6.3 Bisectors of Triangles

Essential Question What conjectures can you make about the perpendicular bisectors and the angle bisectors of a triangle?

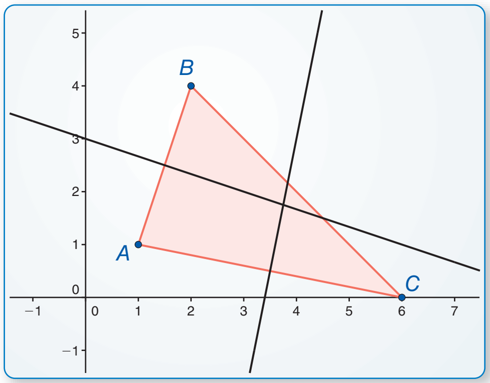
EXPLORATION 1 Properties of the Perpendicular Bisectors of a Triangle

Work with a partner. Use dynamic geometry software. Draw any $\triangle ABC$.

a. Construct the perpendicular bisectors of all three sides of $\triangle ABC$. Then drag the vertices to change $\triangle ABC$. What do you notice about the perpendicular bisectors?

b. Label a point D at the intersection of the perpendicular bisectors.

c. Draw the circle with center D through vertex A of $\triangle ABC$. Then drag the vertices to change $\triangle ABC$. What do you notice?



Sample
Points
$A(1, 1)$
$B(2, 4)$
$C(6, 0)$
Segments
$BC = 5.66$
$AC = 5.10$
$AB = 3.16$
Lines
$x + 3y = 9$
$-5x + y = -17$

LOOKING FOR STRUCTURE

To be proficient in math, you need to see complicated things as single objects or as being composed of several objects.

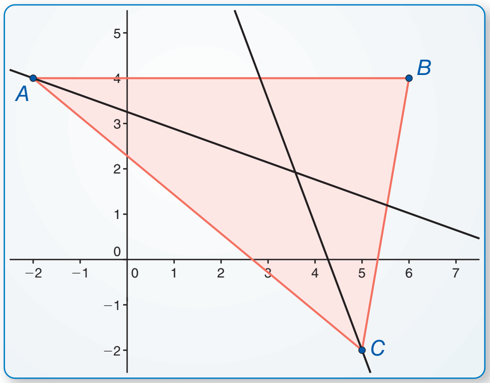
EXPLORATION 2 Properties of the Angle Bisectors of a Triangle

Work with a partner. Use dynamic geometry software. Draw any $\triangle ABC$.

a. Construct the angle bisectors of all three angles of $\triangle ABC$. Then drag the vertices to change $\triangle ABC$. What do you notice about the angle bisectors?

b. Label a point D at the intersection of the angle bisectors.

c. Find the distance between D and $\overline{AB}$. Draw the circle with center D and this distance as a radius. Then drag the vertices to change $\triangle ABC$. What do you notice?



Sample
Points
$A(-2, 4)$
$B(6, 4)$
$C(5, -2)$
Segments
$BC = 6.08$
$AC = 9.22$
$AB = 8$
Lines
$0.35x + 0.94y = 3.06$
$-0.94x - 0.34y = -4.02$

Communicate Your Answer

3. What conjectures can you make about the perpendicular bisectors and the angle bisectors of a triangle?

6.3 Lesson

Core Vocabulary

concurrent, *p. 352*
point of concurrency, *p. 352*
circumcenter, *p. 352*
incenter, *p. 355*

Previous
perpendicular bisector
angle bisector

What You Will Learn

- Use and find the circumcenter of a triangle.
- Use and find the incenter of a triangle.

Using the Circumcenter of a Triangle

When three or more lines, rays, or segments intersect in the same point, they are called **concurrent** lines, rays, or segments. The point of intersection of the lines, rays, or segments is called the **point of concurrency**.

In a triangle, the three perpendicular bisectors are concurrent. The point of concurrency is the **circumcenter** of the triangle.

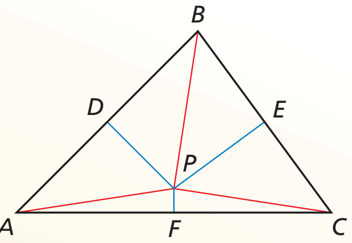
Theorems

Circumcenter Theorem

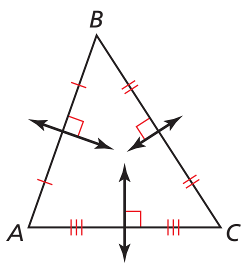
The circumcenter of a triangle is equidistant from the vertices of the triangle.

