5 Probability

Class Ring *(p. 323)*

Horse Racing *(p. 313)*

SEE the Big Idea
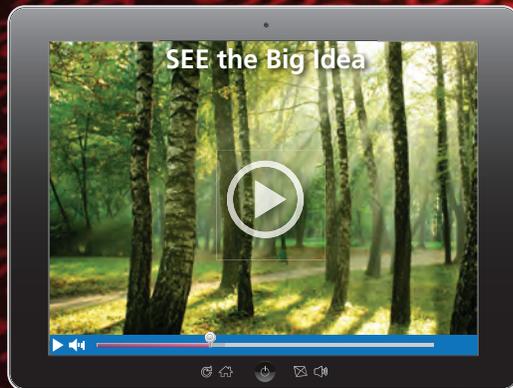
Tree Growth *(p. 310)*

Jogging *(p. 299)*

Coaching *(p. 294)*

Maintaining Mathematical Proficiency

Finding a Percent

Example 1 What percent of 12 is 9?

$$\frac{a}{w} = \frac{p}{100}$$ Write the percent proportion.

$$\frac{9}{12} = \frac{p}{100}$$ Substitute 9 for a and 12 for w.

$$100 \cdot \frac{9}{12} = 100 \cdot \frac{p}{100}$$ Multiplication Property of Equality.

$$75 = p$$ Simplify.

▶ So, 9 is 75% of 12.

Write and solve a proportion to answer the question.

1. What percent of 30 is 6?   **2.** What number is 68% of 25?   **3.** 34.4 is what percent of 86?
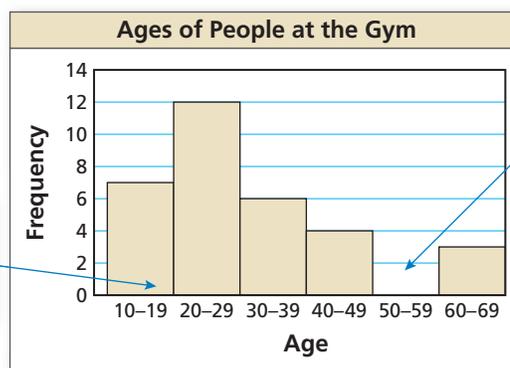
Making a Histogram

Example 2 The frequency table shows the ages of people at a gym. Display the data in a histogram.

Age	Frequency
10–19	7
20–29	12
30–39	6
40–49	4
50–59	0
60–69	3

Step 1 Draw and label the axes.

Step 2 Draw a bar to represent the frequency of each interval.

Ages of People at the Gym

There is no space between the bars of a histogram.

Include any interval with a frequency of 0. The bar height is 0.

Display the data in a histogram.

4.

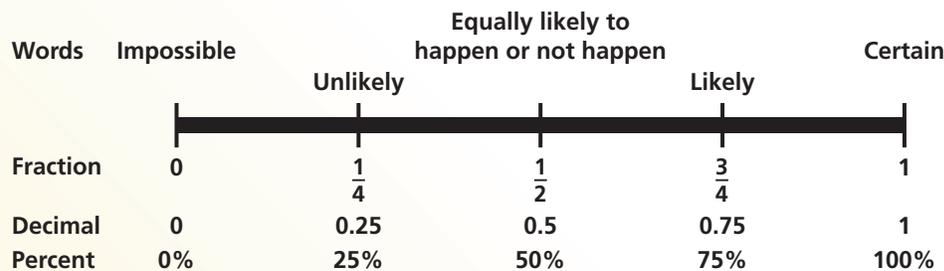| Movies Watched per Week			
Movies	0–1	2–3	4–5
Frequency	35	11	6

5. ABSTRACT REASONING You want to purchase either a sofa or an arm chair at a furniture store. Each item has the same retail price. The sofa is 20% off. The arm chair is 10% off, and you have a coupon to get an additional 10% off the discounted price of the chair. Are the items equally priced after the discounts are applied? Explain.

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Mathematical Practices

Mathematically proficient students apply the mathematics they know to solve real-life problems.

Modeling with Mathematics

Core Concept

Likelihoods and Probabilities

The **probability of an event** is a measure of the likelihood that the event will occur. Probability is a number from 0 to 1, including 0 and 1. The diagram relates *likelihoods* (described in words) and probabilities.

Words	Impossible	Unlikely	Equally likely to happen or not happen	Likely	Certain
Fraction	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
Decimal	0	0.25	0.5	0.75	1
Percent	0%	25%	50%	75%	100%

EXAMPLE 1   Describing Likelihoods

Describe the likelihood of each event.

Probability of an Asteroid or a Meteoroid Hitting Earth			
Name	Diameter	Probability of impact	Flyby date
a. Meteoroid	6 in.	0.75	Any day
b. Apophis	886 ft	0	2029
c. 2000 SG344	121 ft	$\frac{1}{435}$	2068–2110

SOLUTION

a. On any given day, it is *likely* that a meteoroid of this size will enter Earth's atmosphere. If you have ever seen a "shooting star," then you have seen a meteoroid.

b. A probability of 0 means this event is *impossible*.

c. With a probability of $\frac{1}{435} \approx 0.23\%$, this event is very *unlikely*. Of 435 identical asteroids, you would expect only one of them to hit Earth.

Monitoring Progress

In Exercises 1 and 2, describe the event as unlikely, equally likely to happen or not happen, or likely. Explain your reasoning.

1. The oldest child in a family is a girl.

2. The two oldest children in a family with three children are girls.

3. Give an example of an event that is certain to occur.

5.1 Sample Spaces and Probability

Essential Question How can you list the possible outcomes in the sample space of an experiment?

The **sample space** of an experiment is the set of all possible outcomes for that experiment.

EXPLORATION 1   **Finding the Sample Space of an Experiment**

Work with a partner. In an experiment, three coins are flipped. List the possible outcomes in the sample space of the experiment.

EXPLORATION 2   **Finding the Sample Space of an Experiment**

Work with a partner. List the possible outcomes in the sample space of the experiment.
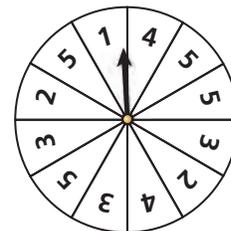
a. One six-sided die is rolled.   **b.** Two six-sided dice are rolled.

EXPLORATION 3   **Finding the Sample Space of an Experiment**

Work with a partner. In an experiment, a spinner is spun.

a. How many ways can you spin a 1? 2? 3? 4? 5?

b. List the sample space.

c. What is the total number of outcomes?

EXPLORATION 4   **Finding the Sample Space of an Experiment**

Work with a partner. In an experiment, a bag contains 2 blue marbles and 5 red marbles. Two marbles are drawn from the bag.

a. How many ways can you choose two blue? a red then blue? a blue then red? two red?

b. List the sample space.

c. What is the total number of outcomes?

LOOKING FOR A PATTERN

To be proficient in math, you need to look closely to discern a pattern or structure.

Communicate Your Answer

5. How can you list the possible outcomes in the sample space of an experiment?

6. For Exploration 3, find the ratio of the number of each possible outcome to the total number of outcomes. Then find the sum of these ratios. Repeat for Exploration 4. What do you observe?

What You Will Learn

▶ Find sample spaces.
▶ Find theoretical probabilities.
▶ Find experimental probabilities.
▶ Find odds.

Sample Spaces

A **probability experiment** is an action, or trial, that has varying results. The possible results of a probability experiment are **outcomes**. For instance, when you roll a six-sided die, there are 6 possible outcomes: 1, 2, 3, 4, 5, or 6. A collection of one or more outcomes is an **event**, such as rolling an odd number. The set of all possible outcomes is called a **sample space**.

EXAMPLE 1   **Finding a Sample Space**

You flip a coin and roll a six-sided die. How many possible outcomes are in the sample space? List the possible outcomes.

SOLUTION

Use a tree diagram to find the outcomes in the sample space.

Coin flip   Heads   Tails

Die roll   1 2 3 4 5 6   1 2 3 4 5 6
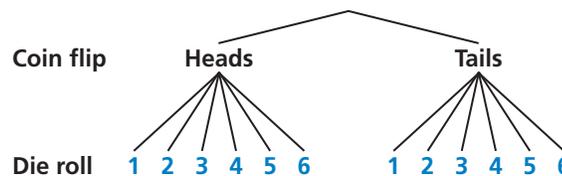
▶ The sample space has 12 possible outcomes. They are listed below.

Heads, 1   Heads, 2   Heads, 3   Heads, 4   Heads, 5   Heads, 6

Tails, 1   Tails, 2   Tails, 3   Tails, 4   Tails, 5   Tails, 6

ANOTHER WAY

Using H for "heads" and T for "tails," you can list the outcomes as shown below.

H1   H2   H3   H4   H5   H6
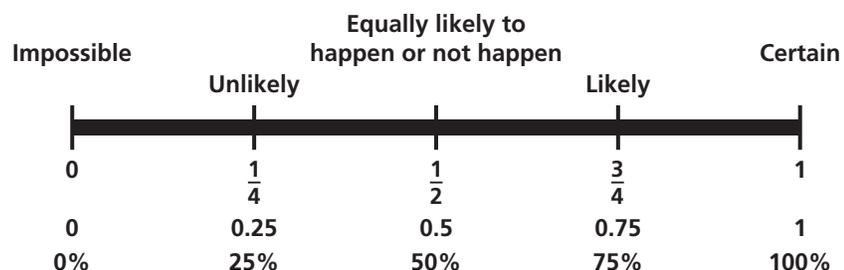T1   T2   T3   T4   T5   T6

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Find the number of possible outcomes in the sample space. Then list the possible outcomes.

