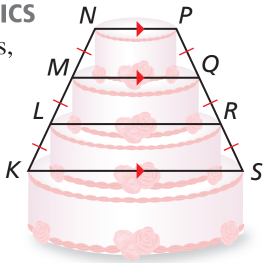
27.



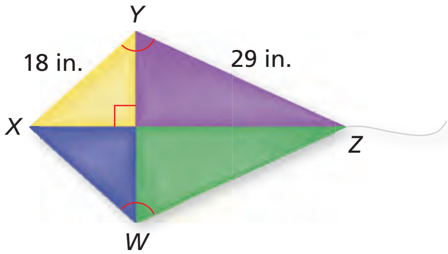
28.



29. **MODELING WITH MATHEMATICS** In the diagram, $NP = 8$ inches, and $LR = 20$ inches. What is the diameter of the bottom layer of the cake?

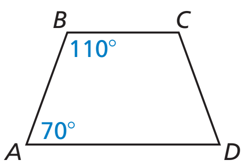


30. **PROBLEM SOLVING** You and a friend are building a kite. You need a stick to place from *X* to *W* and a stick to place from *W* to *Z* to finish constructing the frame. You want the kite to have the geometric shape of a kite. How long does each stick need to be? Explain your reasoning.

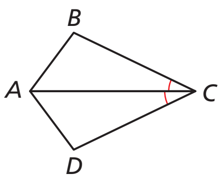


REASONING **In Exercises 31–34, determine which pairs of segments or angles must be congruent so that you can prove that *ABCD* is the indicated quadrilateral. Explain your reasoning. (There may be more than one right answer.)**

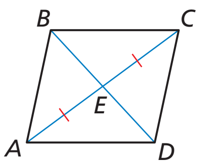
31. isosceles trapezoid



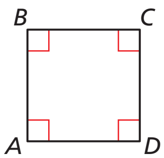
32. kite



33. parallelogram



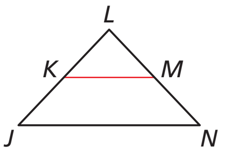
34. square



35. PROOF Write a proof.

Given $\overline{JL} \cong \overline{LN}$, $\overline{KM}$ is a midsegment of $\triangle JLN$.

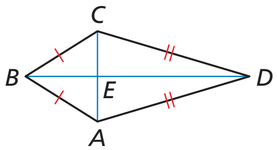
Prove Quadrilateral *JKMN* is an isosceles trapezoid.



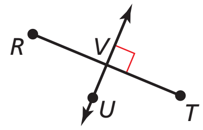
36. PROOF Write a proof.

Given *ABCD* is a kite.
$\overline{AB} \cong \overline{CB}$, $\overline{AD} \cong \overline{CD}$

Prove $\overline{CE} \cong \overline{AE}$

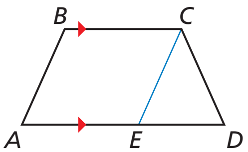


37. ABSTRACT REASONING Point *U* lies on the perpendicular bisector of $\overline{RT}$. Describe the set of points *S* for which *RSTU* is a kite.



38. REASONING Determine whether the points $A(4, 5)$, $B(-3, 3)$, $C(-6, -13)$, and $D(6, -2)$ are the vertices of a kite. Explain your reasoning,

PROVING A THEOREM In Exercises 39 and 40, use the diagram to prove the given theorem. In the diagram, $\overline{EC}$ is drawn parallel to $\overline{AB}$.



39. Isosceles Trapezoid Base Angles Theorem

Given *ABCD* is an isosceles trapezoid.
$\overline{BC} \parallel \overline{AD}$

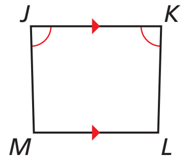
Prove $\angle A \cong \angle D$, $\angle B \cong \angle BCD$

40. Isosceles Trapezoid Base Angles Converse

Given *ABCD* is a trapezoid.
$\angle A \cong \angle D$, $\overline{BC} \parallel \overline{AD}$

Prove *ABCD* is an isosceles trapezoid.

41. MAKING AN ARGUMENT Your cousin claims there is enough information to prove that *JKLM* is an isosceles trapezoid. Is your cousin correct? Explain.



42. MATHEMATICAL CONNECTIONS The bases of a trapezoid lie on the lines $y = 2x + 7$ and $y = 2x - 5$. Write the equation of the line that contains the midsegment of the trapezoid.

43. CONSTRUCTION $\overline{AC}$ and $\overline{BD}$ bisect each other.

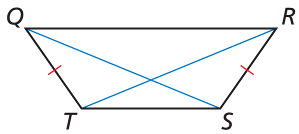
a. Construct quadrilateral *ABCD* so that $\overline{AC}$ and $\overline{BD}$ are congruent, but not perpendicular. Classify the quadrilateral. Justify your answer.

b. Construct quadrilateral *ABCD* so that $\overline{AC}$ and $\overline{BD}$ are perpendicular, but not congruent. Classify the quadrilateral. Justify your answer.

44. PROOF Write a proof.

Given *QRST* is an isosceles trapezoid.