If $\overline{PD}$, $\overline{PE}$, and $\overline{PF}$ are perpendicular bisectors, then $PA = PB = PC$.

Proof p. 352



PROOF Circumcenter Theorem



Given $\triangle ABC$; the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$

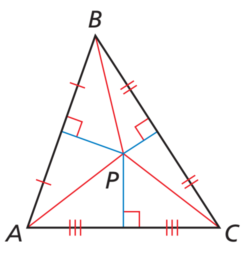
Prove The perpendicular bisectors intersect in a point; that point is equidistant from A, B, and C.

Plan for Proof Show that P, the point of intersection of the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$, also lies on the perpendicular bisector of $\overline{AC}$. Then show that point P is equidistant from the vertices of the triangle.

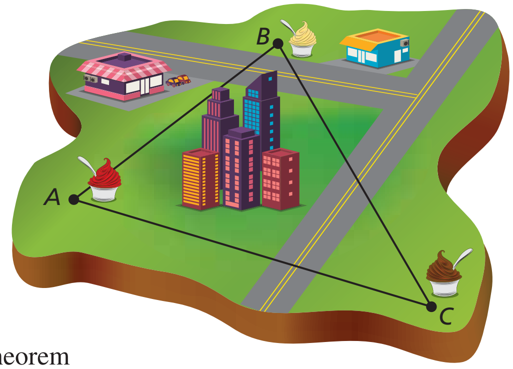
Plan in Action

STATEMENTS	REASONS
1. $\triangle ABC$; the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$	**1.** Given
2. The perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ intersect at some point P.	**2.** Because the sides of a triangle cannot be parallel, these perpendicular bisectors must intersect in some point. Call it P.
3. Draw $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$.	**3.** Two Point Postulate
4. $PA = PB$, $PB = PC$	**4.** Perpendicular Bisector Theorem
5. $PA = PC$	**5.** Transitive Property of Equality
6. P is on the perpendicular bisector of $\overline{AC}$.	**6.** Converse of the Perpendicular Bisector Theorem
7. $PA = PB = PC$. So, P is equidistant from the vertices of the triangle.	**7.** From the results of Steps 4 and 5 and the definition of equidistant

STUDY TIP
Use diagrams like the one below to help visualize your proof.



EXAMPLE 1 Solving a Real-Life Problem



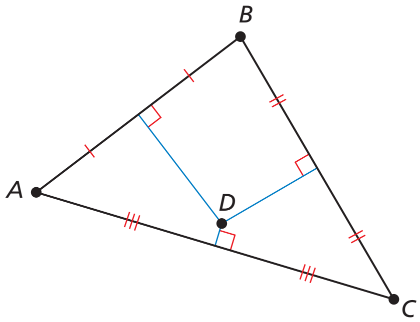
Three snack carts sell frozen yogurt from points A, B, and C outside a city. Each of the three carts is the same distance from the frozen yogurt distributor.

Find the location of the distributor.

SOLUTION

The distributor is equidistant from the three snack carts. The Circumcenter Theorem shows that you can find a point equidistant from three points by using the perpendicular bisectors of the triangle formed by those points.

Copy the positions of points A, B, and C and connect the points to draw $\triangle ABC$. Then use a ruler and protractor to draw the three perpendicular bisectors of $\triangle ABC$. The circumcenter D is the location of the distributor.



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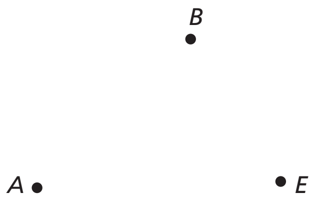
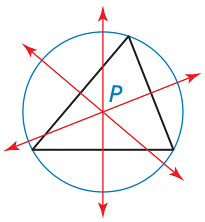
1. Three snack carts sell hot pretzels from points A, B, and E. What is the location of the pretzel distributor if it is equidistant from the three carts? Sketch the triangle and show the location.



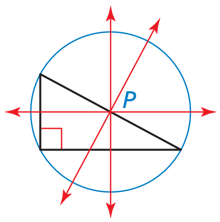
READING

The prefix *circum-* means "around" or "about," as in *circumference* (distance around a circle).