1. You flip two coins.   **2.** You flip two coins and roll a six-sided die.

Theoretical Probabilities

The **probability of an event** is a measure of the likelihood, or chance, that the event will occur. Probability is a number from 0 to 1, including 0 and 1, and can be expressed as a decimal, fraction, or percent.

		Equally likely to		
Impossible		happen or not happen		Certain
	Unlikely		Likely	
0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
0	0.25	0.5	0.75	1
0%	25%	50%	75%	100%

The outcomes for a specified event are called *favorable outcomes*. When all outcomes are equally likely, the **theoretical probability** of the event can be found using the following.

$$\text{Theoretical probability} = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

The probability of event A is written as $P(A)$.

EXAMPLE 2   Finding a Theoretical Probability

A student taking a quiz randomly guesses the answers to four true-false questions. What is the probability of the student guessing exactly two correct answers?

SOLUTION

Step 1 Find the outcomes in the sample space. Let C represent a correct answer and I represent an incorrect answer. The possible outcomes are:

Number correct	Outcome
0	IIII
1	CIII ICII IICI IIIC
2	IICC ICIC ICCI CIIC CICI CCII
3	ICCC CICC CCIC CCCI
4	CCCC

exactly two correct → 2

Step 2 Identify the number of favorable outcomes and the total number of outcomes. There are 6 favorable outcomes with exactly two correct answers and the total number of outcomes is 16.

Step 3 Find the probability of the student guessing exactly two correct answers. Because the student is randomly guessing, the outcomes should be equally likely. So, use the theoretical probability formula.

$$P(\text{exactly two correct answers}) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$
$$= \frac{6}{16}$$
$$= \frac{3}{8}$$

▶ The probability of the student guessing exactly two correct answers is $\frac{3}{8}$, or 37.5%.

The sum of the probabilities of all outcomes in a sample space is 1. So, when you know the probability of event A, you can find the probability of the *complement* of event A. The *complement* of event A consists of all outcomes that are not in A and is denoted by $\overline{A}$. The notation $\overline{A}$ is read as "A bar." You can use the following formula to find $P(\overline{A})$.

Core Concept

Probability of the Complement of an Event

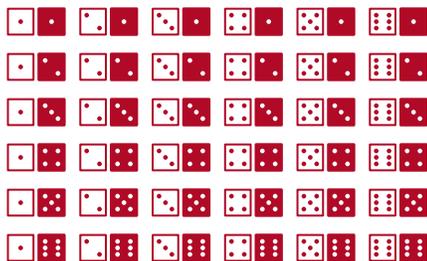The probability of the complement of event A is

$$P(\overline{A}) = 1 - P(A).$$

EXAMPLE 3 Finding Probabilities of Complements

You roll two six-sided dice. Find the probability of each event.

a. The sum is not 6.   **b.** The sum is less than or equal to 9.

SOLUTION

Find the outcomes in the sample space, as shown. There are 36 possible outcomes.

a. $P(\text{sum is not 6}) = 1 - P(\text{sum is 6}) = 1 - \frac{5}{36} = \frac{31}{36} \approx 0.861$

b. $P(\text{sum} \leq 9) = 1 - P(\text{sum} > 9) = 1 - \frac{6}{36} = \frac{30}{36} = \frac{5}{6} \approx 0.833$

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3. You flip a coin and roll a six-sided die. What is the probability that the coin shows tails and the die shows 4?

Find $P(\overline{A})$.

4. $P(A) = 0.45$ **5.** $P(A) = \frac{1}{4}$ **6.** $P(A) = 1$ **7.** $P(A) = 0.03$

Experimental Probabilities

An **experimental probability** is based on repeated *trials* of a probability experiment. The number of trials is the number of times the probability experiment is performed. Each trial in which a favorable outcome occurs is called a *success*. The experimental probability can be found using the following.

$$\text{Experimental probability} = \frac{\text{Number of successes}}{\text{Number of trials}}$$

EXAMPLE 4 Finding an Experimental Probability

Spinner Results			
red	green	blue	yellow
5	9	3	3

Each section of the spinner shown has the same area. The spinner was spun 20 times. The table shows the results. For which color is the experimental probability of stopping on the color the same as the theoretical probability?

SOLUTION

The theoretical probability of stopping on each of the four colors is $\frac{1}{4}$. Use the outcomes in the table to find the experimental probabilities.

$P(\text{red}) = \frac{5}{20} = \frac{1}{4}$ $P(\text{green}) = \frac{9}{20}$ $P(\text{blue}) = \frac{3}{20}$ $P(\text{yellow}) = \frac{3}{20}$

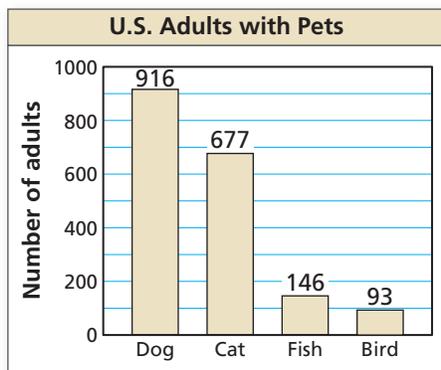▶ The experimental probability of stopping on red is the same as the theoretical probability.

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8. For which color is the experimental probability of stopping on the color greater than the theoretical probability?

EXAMPLE 5 Solving a Real-Life Problem

U.S. Adults with Pets

(bar graph)
- Dog: 916
- Cat: 677
- Fish: 146
- Bird: 93

Y-axis: Number of adults (0 to 1000)

In the United States, a survey of 2184 adults ages 18 and over found that 1328 of them have at least one pet. The types of pets these adults have are shown in the figure. What is the probability that a pet-owning adult chosen at random has a dog?

SOLUTION

The number of trials is the number of pet-owning adults, 1328. A success is a pet-owning adult who has a dog. From the graph, there are 916 adults who have a dog.

$$P(\text{pet-owning adult has a dog}) = \frac{916}{1328} = \frac{229}{332} \approx 0.690$$

▶ The probability that a pet-owning adult chosen at random has a dog is about 69%.

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9. What is the probability that a pet-owning adult chosen at random owns a fish?

Odds

The **odds** of an event compare the number of favorable and unfavorable outcomes when all outcomes are equally likely.

$$\text{Odds } in\ favor = \frac{\text{Number of favorable outcomes}}{\text{Number of unfavorable outcomes}} \qquad \text{Odds } against = \frac{\text{Number of unfavorable outcomes}}{\text{Number of favorable outcomes}}$$

EXAMPLE 6 Finding Odds

In Example 4, find the odds in favor of stopping on yellow.

READING

Odds are read as the ratio of one number to another.

SOLUTION

The 4 possible outcomes are all equally likely. Yellow is the 1 favorable outcome and the other 3 colors are unfavorable outcomes.

$$\text{Odds in favor of yellow} = \frac{\text{Number of favorable outcomes}}{\text{Number of unfavorable outcomes}} = \frac{1}{3}, \text{ or } 1:3$$

EXAMPLE 7 Making a Decision

STUDY TIP

Using the given odds, there is 1 favorable outcome and 5 unfavorable outcomes. So, the total number of outcomes is 6.

The probability of winning a balloon dart game is 0.2. The odds of winning a ring toss game are 1 : 5. Which game gives you a better chance of winning?

SOLUTION

Find the probability of winning the ring toss game. Then compare the probabilities.

$$P(\text{winning ring toss game}) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{1}{6} \approx 0.17$$

▶ Because 0.2 > 0.17, the balloon dart game gives you a better chance of winning.

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10. In Example 4, find the odds against stopping on red.

11. **WHAT IF?** The odds of winning the ring toss game are 2 : 7. Does this change your answer in Example 7? Explain your reasoning.

Vocabulary and Core Concept Check

1. **WRITING** Describe the difference between theoretical probability and experimental probability.

2. **WRITING** Explain how the probability of an event differs from the odds in favor of the event when all outcomes are equally likely.

Monitoring Progress and Modeling with Mathematics

**In Exercises 3–6, find the number of possible outcomes in the sample space. Then list the possible outcomes.** *(See Example 1.)*

3. You roll a die and flip three coins.

4. You flip a coin and draw a marble at random from a bag containing two purple marbles and one white marble.

5. A bag contains four red cards numbered 1 through 4, four white cards numbered 1 through 4, and four black cards numbered 1 through 4. You choose a card at random.

6. You draw two marbles without replacement from a bag containing three green marbles and four black marbles.

7. **PROBLEM SOLVING** A game show airs on television five days per week. Each day, a prize is randomly placed behind one of two doors. The contestant wins the prize by selecting the correct door. What is the probability that exactly two of the five contestants win a prize during a week? *(See Example 2.)*

8. **PROBLEM SOLVING** Your friend has two standard decks of 52 playing cards and asks you to randomly draw one card from each deck. What is the probability that you will draw two spades?

9. **PROBLEM SOLVING** You roll two six-sided dice. Find the probability that (a) the sum is not 4 and (b) the sum is greater than 5. *(See Example 3.)*

10. **PROBLEM SOLVING** The age distribution of a population is shown. Find the probability of each event.