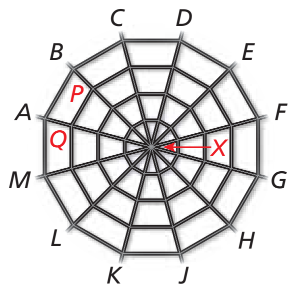
Prove $\angle TQS \cong \angle SRT$



45. MODELING WITH MATHEMATICS A plastic spiderweb is made in the shape of a regular dodecagon (12-sided polygon). $\overline{AB} \parallel \overline{PQ}$, and *X* is equidistant from the vertices of the dodecagon.

a. Are you given enough information to prove that *ABPQ* is an isosceles trapezoid?

b. What is the measure of each interior angle of *ABPQ*?



46. ATTENDING TO PRECISION In trapezoid *PQRS*, $\overline{PQ} \parallel \overline{RS}$ and $\overline{MN}$ is the midsegment of *PQRS*. If $RS = 5 \cdot PQ$, what is the ratio of *MN* to *RS*?

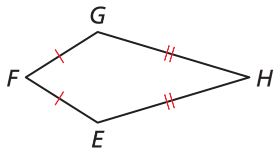
Ⓐ 3 : 5 Ⓑ 5 : 3

Ⓒ 1 : 2 Ⓓ 3 : 1

47. PROVING A THEOREM Use the plan for proof below to write a paragraph proof of the Kite Opposite Angles Theorem.

Given *EFGH* is a kite.
$\overline{EF} \cong \overline{FG}, \overline{EH} \cong \overline{GH}$

Prove $\angle E \cong \angle G, \angle F \ncong \angle H$

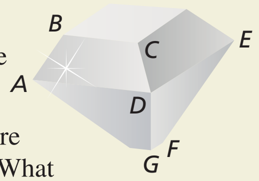


Plan for Proof First show that $\angle E \cong \angle G$. Then use an indirect argument to show that $\angle F \ncong \angle H$.

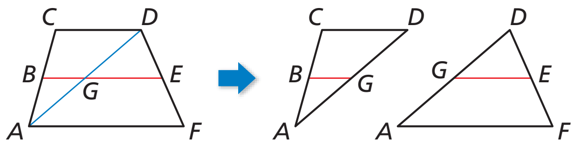
48. HOW DO YOU SEE IT? One of the earliest shapes used for cut diamonds is called the *table cut*, as shown in the figure. Each face of a cut gem is called a *facet*.

a. $\overline{BC} \parallel \overline{AD}$, and $\overline{AB}$ and $\overline{DC}$ are not parallel. What shape is the facet labeled *ABCD*?

b. $\overline{DE} \parallel \overline{GF}$, and $\overline{DG}$ and $\overline{EF}$ are congruent but not parallel. What shape is the facet labeled *DEFG*?



49. PROVING A THEOREM In the diagram below, $\overline{BG}$ is the midsegment of $\triangle ACD$, and $\overline{GE}$ is the midsegment of $\triangle ADF$. Use the diagram to prove the Trapezoid Midsegment Theorem.



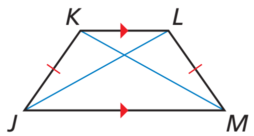
50. THOUGHT PROVOKING Is SSASS a valid congruence theorem for kites? Justify your answer.

51. PROVING A THEOREM To prove the biconditional statement in the Isosceles Trapezoid Diagonals Theorem, you must prove both parts separately.

a. Prove part of the Isosceles Trapezoid Diagonals Theorem.

Given *JKLM* is an isosceles trapezoid.
$\overline{KL} \parallel \overline{JM}, \overline{JK} \cong \overline{LM}$

Prove $\overline{JL} \cong \overline{KM}$

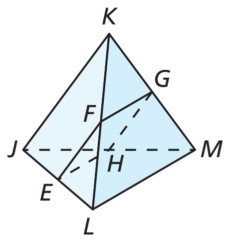


b. Write the other part of the Isosceles Trapezoid Diagonals Theorem as a conditional. Then prove the statement is true.

52. PROOF What special type of quadrilateral is *EFGH*? Write a proof to show that your answer is correct.

Given In the three-dimensional figure, $\overline{JK} \cong \overline{LM}$. *E*, *F*, *G*, and *H* are the midpoints of $\overline{JL}$, $\overline{KL}$, $\overline{KM}$, and $\overline{JM}$, respectively.

Prove *EFGH* is a ________.



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Graph $\triangle PQR$ with vertices $P(-3, 2)$, $Q(2, 3)$, and $R(4, -2)$ and its image after the translation. *(Skills Review Handbook)*

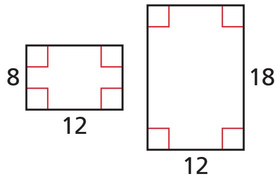
53. $(x, y) \rightarrow (x + 5, y + 8)$

54. $(x, y) \rightarrow (x + 6, y - 3)$

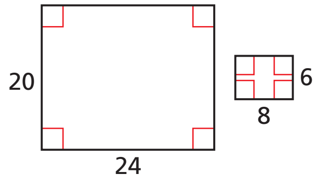
55. $(x, y) \rightarrow (x - 4, y - 7)$

Tell whether the two figures are similar. Explain your reasoning. *(Skills Review Handbook)*

56.



57.



58.