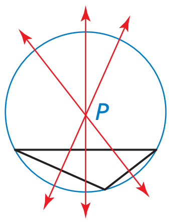
The circumcenter P is equidistant from the three vertices, so P is the center of a circle that passes through all three vertices. As shown below, the location of P depends on the type of triangle. The circle with center P is said to be *circumscribed* about the triangle.



Acute triangle
P is inside triangle.



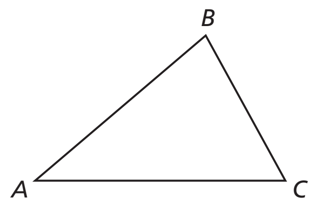
Right triangle
P is on triangle.



Obtuse triangle
P is outside triangle.

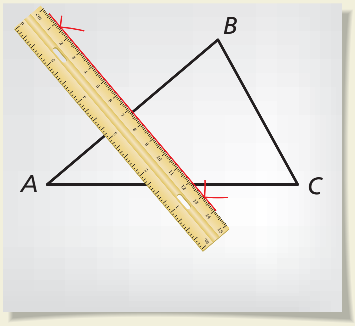
CONSTRUCTION Circumscribing a Circle About a Triangle

Use a compass and straightedge to construct a circle that is circumscribed about $\triangle ABC$.



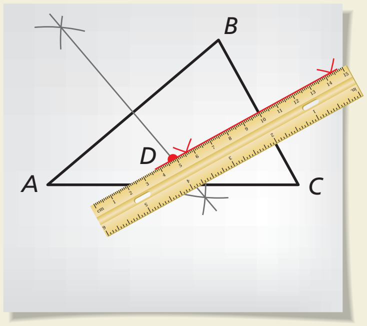
SOLUTION

Step 1



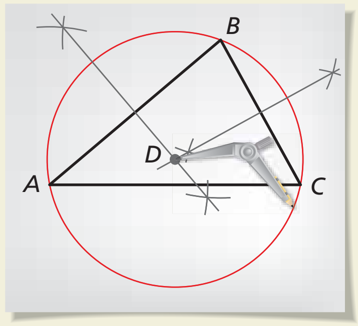
Draw a bisector Draw the perpendicular bisector of $\overline{AB}$.

Step 2



Draw a bisector Draw the perpendicular bisector of $\overline{BC}$. Label the intersection of the bisectors D. This is the circumcenter.

Step 3



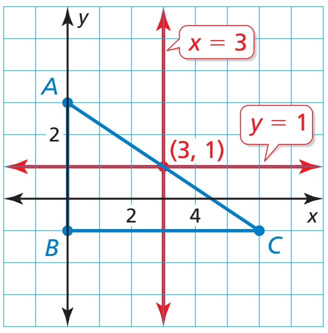
Draw a circle Place the compass at D. Set the width by using any vertex of the triangle. This is the radius of the *circumcircle*. Draw the circle. It should pass through all three vertices A, B, and C.

STUDY TIP

Note that you only need to find the equations for *two* perpendicular bisectors. You can use the perpendicular bisector of the third side to verify your result.

EXAMPLE 2 Finding the Circumcenter of a Triangle

Find the coordinates of the circumcenter of $\triangle ABC$ with vertices $A(0, 3)$, $B(0, -1)$, and $C(6, -1)$.



SOLUTION

Step 1 Graph $\triangle ABC$.

Step 2 Find equations for two perpendicular bisectors. Use the Slopes of Perpendicular Lines Theorem, which states that horizontal lines are perpendicular to vertical lines.

The midpoint of $\overline{AB}$ is $(0, 1)$. The line through $(0, 1)$ that is perpendicular to $\overline{AB}$ is $y = 1$.

The midpoint of $\overline{BC}$ is $(3, -1)$. The line through $(3, -1)$ that is perpendicular to $\overline{BC}$ is $x = 3$.

Step 3 Find the point where $x = 3$ and $y = 1$ intersect. They intersect at $(3, 1)$.

▶ So, the coordinates of the circumcenter are $(3, 1)$.

MAKING SENSE OF PROBLEMS

Because $\triangle ABC$ is a right triangle, the circumcenter lies on the triangle.

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Find the coordinates of the circumcenter of the triangle with the given vertices.

2. $R(-2, 5)$, $S(-6, 5)$, $T(-2, -1)$

3. $W(-1, 4)$, $X(1, 4)$, $Y(1, -6)$

Using the Incenter of a Triangle

Just as a triangle has three perpendicular bisectors, it also has three angle bisectors. The angle bisectors of a triangle are also concurrent. This point of concurrency is the **incenter** of the triangle. For any triangle, the incenter always lies inside the triangle.