Age Distribution

Under 5: 7%
65+: 13%
5–14: 13%
15–24: 14%
25–34: 13%
35–44: 13%
45–54: 15%
55–64: 12%
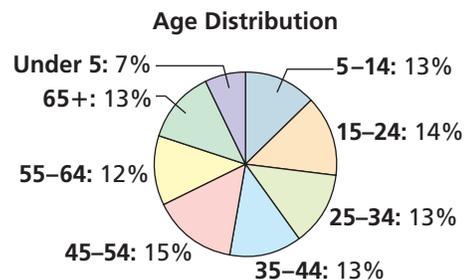
a. A person chosen at random is at least 15 years old.

b. A person chosen at random is from 25 to 44 years old.

11. **ERROR ANALYSIS** A student randomly guesses the answers to two true-false questions. Describe and correct the error in finding the probability of the student guessing both answers correctly.

> ✗ The student can either guess two incorrect answers, two correct answers, or one of each. So the probability of guessing both answers correctly is $\frac{1}{3}$.

12. **ERROR ANALYSIS** A student randomly draws a number between 1 and 30. Describe and correct the error in finding the probability that the number drawn is greater than 4.

> ✗ The probability that the number is less than 4 is $\frac{3}{30}$, or $\frac{1}{10}$. So, the probability that the number is greater than 4 is $1 - \frac{1}{10}$, or $\frac{9}{10}$.

13. **DRAWING CONCLUSIONS** You roll a six-sided die 60 times. The table shows the results. For which number is the experimental probability of rolling the number the same as the theoretical probability? *(See Example 4.)*

Six-sided Die Results					
⚀	⚁	⚂	⚃	⚄	⚅
11	14	7	10	6	12

14. **DRAWING CONCLUSIONS** A bag contains 5 marbles that are each a different color. A marble is drawn, its color is recorded, and then the marble is placed back in the bag. This process is repeated until 30 marbles have been drawn. The table shows the results. For which marble is the experimental probability of drawing the marble the same as the theoretical probability?
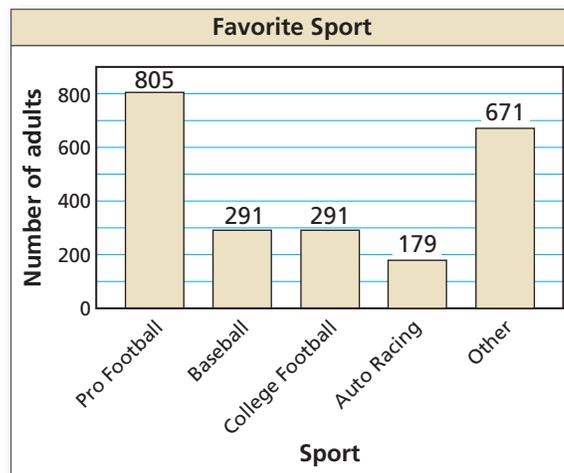
Drawing Results				
white	black	red	green	blue
5	6	8	2	9

15. **REASONING** Refer to the spinner shown. The spinner is divided into sections with the same area.
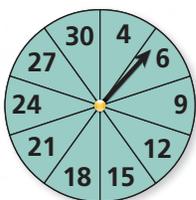

a. What is the theoretical probability that the spinner stops on a multiple of 3?

b. You spin the spinner 30 times. It stops on a multiple of 3 twenty times. What is the experimental probability of stopping on a multiple of 3?

c. Explain why the probability you found in part (b) is different than the probability you found in part(a).

d. You spin the spinner 60 times. It stops on a multiple of 10 six times. How does the experimental probability of stopping on a multiple of 10 compare to the theoretical probability that the spinner stops on a multiple of 10?

16. **OPEN-ENDED** Describe a real-life event that has a probability of 0. Then describe a real-life event that has a probability of 1.

17. **DRAWING CONCLUSIONS** A survey of 2237 adults ages 18 and over asked which sport is their favorite. The results are shown in the figure. What is the probability that an adult chosen at random prefers auto racing? *(See Example 5.)*

18. **DRAWING CONCLUSIONS** A survey of 2392 adults ages 18 and over asked what type of food they would be most likely to choose at a restaurant. The results are shown in the figure. What is the probability that an adult chosen at random prefers Italian food?
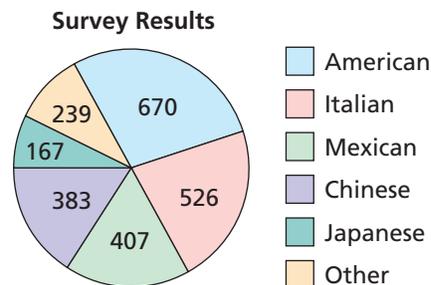
In Exercises 19–22, use the spinner in Exercise 15 to find the given odds. *(See Example 6.)*

19. in favor of stopping on a multiple of 6

20. in favor of stopping on a number greater than 10

21. against stopping on a number less than 9

22. against stopping on a number that is divisible by 7

23. **PROBLEM SOLVING** The odds of winning a lottery game are 3 : 5. The probability of winning a bingo game is 35%. Which game gives you a better chance of winning? Explain. *(See Example 7.)*

24. **PROBLEM SOLVING** The probability of winning a carnival game is $\frac{1}{4}$. The odds of winning a raffle are 1 : 3. For which do you have a better chance of winning? Explain.

25. REASONING Use the spinner in Exercise 15 to give an example of an event whose odds are 1 : 1.

26. ANALYZING RELATIONSHIPS Refer to the chart below.

Four-Day Forecast			
Friday	Saturday	Sunday	Monday
Chance of Rain 5%	Chance of Rain 30%	Chance of Rain 80%	Chance of Rain 90%

a. Order the following events from least likely to most likely.

   A. It rains on Sunday.

   B. It does not rain on Saturday.

   C. It rains on Monday.

   D. It does not rain on Friday.

b. Find the odds in favor of rain for each day.

27. USING TOOLS Use the figure in Example 3 to answer each question.

a. List the possible sums that result from rolling two six-sided dice.

b. Find the theoretical probability of rolling each sum.

c. The table below shows a simulation of rolling two six-sided dice three times. Use a random number generator to simulate rolling two six-sided dice 50 times. Compare the experimental probabilities of rolling each sum with the theoretical probabilities.

|	A	B	C	
1	First Die	Second Die	Sum	
2	4	6	10	
3	3	5	8	
4	1	6	7	
5				

28. MAKING AN ARGUMENT You flip a coin three times. It lands on heads twice and on tails once. Your friend concludes that the theoretical probability of the coin landing heads up is $P(\text{heads up}) = \frac{2}{3}$. Is your friend correct? Explain your reasoning.

29. CRITICAL THINKING You randomly draw a marble from a bag containing red, white, and blue marbles. The odds against drawing a white marble are 47 : 3.

   a. There are less than 100 marbles in the bag. How many marbles are in the bag? Justify your answer.

   b. The probability of drawing a red marble is 0.5. What is the probability of drawing a blue marble? Explain your reasoning.

30. HOW DO YOU SEE IT? Consider the graph of f shown. What is the probability that the graph of $y = f(x) + c$ intersects the x-axis when c is a randomly chosen integer from 1 to 6? Explain.
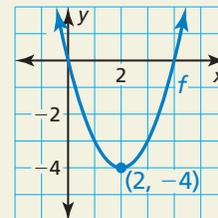
31. DRAWING CONCLUSIONS A manufacturer tests 1200 computers and finds that 9 of them have defects. Find the probability that a computer chosen at random has a defect. Predict the number of computers with defects in a shipment of 15,000 computers. Explain your reasoning.

32. THOUGHT PROVOKING The tree diagram shows a sample space. Write a probability problem that can be represented by the sample space. Then write the answer(s) to the problem.

Box A	Box B	Outcomes	Sum	Product
1	1	(1, 1)	2	1
	2	(1, 2)	3	2
2	1	(2, 1)	3	2
	2	(2, 2)	4	4
3	1	(3, 1)	4	3
	2	(3, 2)	5	6

Maintaining Mathematical Proficiency
Reviewing what you learned in previous grades and lessons

Factor the polynomial. *(Section 2.6)*

33. $3x^2 + 12x - 36$

34. $2x^2 - 11x + 9$

35. $4x^2 + 7x - 15$

Solve the equation using square roots. *(Section 4.3)*

36. $x^2 - 36 = 0$

37. $5x^2 - 20 = 0$

38. $(x + 4)^2 = 81$

39. $25(x - 2)^2 = 9$

5.2 Independent and Dependent Events

Essential Question How can you determine whether two events are independent or dependent?

Two events are **independent events** when the occurrence of one event does not affect the occurrence of the other event. Two events are **dependent events** when the occurrence of one event *does* affect the occurrence of the other event.

EXPLORATION 1 **Identifying Independent and Dependent Events**

REASONING ABSTRACTLY

To be proficient in math, you need to make sense of quantities and their relationships in problem situations.

Work with a partner. Determine whether the events are independent or dependent. Explain your reasoning.

a. Two six-sided dice are rolled.

b. Six pieces of paper, numbered 1 through 6, are in a bag. Two pieces of paper are selected one at a time without replacement.

EXPLORATION 2 **Finding Experimental Probabilities**

Work with a partner.

a. In Exploration 1(a), experimentally estimate the probability that the sum of the two numbers rolled is 7. Describe your experiment.

b. In Exploration 1(b), experimentally estimate the probability that the sum of the two numbers selected is 7. Describe your experiment.