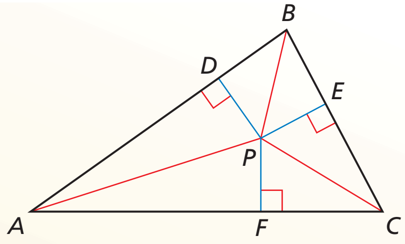
Theorem

Incenter Theorem

The incenter of a triangle is equidistant from the sides of the triangle.

If $\overline{AP}$, $\overline{BP}$, and $\overline{CP}$ are angle bisectors of $\triangle ABC$, then $PD = PE = PF$.

Proof Ex. 38, p. 359

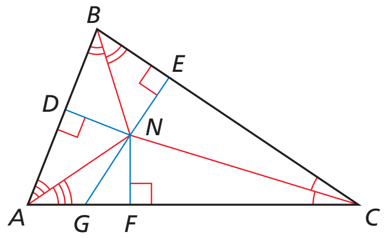


EXAMPLE 3 Using the Incenter of a Triangle

In the figure shown, $ND = 5x - 1$ and $NE = 2x + 11$.

a. Find NF.

b. Can NG be equal to 18? Explain your reasoning.



SOLUTION

a. N is the incenter of $\triangle ABC$ because it is the point of concurrency of the three angle bisectors. So, by the Incenter Theorem, $ND = NE = NF$.

Step 1 Solve for x.

$ND = NE$	Incenter Theorem
$5x - 1 = 2x + 11$	Substitute.
$x = 4$	Solve for x.

Step 2 Find ND (or NE).

$$ND = 5x - 1 = 5(4) - 1 = 19$$

▶ So, because $ND = NF$, $NF = 19$.

b. Recall that the shortest distance between a point and a line is a perpendicular segment. In this case, the perpendicular segment is $\overline{NF}$, which has a length of 19. Because $18 < 19$, NG cannot be equal to 18.

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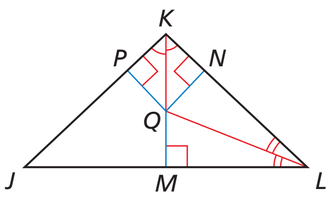
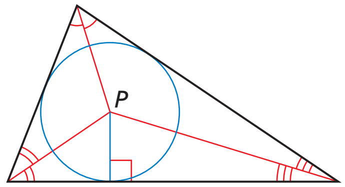
4. In the figure shown, $QM = 3x + 8$ and $QN = 7x + 2$. Find QP.

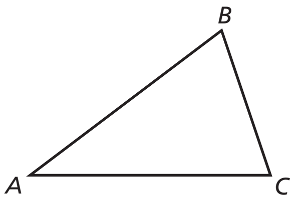




Because the incenter P is equidistant from the three sides of the triangle, a circle drawn using P as the center and the distance to one side of the triangle as the radius will just touch the other two sides of the triangle. The circle is said to be *inscribed* within the triangle.

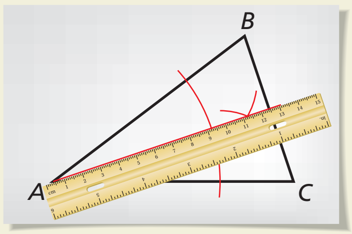
CONSTRUCTION Inscribing a Circle Within a Triangle

Use a compass and straightedge to construct a circle that is inscribed within $\triangle ABC$.



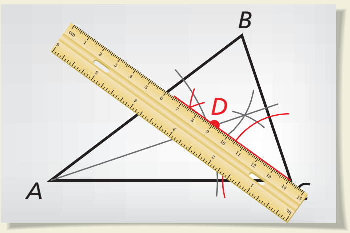
SOLUTION

Step 1



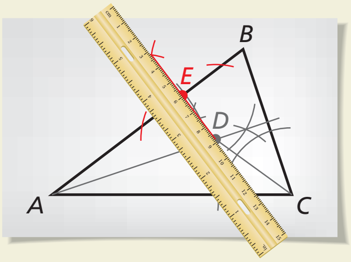
Draw a bisector Draw the angle bisector of $\angle A$.

Step 2



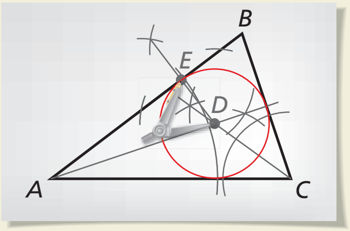
Draw a bisector Draw the angle bisector of $\angle C$. Label the intersection of the bisectors D. This is the incenter.