EXPLORATION 3 **Finding Theoretical Probabilities**

Work with a partner.

a. In Exploration 1(a), find the theoretical probability that the sum of the two numbers rolled is 7. Then compare your answer with the experimental probability you found in Exploration 2(a).

b. In Exploration 1(b), find the theoretical probability that the sum of the two numbers selected is 7. Then compare your answer with the experimental probability you found in Exploration 2(b).

c. Compare the probabilities you obtained in parts (a) and (b).

Communicate Your Answer

4. How can you determine whether two events are independent or dependent?

5. Determine whether the events are independent or dependent. Explain your reasoning.

  a. You roll a 4 on a six-sided die and spin red on a spinner.

  b. Your teacher chooses a student to lead a group, chooses another student to lead a second group, and chooses a third student to lead a third group.

Core Vocabulary

independent events, *p. 288*
dependent events, *p. 289*
conditional probability, *p. 289*

Previous
probability
sample space

What You Will Learn

▶ Determine whether events are independent events.
▶ Find probabilities of independent and dependent events.
▶ Find conditional probabilities.

Determining Whether Events Are Independent

Two events are **independent events** when the occurrence of one event does not affect the occurrence of the other event.

> ## 🔄 Core Concept
>
> **Probability of Independent Events**
>
> **Words**   Two events A and B are independent events if and only if the probability that both events occur is the product of the probabilities of the events.
>
> **Symbols** $P(A \text{ and } B) = P(A) \cdot P(B)$

EXAMPLE 1   Determining Whether Events Are Independent

A student taking a quiz randomly guesses the answers to four true-false questions. Use a sample space to determine whether guessing Question 1 correctly and guessing Question 2 correctly are independent events.

SOLUTION

Using the sample space in Example 2 on page 281:

$$P(\text{correct on Question 1}) = \tfrac{8}{16} = \tfrac{1}{2} \qquad P(\text{correct on Question 2}) = \tfrac{8}{16} = \tfrac{1}{2}$$

$$P(\text{correct on Question 1 and correct on Question 2}) = \tfrac{4}{16} = \tfrac{1}{4}$$

▶ Because $\tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}$, the events are independent.

EXAMPLE 2   Determining Whether Events Are Independent

A group of four students includes one boy and three girls. The teacher randomly selects one of the students to be the speaker and a different student to be the recorder. Use a sample space to determine whether randomly selecting a girl first and randomly selecting a girl second are independent events.

SOLUTION

Let B represent the boy. Let G_1, G_2, and G_3 represent the three girls. Use a table to list the outcomes in the sample space.

Number of girls	Outcome	
1	G_1B	BG_1
1	G_2B	BG_2
1	G_3B	BG_3
2	G_1G_2	G_2G_1
2	G_1G_3	G_3G_1
2	G_2G_3	G_3G_2

Using the sample space:

$$P(\text{girl first}) = \tfrac{9}{12} = \tfrac{3}{4} \qquad\qquad P(\text{girl second}) = \tfrac{9}{12} = \tfrac{3}{4}$$

$$P(\text{girl first and girl second}) = \tfrac{6}{12} = \tfrac{1}{2}$$

▶ Because $\tfrac{3}{4} \cdot \tfrac{3}{4} \neq \tfrac{1}{2}$, the events are not independent.

1. In Example 1, determine whether guessing Question 1 incorrectly and guessing Question 2 correctly are independent events.

2. In Example 2, determine whether randomly selecting a girl first and randomly selecting a boy second are independent events.

Finding Probabilities of Events

In Example 1, it makes sense that the events are independent because the second guess should not be affected by the first guess. In Example 2, however, the selection of the second person *depends* on the selection of the first person because the same person cannot be selected twice. These events are *dependent*. Two events are **dependent events** when the occurrence of one event *does* affect the occurrence of the other event.

The probability that event B occurs given that event A has occurred is called the **conditional probability** of B given A and is written as $P(B|A)$.

MAKING SENSE OF PROBLEMS

One way that you can find $P($girl second $|$ girl first$)$ is to list the 9 outcomes in which a girl is chosen first and then find the fraction of these outcomes in which a girl is chosen second:

G$_1$B   G$_2$B   G$_3$B
G$_1$G$_2$ G$_2$G$_1$ G$_3$G$_1$
G$_1$G$_3$ G$_2$G$_3$ G$_3$G$_2$

🜂 Core Concept

Probability of Dependent Events

Words   If two events A and B are dependent events, then the probability that both events occur is the product of the probability of the first event and the conditional probability of the second event given the first event.

Symbols $P(A \text{ and } B) = P(A) \cdot P(B|A)$

Example   Using the information in Example 2:

$P($girl first and girl second$) = P($girl first$) \cdot P($girl second $|$ girl first$)$

$$= \frac{9}{12} \cdot \frac{6}{9} = \frac{1}{2}$$

EXAMPLE 3   **Finding the Probability of Independent Events**

As part of a board game, you need to spin the spinner, which is divided into equal parts. Find the probability that you get a 5 on your first spin and a number greater than 3 on your second spin.

SOLUTION

Let event A be "5 on first spin" and let event B be "greater than 3 on second spin."

The events are independent because the outcome of your second spin is not affected by the outcome of your first spin. Find the probability of each event and then multiply the probabilities.

$P(A) = \frac{1}{8}$ 1 of the 8 sections is a "5."

$P(B) = \frac{5}{8}$ 5 of the 8 sections (4, 5, 6, 7, 8) are greater than 3.

$P(A \text{ and } B) = P(A) \cdot P(B) = \frac{1}{8} \cdot \frac{5}{8} = \frac{5}{64} \approx 0.078$

▶   So, the probability that you get a 5 on your first spin and a number greater than 3 on your second spin is about 7.8%.

EXAMPLE 4   **Finding the Probability of Dependent Events**

A bag contains twenty $1 bills and five $100 bills. You randomly draw a bill from the bag, set it aside, and then randomly draw another bill from the bag. Find the probability that both events *A* and *B* will occur.

Event *A*: The first bill is $100.   **Event *B*:** The second bill is $100.

SOLUTION

The events are dependent because there is one less bill in the bag on your second draw than on your first draw. Find $P(A)$ and $P(B|A)$. Then multiply the probabilities.

$$P(A) = \frac{5}{25} \qquad \text{5 of the 25 bills are \$100 bills.}$$

$$P(B|A) = \frac{4}{24} \qquad \text{4 of the remaining 24 bills are \$100 bills.}$$

$$P(A \text{ and } B) = P(A) \cdot P(B|A) = \frac{5}{25} \cdot \frac{4}{24} = \frac{1}{5} \cdot \frac{1}{6} = \frac{1}{30} \approx 0.033.$$

▶ So, the probability that you draw two $100 bills is about 3.3%.

EXAMPLE 5   **Comparing Independent and Dependent Events**

You randomly select 3 cards from a standard deck of 52 playing cards. What is the probability that all 3 cards are hearts when (a) you replace each card before selecting the next card, and (b) you do not replace each card before selecting the next card? Compare the probabilities.

STUDY TIP

The formulas for finding probabilities of independent and dependent events can be extended to three or more events.

SOLUTION

Let event *A* be "first card is a heart," event *B* be "second card is a heart," and event *C* be "third card is a heart."

a. Because you replace each card before you select the next card, the events are independent. So, the probability is

$$P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B) \cdot P(C) = \frac{13}{52} \cdot \frac{13}{52} \cdot \frac{13}{52} = \frac{1}{64} \approx 0.016.$$

b. Because you do not replace each card before you select the next card, the events are dependent. So, the probability is

$$P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B|A) \cdot P(C|A \text{ and } B)$$

$$= \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} = \frac{11}{850} \approx 0.013.$$

▶ So, you are $\frac{1}{64} \div \frac{11}{850} \approx 1.2$ times more likely to select 3 hearts when you replace each card before you select the next card.

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3. In Example 3, what is the probability that you spin an even number and then an odd number?

4. In Example 4, what is the probability that both bills are $1 bills?

5. In Example 5, what is the probability that none of the cards drawn are hearts when (a) you replace each card, and (b) you do not replace each card? Compare the probabilities.

Finding Conditional Probabilities

EXAMPLE 6 **Using a Table to Find Conditional Probabilities**

	Pass	Fail
Defective	3	36
Non-defective	450	11

A quality-control inspector checks for defective parts. The table shows the results of the inspector's work. Find (a) the probability that a defective part "passes," and (b) the probability that a non-defective part "fails."

SOLUTION

a. $P(\text{pass}\,|\,\text{defective}) = \dfrac{\text{Number of defective parts "passed"}}{\text{Total number of defective parts}}$

$= \dfrac{3}{3+36} = \dfrac{3}{39} = \dfrac{1}{13} \approx 0.077$, or about 7.7%

b. $P(\text{fail}\,|\,\text{non-defective}) = \dfrac{\text{Number of non-defective parts "failed"}}{\text{Total number of non-defective parts}}$

$= \dfrac{11}{450+11} = \dfrac{11}{461} \approx 0.024$, or about 2.4%

STUDY TIP

Note that when A and B are independent, this rule still applies because $P(B) = P(B\,|\,A)$.

You can rewrite the formula for the probability of dependent events to write a rule for finding conditional probabilities.

$P(A) \cdot P(B\,|\,A) = P(A \text{ and } B)$ Write formula.

$P(B\,|\,A) = \dfrac{P(A \text{ and } B)}{P(A)}$ Divide each side by P(A).

EXAMPLE 7 **Finding a Conditional Probability**

At a school, 60% of students buy a school lunch. Only 10% of students buy lunch and dessert. What is the probability that a student who buys lunch also buys dessert?