Step 3



Draw a perpendicular line Draw the perpendicular line from D to $\overline{AB}$. Label the point where it intersects $\overline{AB}$ as E.

Step 4



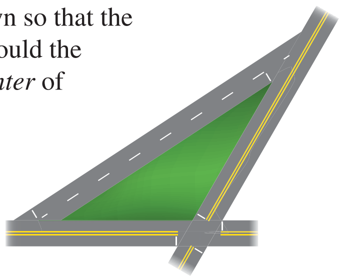
Draw a circle Place the compass at D. Set the width to E. This is the radius of the *incircle*. Draw the circle. It should touch each side of the triangle.

EXAMPLE 4 Solving a Real-Life Problem

ATTENDING TO PRECISION

Pay close attention to how a problem is stated. The city wants the lamppost to be the *same distance* from the three streets, not from where the streets intersect.

A city wants to place a lamppost on the boulevard shown so that the lamppost is the same distance from all three streets. Should the location of the lamppost be at the *circumcenter* or *incenter* of the triangular boulevard? Explain.



SOLUTION

Because the shape of the boulevard is an obtuse triangle, its circumcenter lies outside the triangle. So, the location of the lamppost cannot be at the circumcenter. The city wants the lamppost to be the same distance from all three streets. By the Incenter Theorem, the incenter of a triangle is equidistant from the sides of a triangle.

▶ So, the location of the lamppost should be at the incenter of the boulevard.

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5. Draw a sketch to show the location L of the lamppost in Example 4.

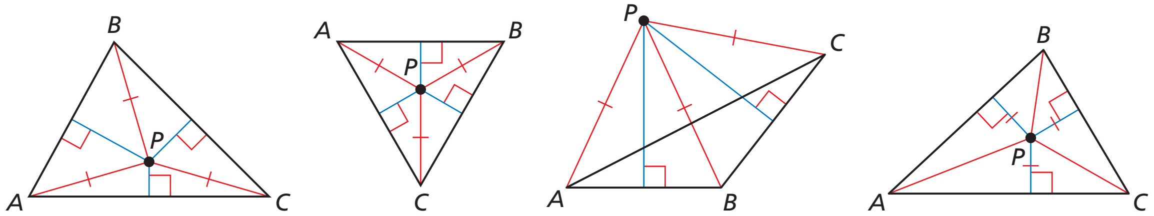
6.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. VOCABULARY When three or more lines, rays, or segments intersect in the same point, they are called ___________ lines, rays, or segments.

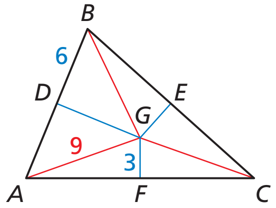
2. WHICH ONE DOESN'T BELONG? Which triangle does *not* belong with the other three? Explain your reasoning.



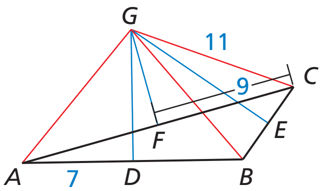
Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, the perpendicular bisectors of △*ABC* intersect at point *G* and are shown in blue. Find the indicated measure.

3. Find *BG*.

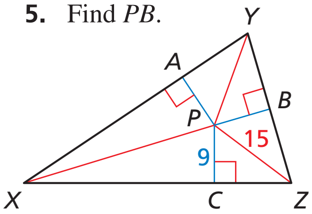


4. Find *GA*.

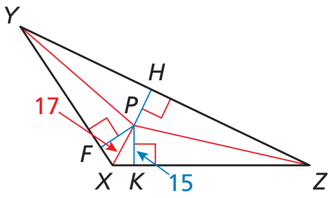


In Exercises 5 and 6, the angle bisectors of △*XYZ* intersect at point *P* and are shown in red. Find the indicated measure.