SOLUTION

Let event A be "buys lunch" and let event B be "buys dessert." You are given $P(A) = 0.6$ and $P(A \text{ and } B) = 0.1$. Use the formula to find $P(B\,|\,A)$.

$P(B\,|\,A) = \dfrac{P(A \text{ and } B)}{P(A)}$ Write formula for conditional probability.

$= \dfrac{0.1}{0.6}$ Substitute 0.1 for P(A and B) and 0.6 for P(A).

$= \dfrac{1}{6} \approx 0.167$ Simplify.

▶ So, the probability that a student who buys lunch also buys dessert is about 16.7%.

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6. In Example 6, find (a) the probability that a non-defective part "passes," and (b) the probability that a defective part "fails."

7. At a coffee shop, 80% of customers order coffee. Only 15% of customers order coffee and a bagel. What is the probability that a customer who orders coffee also orders a bagel?

Vocabulary and Core Concept Check

1. **WRITING** Explain the difference between dependent events and independent events, and give an example of each.

2. **COMPLETE THE SENTENCE** The probability that event *B* will occur given that event *A* has occurred is called the _____ of *B* given *A* and is written as _____.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, tell whether the events are independent or dependent. Explain your reasoning.

3. A box of granola bars contains an assortment of flavors. You randomly choose a granola bar and eat it. Then you randomly choose another bar.

   Event A: You choose a coconut almond bar first.
   Event B: You choose a cranberry almond bar second.

4. You roll a six-sided die and flip a coin.

   Event A: You get a 4 when rolling the die.
   **Event B:** You get tails when flipping the coin.
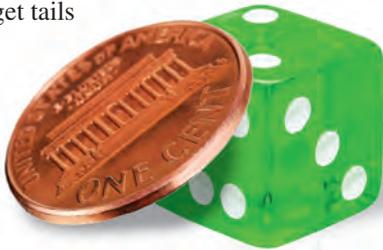
5. Your MP3 player contains hip-hop and rock songs. You randomly choose a song. Then you randomly choose another song without repeating song choices.

   Event A: You choose a hip-hop song first.
   Event B: You choose a rock song second.
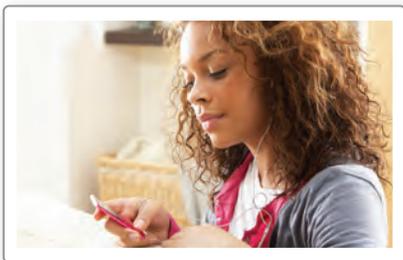
6. There are 22 novels of various genres on a shelf. You randomly choose a novel and put it back. Then you randomly choose another novel.

   Event A: You choose a mystery novel.
   Event B: You choose a science fiction novel.

In Exercises 7–10, determine whether the events are independent. *(See Examples 1 and 2.)*

7. You play a game that involves spinning a wheel. Each section of the wheel shown has the same area. Use a sample space to determine whether randomly spinning blue and then green are independent events.
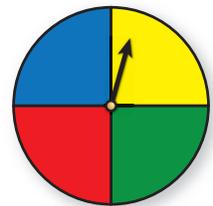
8. You have one red apple and three green apples in a bowl. You randomly select one apple to eat now and another apple for your lunch. Use a sample space to determine whether randomly selecting a green apple first and randomly selecting a green apple second are independent events.

9. A student is taking a multiple-choice test where each question has four choices. The student randomly guesses the answers to the five-question test. Use a sample space to determine whether guessing Question 1 correctly and Question 2 correctly are independent events.

10. A vase contains four white roses and one red rose. You randomly select two roses to take home. Use a sample space to determine whether randomly selecting a white rose first and randomly selecting a white rose second are independent events.

11. **PROBLEM SOLVING** You play a game that involves spinning the money wheel shown. You spin the wheel twice. Find the probability that you get more than $500 on your first spin and then go bankrupt on your second spin. *(See Example 3.)*
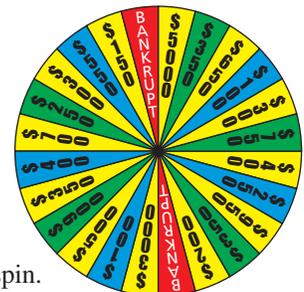
12. PROBLEM SOLVING You play a game that involves drawing two numbers from a hat. There are 25 pieces of paper numbered from 1 to 25 in the hat. Each number is replaced after it is drawn. Find the probability that you will draw the 3 on your first draw and a number greater than 10 on your second draw.

13. PROBLEM SOLVING A drawer contains 12 white socks and 8 black socks. You randomly choose 1 sock and do not replace it. Then you randomly choose another sock. Find the probability that both events A and B will occur. *(See Example 4.)*

Event A: The first sock is white.

Event B: The second sock is white.

14. PROBLEM SOLVING A word game has 100 tiles, 98 of which are letters and 2 of which are blank. The numbers of tiles of each letter are shown. You randomly draw 1 tile, set it aside, and then randomly draw another tile. Find the probability that both events A and B will occur.

Event A:
The first tile is a consonant.

Event B:
The second tile is a vowel.

A – 9	H – 2	O – 8	V – 2
B – 2	I – 9	P – 2	W – 2
C – 2	J – 1	Q – 1	X – 1
D – 4	K – 1	R – 6	Y – 2
E – 12	L – 4	S – 4	Z – 1
F – 2	M – 2	T – 6	– 2
G – 3	N – 6	U – 4	Blank

15. ERROR ANALYSIS Events A and B are independent. Describe and correct the error in finding $P(A \text{ and } B)$.

> ✗ $P(A) = 0.6 \qquad P(B) = 0.2$
> $P(A \text{ and } B) = 0.6 + 0.2 = 0.8$

16. ERROR ANALYSIS A shelf contains 3 fashion magazines and 4 health magazines. You randomly choose one to read, set it aside, and randomly choose another for your friend to read. Describe and correct the error in finding the probability that both events A and B occur.

Event A: The first magazine is fashion.

Event B: The second magazine is health.

> ✗ $P(A) = \frac{3}{7} \qquad P(B|A) = \frac{4}{7}$
> $P(A \text{ and } B) = \frac{3}{7} \cdot \frac{4}{7} = \frac{12}{49} \approx 0.245$

17. NUMBER SENSE Events A and B are independent. Suppose $P(B) = 0.4$ and $P(A \text{ and } B) = 0.13$. Find $P(A)$.

18. NUMBER SENSE Events A and B are dependent. Suppose $P(B|A) = 0.6$ and $P(A \text{ and } B) = 0.15$. Find $P(A)$.

19. ANALYZING RELATIONSHIPS You randomly select three cards from a standard deck of 52 playing cards. What is the probability that all three cards are face cards when (a) you replace each card before selecting the next card, and (b) you do not replace each card before selecting the next card? Compare the probabilities. *(See Example 5.)*

20. ANALYZING RELATIONSHIPS A bag contains 9 red marbles, 4 blue marbles, and 7 yellow marbles. You randomly select three marbles from the bag. What is the probability that all three marbles are red when (a) you replace each marble before selecting the next marble, and (b) you do not replace each marble before selecting the next marble? Compare the probabilities.

21. ATTEND TO PRECISION The table shows the number of species in the United States listed as endangered and threatened. Find (a) the probability that a randomly selected endangered species is a bird, and (b) the probability that a randomly selected mammal is endangered. *(See Example 6.)*

	Endangered	Threatened
Mammals	70	16
Birds	80	16
Other	318	142

22. ATTEND TO PRECISION The table shows the number of tropical cyclones that formed during the hurricane seasons over a 12-year period. Find (a) the probability to predict whether a future tropical cyclone in the Northern Hemisphere is a hurricane, and (b) the probability to predict whether a hurricane is in the Southern Hemisphere.

| Type of Tropical Cyclone	Northern Hemisphere	Southern Hemisphere
tropical depression	100	107
tropical storm	342	487
hurricane	379	525

23. PROBLEM SOLVING At a school, 43% of students attend the homecoming football game. Only 23% of students go to the game and the homecoming dance. What is the probability that a student who attends the football game also attends the dance? *(See Example 7.)*

24. PROBLEM SOLVING At a gas station, 84% of customers buy gasoline. Only 5% of customers buy gasoline and a beverage. What is the probability that a customer who buys gasoline also buys a beverage?

25. PROBLEM SOLVING You and 19 other students volunteer to present the "Best Teacher" award at a school banquet. One student volunteer will be chosen to present the award. Each student worked at least 1 hour in preparation for the banquet. You worked for 4 hours, and the group worked a combined total of 45 hours. For each situation, describe a process that gives you a "fair" chance to be chosen, and find the probability that you are chosen.

   a. "Fair" means equally likely.

   b. "Fair" means proportional to the number of hours each student worked in preparation.

26. HOW DO YOU SEE IT? A bag contains one red marble and one blue marble. The diagrams show the possible outcomes of randomly choosing two marbles using different methods. For each method, determine whether the marbles were selected with or without replacement.

a.
1st Draw	2nd Draw

b.
1st Draw	2nd Draw
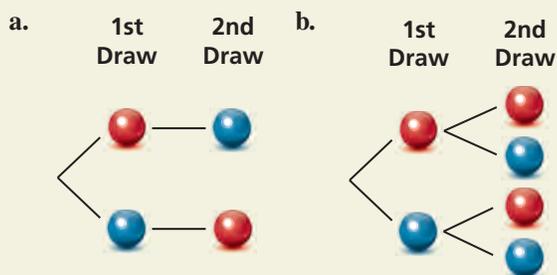
27. MAKING AN ARGUMENT A meteorologist claims that there is a 70% chance of rain. When it rains, there is a 75% chance that your softball game will be rescheduled. Your friend believes the game is more likely to be rescheduled than played. Is your friend correct? Explain your reasoning.