5. Find *PB*.



6. Find *HP*.



In Exercises 7–10, find the coordinates of the circumcenter of the triangle with the given vertices. *(See Example 2.)*

7. $A(2, 6)$, $B(8, 6)$, $C(8, 10)$

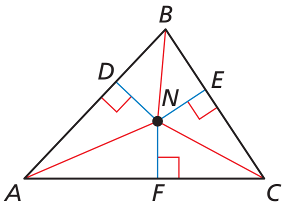
8. $D(-7, -1)$, $E(-1, -1)$, $F(-7, -9)$

9. $H(-10, 7)$, $J(-6, 3)$, $K(-2, 3)$

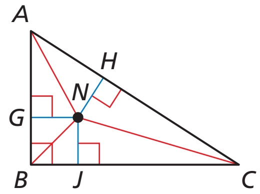
10. $L(3, -6)$, $M(5, -3)$, $N(8, -6)$

In Exercises 11–14, *N* is the incenter of △*ABC*. Use the given information to find the indicated measure. *(See Example 3.)*

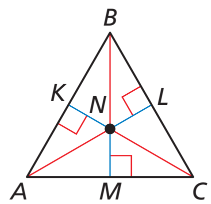
11. $ND = 6x - 2$
$NE = 3x + 7$
Find *NF*.



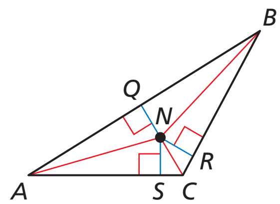
12. $NG = x + 3$
$NH = 2x - 3$
Find *NJ*.



13. $NK = 2x - 2$
$NL = -x + 10$
Find *NM*.



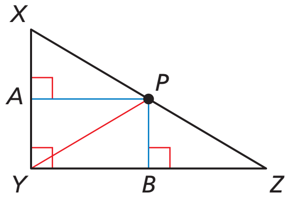
14. $NQ = 2x$
$NR = 3x - 2$
Find *NS*.



15. *P* is the circumcenter of △*XYZ*. Use the given information to find *PZ*.

$PX = 3x + 2$

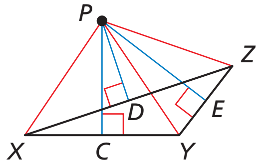
$PY = 4x - 8$



16. P is the circumcenter of $\triangle XYZ$. Use the given information to find PY.

$PX = 4x + 3$

$PZ = 6x - 11$



CONSTRUCTION **In Exercises 17–20, draw a triangle of the given type. Find the circumcenter. Then construct the circumscribed circle.**

17. right

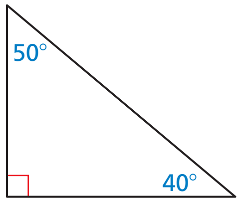
18. obtuse

19. acute isosceles

20. equilateral

CONSTRUCTION **In Exercises 21–24, copy the triangle with the given angle measures. Find the incenter. Then construct the inscribed circle.**

21.



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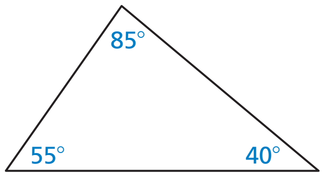

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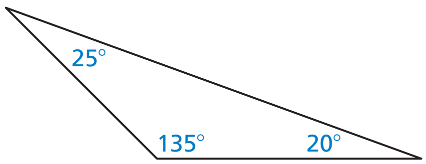

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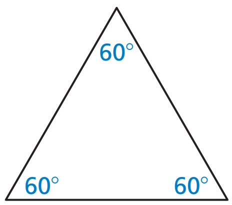
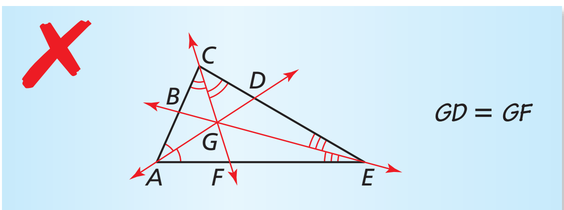
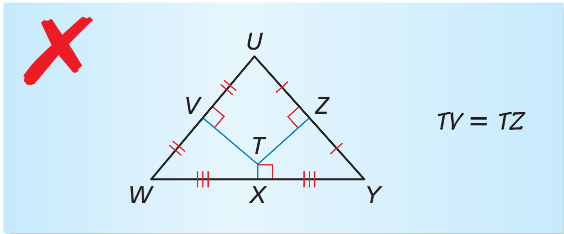


ERROR ANALYSIS **In Exercises 25 and 26, describe and correct the error in identifying equal distances inside the triangle.**

25.

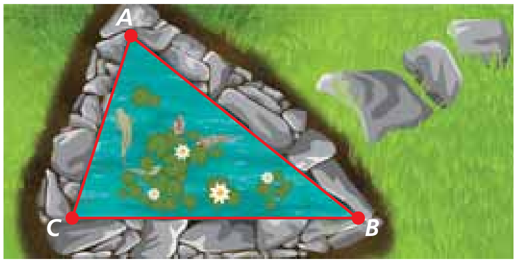


26.