28. THOUGHT PROVOKING Two six-sided dice are rolled once. Events A and B are represented by the diagram. Describe each event. Are the two events dependent or independent? Justify your reasoning.
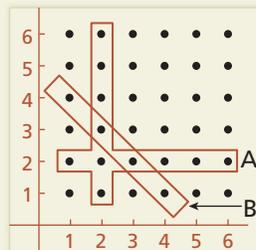
29. MODELING WITH MATHEMATICS A football team is losing by 14 points near the end of a game. The team scores two touchdowns (worth 6 points each) before the end of the game. After each touchdown, the coach must decide whether to go for 1 point with a kick (which is successful 99% of the time) or 2 points with a run or pass (which is successful 45% of the time).

   a. If the team goes for 1 point after each touchdown, what is the probability that the team wins? loses? ties?

   b. If the team goes for 2 points after each touchdown, what is the probability that the team wins? loses? ties?

   c. Can you develop a strategy so that the coach's team has a probability of winning the game that is greater than the probability of losing? If so, explain your strategy and calculate the probabilities of winning and losing the game.

30. ABSTRACT REASONING Assume that A and B are independent events.

   a. Explain why $P(B) = P(B|A)$ and $P(A) = P(A|B)$.

   b. Can $P(A \text{ and } B)$ also be defined as $P(B) \cdot P(A|B)$? Justify your reasoning.

Maintaining Mathematical Proficiency
Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution. *(Skills Review Handbook)*

31. $\frac{9}{10}x = 0.18$

**32.** $\frac{1}{4}x + 0.5x = 1.5$

33. $0.3x - \frac{3}{5}x + 1.6 = 1.555$

5.3 Two-Way Tables and Probability

Essential Question How can you construct and interpret a two-way table?

EXPLORATION 1 **Completing and Using a Two-Way Table**

Work with a partner. A *two-way table* displays the same information as a Venn diagram. In a two-way table, one category is represented by the rows and the other category is represented by the columns.
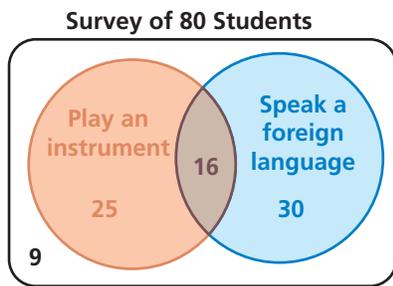
The Venn diagram shows the results of a survey in which 80 students were asked whether they play a musical instrument and whether they speak a foreign language. Use the Venn diagram to complete the two-way table. Then use the two-way table to answer each question.

Survey of 80 Students

	Play an Instrument	Do Not Play an Instrument	Total
Speak a Foreign Language			
Do Not Speak a Foreign Language			
Total			

a. How many students play an instrument?

b. How many students speak a foreign language?

c. How many students play an instrument and speak a foreign language?

d. How many students do not play an instrument and do not speak a foreign language?

e. How many students play an instrument and do not speak a foreign language?

EXPLORATION 2 **Two-Way Tables and Probability**

Work with a partner. In Exploration 1, one student is selected at random from the 80 students who took the survey. Find the probability that the student

a. plays an instrument.

b. speaks a foreign language.

c. plays an instrument and speaks a foreign language.

d. does not play an instrument and does not speak a foreign language.

e. plays an instrument and does not speak a foreign language.

EXPLORATION 3 **Conducting a Survey**

Work with your class. Conduct a survey of the students in your class. Choose two categories that are different from those given in Explorations 1 and 2. Then summarize the results in both a Venn diagram and a two-way table. Discuss the results.

MODELING WITH MATHEMATICS

To be proficient in math, you need to identify important quantities in a practical situation and map their relationships using such tools as diagrams and two-way tables.

Communicate Your Answer

4. How can you construct and interpret a two-way table?

5. How can you use a two-way table to determine probabilities?

Core Vocabulary
two-way table, *p. 296*
joint frequency, *p. 296*
marginal frequency, *p. 296*
joint relative frequency,
    p. 297
marginal relative frequency,
    p. 297
conditional relative frequency,
    p. 297

Previous
conditional probability

READING

A two-way table is also called a *contingency table*, or a *two-way frequency table*.

What You Will Learn

▶ Make two-way tables.

▶ Find relative and conditional relative frequencies.

▶ Use conditional relative frequencies to find conditional probabilities.

Making Two-Way Tables

A **two-way table** is a frequency table that displays data collected from one source that belong to two different categories. One category of data is represented by rows and the other is represented by columns. Suppose you randomly survey freshmen and sophomores about whether they are attending a school concert. A two-way table is one way to organize your results.

Each entry in the table is called a **joint frequency**. The sums of the rows and columns are called **marginal frequencies**, which you will find in Example 1.

		Attendance	
		Attending	**Not Attending**
Class	**Freshman**	25	44
	Sophomore	80	32

joint frequency

EXAMPLE 1 Making a Two-Way Table

In another survey similar to the one above, 106 juniors and 114 seniors respond. Of those, 42 juniors and 77 seniors plan on attending. Organize these results in a two-way table. Then find and interpret the marginal frequencies.

SOLUTION

Step 1 Find the joint frequencies. Because 42 of the 106 juniors are attending, $106 - 42 = 64$ juniors are not attending. Because 77 of the 114 seniors are attending, $114 - 77 = 37$ seniors are not attending. Place each joint frequency in its corresponding cell.

Step 2 Find the marginal frequencies. Create a new column and row for the sums. Then add the entries and interpret the results.

		Attendance		
		Attending	**Not Attending**	**Total**
Class	**Junior**	42	64	106
	Senior	77	37	114
	Total	119	101	220

106 juniors responded.
114 seniors responded.
220 students were surveyed.
119 students are attending.
101 students are not attending.

Step 3 Find the sums of the marginal frequencies. Notice the sums $106 + 114 = 220$ and $119 + 101 = 220$ are equal. Place this value at the bottom right.

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1. You randomly survey students about whether they are in favor of planting a community garden at school. Of 96 boys surveyed, 61 are in favor. Of 88 girls surveyed, 17 are against. Organize the results in a two-way table. Then find and interpret the marginal frequencies.

Finding Relative and Conditional Relative Frequencies

You can display values in a two-way table as frequency counts (as in Example 1) or as *relative frequencies*.

Core Concept

Relative and Conditional Relative Frequencies

A **joint relative frequency** is the ratio of a frequency that is not in the total row or the total column to the total number of values or observations.

A **marginal relative frequency** is the sum of the joint relative frequencies in a row or a column.

A **conditional relative frequency** is the ratio of a joint relative frequency to the marginal relative frequency. You can find a conditional relative frequency using a row total or a column total of a two-way table.

EXAMPLE 2   Finding Joint and Marginal Relative Frequencies

Use the survey results in Example 1 to make a two-way table that shows the joint and marginal relative frequencies.

INTERPRETING MATHEMATICAL RESULTS

Relative frequencies can be interpreted as probabilities. The probability that a randomly selected student is a junior and is *not* attending the concert is 29.1%.

SOLUTION

To find the joint relative frequencies, divide each frequency by the total number of students in the survey. Then find the sum of each row and each column to find the marginal relative frequencies.

		Attendance		
		Attending	**Not Attending**	**Total**
Class	**Junior**	$\frac{42}{220} \approx 0.191$	$\frac{64}{220} \approx 0.291$	0.482
	Senior	$\frac{77}{220} = 0.35$	$\frac{37}{220} \approx 0.168$	0.518
	Total	0.541	0.459	1

About 29.1% of the students in the survey are juniors and are *not* attending the concert.

About 51.8% of the students in the survey are seniors.

EXAMPLE 3   Finding Conditional Relative Frequencies

Use the survey results in Example 1 to make a two-way table that shows the conditional relative frequencies based on the row totals.

SOLUTION

Use the marginal relative frequency of each *row* to calculate the conditional relative frequencies.

		Attendance	
		Attending	**Not Attending**
Class	**Junior**	$\frac{0.191}{0.482} \approx 0.396$	$\frac{0.291}{0.482} \approx 0.604$
	Senior	$\frac{0.35}{0.518} \approx 0.676$	$\frac{0.168}{0.518} \approx 0.324$

Given that a student is a senior, the conditional relative frequency that he or she is *not* attending the concert is about 32.4%.

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2. Use the survey results in Monitoring Progress Question 1 to make a two-way table that shows the joint and marginal relative frequencies.

3. Use the survey results in Example 1 to make a two-way table that shows the conditional relative frequencies based on the column totals. Interpret the conditional relative frequencies in the context of the problem.

4. Use the survey results in Monitoring Progress Question 1 to make a two-way table that shows the conditional relative frequencies based on the row totals. Interpret the conditional relative frequencies in the context of the problem.

Finding Conditional Probabilities

You can use conditional relative frequencies to find conditional probabilities.

EXAMPLE 4 **Finding Conditional Probabilities**

A satellite TV provider surveys customers in three cities. The survey asks whether they would recommend the TV provider to a friend. The results, given as joint relative frequencies, are shown in the two-way table.

		Location		
		Glendale	**Santa Monica**	**Long Beach**
Response	**Yes**	0.29	0.27	0.32
	No	0.05	0.03	0.04

a. What is the probability that a randomly selected customer who is located in Glendale will recommend the provider?

b. What is the probability that a randomly selected customer who will not recommend the provider is located in Long Beach?

c. Determine whether recommending the provider to a friend and living in Long Beach are independent events.

SOLUTION

INTERPRETING MATHEMATICAL RESULTS

The probability 0.853 is a conditional relative frequency based on a column total. The condition is that the customer lives in Glendale.

a. $P(\text{yes}\,|\,\text{Glendale}) = \dfrac{P(\text{Glendale and yes})}{P(\text{Glendale})} = \dfrac{0.29}{0.29 + 0.05} \approx 0.853$

▶ So, the probability that a customer who is located in Glendale will recommend the provider is about 85.3%.

b. $P(\text{Long Beach}\,|\,\text{no}) = \dfrac{P(\text{no and Long Beach})}{P(\text{no})} = \dfrac{0.04}{0.05 + 0.03 + 0.04} \approx 0.333$

▶ So, the probability that a customer who will not recommend the provider is located in Long Beach is about 33.3%.

c. Use the formula $P(B) = P(B\,|\,A)$ and compare $P(\text{Long Beach})$ and $P(\text{Long Beach}\,|\,\text{yes})$.

$P(\text{Long Beach}) = 0.32 + 0.04 = 0.36$

$P(\text{Long Beach}\,|\,\text{yes}) = \dfrac{P(\text{Yes and Long Beach})}{P(\text{yes})} = \dfrac{0.32}{0.29 + 0.27 + 0.32} \approx 0.36$

▶ Because $P(\text{Long Beach}) \approx P(\text{Long Beach}\,|\,\text{yes})$, the two events are independent.

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5. In Example 4, what is the probability that a randomly selected customer who is located in Santa Monica will not recommend the provider to a friend?

6. In Example 4, determine whether recommending the provider to a friend and living in Santa Monica are independent events. Explain your reasoning.

EXAMPLE 5 **Comparing Conditional Probabilities**
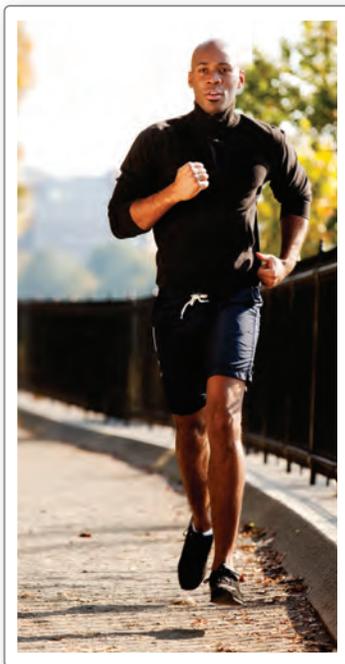
A jogger wants to burn a certain number of calories during his workout. He maps out three possible jogging routes. Before each workout, he randomly selects a route, and then determines the number of calories he burns and whether he reaches his goal. The table shows his findings. Which route should he use?

	Reaches Goal	Does Not Reach Goal
Route A	JHT JHT I	JHT I
Route B	JHT JHT I	IIII
Route C	JHT JHT II	JHT I

SOLUTION

Step 1 Use the findings to make a two-way table that shows the joint and marginal relative frequencies. There are a total of 50 observations in the table.

Step 2 Find the conditional probabilities by dividing each joint relative frequency in the "Reaches Goal" column by the marginal relative frequency in its corresponding row.

		Result		
		Reaches Goal	Does Not Reach Goal	Total
Route	**A**	0.22	0.12	0.34
	B	0.22	0.08	0.30
	C	0.24	0.12	0.36
	Total	0.68	0.32	1

$$P(\text{reaches goal} \mid \text{Route A}) = \frac{P(\text{Route A and reaches goal})}{P(\text{Route A})} = \frac{0.22}{0.34} \approx 0.647$$

$$P(\text{reaches goal} \mid \text{Route B}) = \frac{P(\text{Route B and reaches goal})}{P(\text{Route B})} = \frac{0.22}{0.30} \approx 0.733$$

$$P(\text{reaches goal} \mid \text{Route C}) = \frac{P(\text{Route C and reaches goal})}{P(\text{Route C})} = \frac{0.24}{0.36} \approx 0.667$$

▶ Based on the sample, the probability that he reaches his goal is greatest when he uses Route B. So, he should use Route B.

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7. A manager is assessing three employees in order to offer one of them a promotion. Over a period of time, the manager records whether the employees meet or exceed expectations on their assigned tasks. The table shows the manager's results. Which employee should be offered the promotion? Explain.

	Exceed Expectations	Meet Expectations
Joy	JHT IIII	JHT I
Elena	JHT JHT II	JHT III
Sam	JHT JHT I	JHT II

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A(n) _____ displays data collected from the same source that belongs to two different categories.

2. **WRITING** Compare the definitions of joint relative frequency, marginal relative frequency, and conditional relative frequency.

Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, complete the two-way table.

3.

		Preparation		
		Studied	Did Not Study	Total
Grade	Pass		6	
	Fail			10
	Total	38		50

4.

		Response		
		Yes	No	Total
Role	Student	56		
	Teacher		7	10
	Total		49	

5. **MODELING WITH MATHEMATICS** You survey 171 males and 180 females at Grand Central Station in New York City. Of those, 132 males and 151 females wash their hands after using the public rest rooms. Organize these results in a two-way table. Then find and interpret the marginal frequencies. *(See Example 1.)*

6. **MODELING WITH MATHEMATICS** A survey asks 60 teachers and 48 parents whether school uniforms reduce distractions in school. Of those, 49 teachers and 18 parents say uniforms reduce distractions in school. Organize these results in a two-way table. Then find and interpret the marginal frequencies.

USING STRUCTURE In Exercises 7 and 8, use the two-way table to create a two-way table that shows the joint and marginal relative frequencies.

7.

		Dominant Hand		
		Left	Right	Total
Gender	Female	11	104	115
	Male	24	92	116
	Total	35	196	231

8.

		Gender		
		Male	Female	Total
Experience	Expert	62	6	68
	Average	275	24	299
	Novice	40	3	43
	Total	377	33	410

9. **MODELING WITH MATHEMATICS** Use the survey results from Exercise 5 to make a two-way table that shows the joint and marginal relative frequencies. *(See Example 2.)*

10. **MODELING WITH MATHEMATICS** In a survey, 49 people received a flu vaccine before the flu season and 63 people did not receive the vaccine. Of those who receive the flu vaccine, 16 people got the flu. Of those who did not receive the vaccine, 17 got the flu. Make a two-way table that shows the joint and marginal relative frequencies.
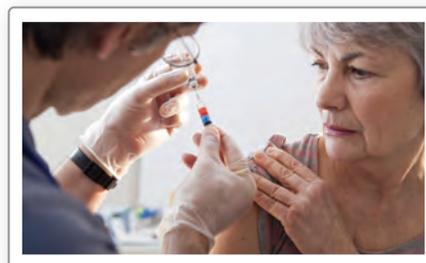
11. **MODELING WITH MATHEMATICS** A survey finds that 110 people ate breakfast and 30 people skipped breakfast. Of those who ate breakfast, 10 people felt tired. Of those who skipped breakfast, 10 people felt tired. Make a two-way table that shows the conditional relative frequencies based on the breakfast totals. *(See Example 3.)*

12. **MODELING WITH MATHEMATICS** Use the survey results from Exercise 10 to make a two-way table that shows the conditional relative frequencies based on the flu vaccine totals.

13. **PROBLEM SOLVING** Three different local hospitals in New York surveyed their patients. The survey asked whether the patient's physician communicated efficiently. The results, given as joint relative frequencies, are shown in the two-way table. *(See Example 4.)*

	Location		
	Glens Falls	**Saratoga**	**Albany**
Yes	0.123	0.288	0.338
No	0.042	0.077	0.131

(Response is the row label on the left.)

a. What is the probability that a randomly selected patient located in Saratoga was satisfied with the communication of the physician?

b. What is the probability that a randomly selected patient who was not satisfied with the physician's communication is located in Glens Falls?

c. Determine whether being satisfied with the communication of the physician and living in Saratoga are independent events.

14. **PROBLEM SOLVING** A researcher surveys a random sample of high school students in seven states. The survey asks whether students plan to stay in their home state after graduation. The results, given as joint relative frequencies, are shown in the two-way table.

	Location		
	Nebraska	**North Carolina**	**Other States**
Yes	0.044	0.051	0.056
No	0.400	0.193	0.256

(Response is the row label on the left.)

a. What is the probability that a randomly selected student who lives in Nebraska plans to stay in his or her home state after graduation?

b. What is the probability that a randomly selected student who does not plan to stay in his or her home state after graduation lives in North Carolina?

c. Determine whether planning to stay in their home state and living in Nebraska are independent events.

ERROR ANALYSIS In Exercises 15 and 16, describe and correct the error in finding the given conditional probability.

	City			
	Tokyo	**London**	**Washington, D.C.**	**Total**
Yes	0.049	0.136	0.171	0.356
No	0.341	0.112	0.191	0.644
Total	0.39	0.248	0.362	1

(Response is the row label on the left.)