27. **MODELING WITH MATHEMATICS** You and two friends plan to meet to walk your dogs together. You want the meeting place to be the same distance from each person's house. Explain how you can use the diagram to locate the meeting place. *(See Example 1.)*

28. **MODELING WITH MATHEMATICS** You are placing a fountain in a triangular koi pond. You want the fountain to be the same distance from each edge of the pond. Where should you place the fountain? Explain your reasoning. Use a sketch to support your answer. *(See Example 4.)*



CRITICAL THINKING **In Exercises 29–32, complete the statement with *always*, *sometimes*, or *never*. Explain your reasoning.**

29. The circumcenter of a scalene triangle is __________ inside the triangle.

30. If the perpendicular bisector of one side of a triangle intersects the opposite vertex, then the triangle is __________ isosceles.

31. The perpendicular bisectors of a triangle intersect at a point that is __________ equidistant from the midpoints of the sides of the triangle.

32. The angle bisectors of a triangle intersect at a point that is __________ equidistant from the sides of the triangle.

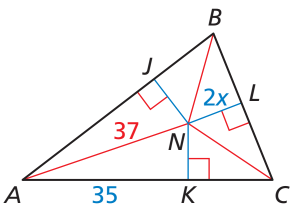
CRITICAL THINKING **In Exercises 33 and 34, find the coordinates of the circumcenter of the triangle with the given vertices.**

33. $A(2, 5)$, $B(6, 6)$, $C(12, 3)$

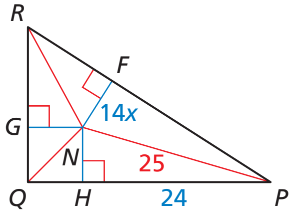
34. $D(-9, -5)$, $E(-5, -9)$, $F(-2, -2)$

MATHEMATICAL CONNECTIONS **In Exercises 35 and 36, find the value of x that makes N the incenter of the triangle.**

35.



36.

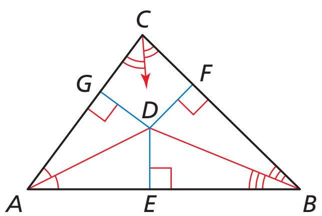


37. PROOF Where is the circumcenter located in any right triangle? Write a coordinate proof of this result.

38. PROVING A THEOREM Write a proof of the Incenter Theorem.

Given $\triangle ABC$, $\overline{AD}$ bisects $\angle CAB$, $\overline{BD}$ bisects $\angle CBA$, $\overline{DE} \perp \overline{AB}$, $\overline{DF} \perp \overline{BC}$, and $\overline{DG} \perp \overline{CA}$

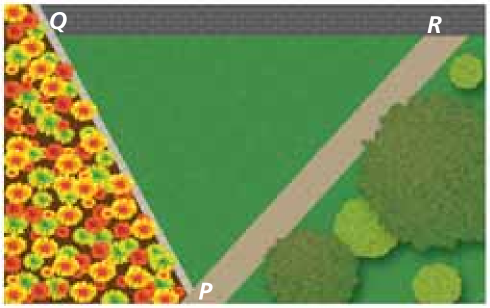
Prove The angle bisectors intersect at D, which is equidistant from $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.



39. WRITING Explain the difference between the circumcenter and the incenter of a triangle.

40. REASONING Is the incenter of a triangle ever located outside the triangle? Explain your reasoning.

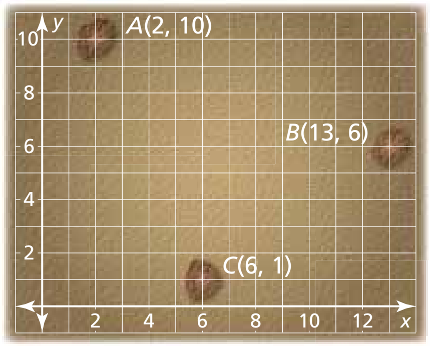
41. MODELING WITH MATHEMATICS You are installing a circular pool in the triangular courtyard shown. You want to have the largest pool possible on the site without extending into the walkway.



a. Copy the triangle and show how to install the pool so that it just touches each edge. Then explain how you can be sure that you could not fit a larger pool on the site.

b. You want to have the largest pool possible while leaving at least 1 foot of space around the pool. Would the center of the pool be in the same position as in part (a)? Justify your answer.

42. MODELING WITH MATHEMATICS Archaeologists find three stones. They believe that the stones were once part of a circle of stones with a community fire pit at its center. They mark the locations of stones A, B, and C on a graph, where distances are measured in feet.



a. Explain how archaeologists can use a sketch to estimate the center of the circle of stones.

b. Copy the diagram and find the approximate coordinates of the point at which the archaeologists should look for the fire pit.

43. REASONING Point P is inside $\triangle ABC$ and is equidistant from points A and B. On which of the following segments must P be located?

Ⓐ $\overline{AB}$

Ⓑ the perpendicular bisector of $\overline{AB}$

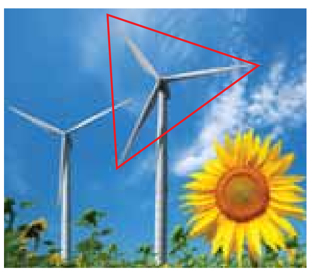
Ⓒ $\overline{AC}$

Ⓓ the perpendicular bisector of $\overline{AC}$

44. **CRITICAL THINKING** A high school is being built for the four towns shown on the map. Each town agrees that the school should be an equal distance from each of the four towns. Is there a single point where they could agree to build the school? If so, find it. If not, explain why not. Justify your answer with a diagram.

45. **MAKING AN ARGUMENT** Your friend says that the circumcenter of an equilateral triangle is also the incenter of the triangle. Is your friend correct? Explain your reasoning.

46. **HOW DO YOU SEE IT?** The arms of the windmill are the angle bisectors of the red triangle. What point of concurrency is the point that connects the three arms?

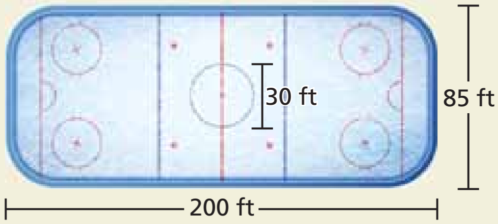


47. **ABSTRACT REASONING** You are asked to draw a triangle and all its perpendicular bisectors and angle bisectors.

a. For which type of triangle would you need the fewest segments? What is the minimum number of segments you would need? Explain.

b. For which type of triangle would you need the most segments? What is the maximum number of segments you would need? Explain.

48. **THOUGHT PROVOKING** The diagram shows an official hockey rink used by the National Hockey League. Create a triangle using hockey players as vertices in which the center circle is inscribed in the triangle. The center dot should be the incenter of your triangle. Sketch a drawing of the locations of your hockey players. Then label the actual lengths of the sides and the angle measures in your triangle.

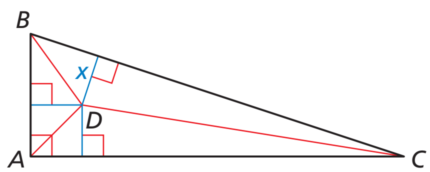


COMPARING METHODS **In Exercises 49 and 50, state whether you would use *perpendicular bisectors* or *angle bisectors*. Then solve the problem.**

49. You need to cut the largest circle possible from an isosceles triangle made of paper whose sides are 8 inches, 12 inches, and 12 inches. Find the radius of the circle.

50. On a map of a camp, you need to create a circular walking path that connects the pool at (10, 20), the nature center at (16, 2), and the tennis court at (2, 4). Find the coordinates of the center of the circle and the radius of the circle.

51. **CRITICAL THINKING** Point D is the incenter of $\triangle ABC$. Write an expression for the length x in terms of the three side lengths AB, AC, and BC.



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

The endpoints of $\overline{AB}$ are given. Find the coordinates of the midpoint M. Then find AB. *(Skills Review Handbook)*

52. $A(-3, 5)$, $B(3, 5)$

53. $A(2, -1)$, $B(10, 7)$

54. $A(-5, 1)$, $B(4, -5)$

55. $A(-7, 5)$, $B(5, 9)$

Write an equation of the line passing through point P that is perpendicular to the given line. Graph the equations of the lines to check that they are perpendicular. *(Skills Review Handbook)*

56. $P(2, 8)$, $y = 2x + 1$

57. $P(6, -3)$, $y = -5$

58. $P(-8, -6)$, $2x + 3y = 18$

59. $P(-4, 1)$, $y + 3 = -4(x + 3)$