15. $P(\text{yes}\,|\,\text{Tokyo})$

$$✗ \quad P(\text{yes}\,|\,\text{Tokyo}) = \frac{P(\text{Tokyo and yes})}{P(\text{Tokyo})}$$
$$= \frac{0.049}{0.356} \approx 0.138$$

16. $P(\text{London}\,|\,\text{no})$

$$✗ \quad P(\text{London}\,|\,\text{no}) = \frac{P(\text{no and London})}{P(\text{London})}$$
$$= \frac{0.112}{0.248} \approx 0.452$$

17. **PROBLEM SOLVING** You want to find the quickest route to school. You map out three routes. Before school, you randomly select a route and record whether you are late or on time. The table shows your findings. Assuming you leave at the same time each morning, which route should you use? Explain. *(See Example 5.)*

	On Time	Late
Route A	卌 ‖	‖‖
Route B	卌 卌 ‖	‖‖
Route C	卌 卌 ‖	‖‖‖

18. **PROBLEM SOLVING** A teacher is assessing three groups of students in order to offer one group a prize. Over a period of time, the teacher records whether the groups meet or exceed expectations on their assigned tasks. The table shows the teacher's results. Which group should be awarded the prize? Explain.

	Exceed Expectations	Meet Expectations
Group 1	卌 卌 ‖	‖‖‖
Group 2	卌 ‖‖	卌
Group 3	卌 ‖‖‖	卌 ‖

19. **OPEN-ENDED** Create and conduct a survey in your class. Organize the results in a two-way table. Then create a two-way table that shows the joint and marginal frequencies.

20. **HOW DO YOU SEE IT?** A research group surveys parents and coaches of high school students about whether competitive sports are important in school. The two-way table shows the results of the survey.

|		Role		
		Parent	**Coach**	**Total**
Important	**Yes**	880	456	1336
	No	120	45	165
	Total	1000	501	1501

a. What does 120 represent?

b. What does 1336 represent?

c. What does 1501 represent?

21. **MAKING AN ARGUMENT** Your friend uses the table below to determine which workout routine is the best. Your friend decides that Routine B is the best option because it has the fewest tally marks in the "Does Not Reach Goal" column. Is your friend correct? Explain your reasoning.

	Reached Goal	**Does Not Reach Goal**
Routine A	JHr	III
Routine B	IIII	II
Routine C	JHr II	IIII

22. **MODELING WITH MATHEMATICS** A survey asks students whether they prefer math class or science class. Of the 150 male students surveyed, 62% prefer math class over science class. Of the female students surveyed, 74% prefer math. Construct a two-way table to show the number of students in each category if 350 students were surveyed.

23. **MULTIPLE REPRESENTATIONS** Use the Venn diagram to construct a two-way table. Then use your table to answer the questions.
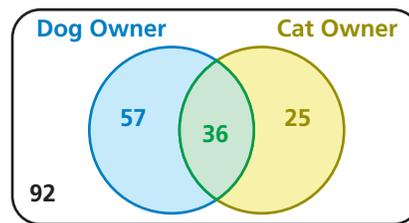
a. What is the probability that a randomly selected person does not own either pet?

b. What is the probability that a randomly selected person who owns a dog also owns a cat?

24. **WRITING** Compare two-way tables and Venn diagrams. Then describe the advantages and disadvantages of each.

25. **PROBLEM SOLVING** A company creates a new snack, N, and tests it against its current leader, L. The table shows the results.

	Prefer L	**Prefer N**
Current L Consumer	72	46
Not Current L Consumer	52	114

The company is deciding whether it should try to improve the snack before marketing it, and to whom the snack should be marketed. Use probability to explain the decisions the company should make when the total size of the snack's market is expected to (a) change very little, and (b) expand very rapidly.

26. **THOUGHT PROVOKING** Bayes' Theorem is given by

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}.$$

Use a two-way table to write an example of Bayes' Theorem.

Maintaining Mathematical Proficiency
Reviewing what you learned in previous grades and lessons

Draw a Venn diagram of the sets described. *(Skills Review Handbook)*

27. Of the positive integers less than 15, set A consists of the factors of 15 and set B consists of all odd numbers.

28. Of the positive integers less than 14, set A consists of all prime numbers and set B consists of all even numbers.

29. Of the positive integers less than 24, set A consists of the multiples of 2 and set B consists of all the multiples of 3.

5.1–5.3 What Did You Learn?

Core Vocabulary

Core Concepts

Section 5.1

Section 5.2

Section 5.3

Mathematical Practices

1. How can you use a number line to analyze the error in Exercise 12 on page 284?

2. Explain how you used probability to correct the flawed logic of your friend in Exercise 21 on page 302.

Making a Mental Cheat Sheet

STUDY SKILLS

- Write down important information on note cards.
- Memorize the information on the note cards, placing the ones containing information you know in one stack and the ones containing information you do not know in another stack. Keep working on the information you do not know.

1. You randomly draw a marble out of a bag containing 8 green marbles, 4 blue marbles, 12 yellow marbles, and 10 red marbles. Find the probability of drawing a marble that is not yellow. *(Section 5.1)*

Find $P(\overline{A})$. *(Section 5.1)*

2. $P(A) = 0.32$

3. $P(A) = \frac{8}{9}$

4. $P(A) = 0.01$

5. You roll a six-sided die 30 times. A 5 is rolled 8 times. What is the theoretical probability of rolling a 5? What is the experimental probability of rolling a 5? *(Section 5.1)*

6. You randomly draw one card from a standard deck of 52 playing cards. *(Section 5.1)*
   a. What are the odds that it is a diamond?
   b. What are the odds that it is a 7?

7. Events A and B are independent. Find the missing probability. *(Section 5.2)*

   $P(A) = 0.25$

   $P(B) = $ _____

   $P(A \text{ and } B) = 0.05$

8. Events A and B are dependent. Find the missing probability. *(Section 5.2)*

   $P(A) = 0.6$

   $P(B|A) = 0.2$

   $P(A \text{ and } B) = $ _____

You roll a red six-sided die and a blue six-sided die. Find the probability. *(Section 5.2)*

9. $P(\text{red odd and blue odd})$

10. $P(\text{red and blue both less than 3})$

11. $P(\text{red 2 and blue greater than 1})$

12. A survey asks 13-year-old and 15-year-old students about their eating habits. Four hundred students are surveyed, 100 male students and 100 female students from each age group. The bar graph shows the number of students who said they eat fruit every day. *(Section 5.2)*
   a. Find the probability that a female student, chosen at random from the students surveyed, eats fruit every day.
   b. Find the probability that a 15-year-old student, chosen at random from the students surveyed, eats fruit every day.

Survey Results

13. At school, 55% of students use public transportation. Twenty-five percent of students use public transportation and participate in an extracurricular activity after school. What is the probability that a student who uses public transportation also participates in an extracurricular activity after school? *(Section 5.2)*

14. There are 14 boys and 18 girls in a class. The teacher allows the students to vote whether they want to take a test on Friday or on Monday. A total of 6 boys and 10 girls vote to take the test on Friday. Organize the information in a two-way table. Then find and interpret the marginal frequencies. *(Section 5.3)*

15. Three schools compete in a cross country invitational. Of the 15 athletes on your team, 9 achieve their goal times. Of the 20 athletes on the home team, 6 achieve their goal times. On your rival's team, 8 of the 13 athletes achieve their goal times. Organize the information in a two-way table. Then determine the probability that a randomly selected runner who achieves his or her goal time is from your school. *(Section 5.3)*

5.4 Probability of Disjoint and Overlapping Events

Essential Question How can you find probabilities of disjoint and overlapping events?

Two events are **disjoint**, or **mutually exclusive**, when they have no outcomes in common. Two events are **overlapping** when they have one or more outcomes in common.

EXPLORATION 1 Disjoint Events and Overlapping Events

Work with a partner. A six-sided die is rolled. Draw a Venn diagram that relates the two events. Then decide whether the events are disjoint or overlapping.

a. Event A: The result is an even number.
   Event B: The result is a prime number.

b. Event A: The result is 2 or 4.
   Event B: The result is an odd number.

EXPLORATION 2 Finding the Probability that Two Events Occur

Work with a partner. A six-sided die is rolled. For each pair of events, find (a) $P(A)$, (b) $P(B)$, (c) $P(A$ and $B)$, and (d) $P(A$ or $B)$.

a. Event A: The result is an even number.
   Event B: The result is a prime number.

b. Event A: The result is 2 or 4.
   Event B: The result is an odd number.

EXPLORATION 3 Discovering Probability Formulas

Work with a partner.

a. In general, if event A and event B are disjoint, then what is the probability that event A or event B will occur? Use a Venn diagram to justify your conclusion.

b. In general, if event A and event B are overlapping, then what is the probability that event A or event B will occur? Use a Venn diagram to justify your conclusion.

c. Conduct an experiment using a six-sided die. Roll the die 50 times and record the results. Then use the results to find the probabilities described in Exploration 2. How closely do your experimental probabilities compare to the theoretical probabilities you found in Exploration 2?

Communicate Your Answer

4. How can you find probabilities of disjoint and overlapping events?

5. Give examples of disjoint events and overlapping events that do not involve dice.

Core Vocabulary

compound event, *p. 306*
overlapping events, *p. 306*
disjoint or mutually exclusive
   events, *p. 306*

Previous
Venn diagram

What You Will Learn

▶ Find probabilities of compound events.

▶ Use more than one probability rule to solve real-life problems.
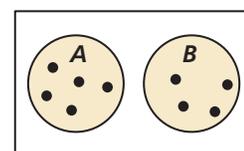
Compound Events

When you consider all the outcomes for either of two events *A* and *B*, you form the *union* of *A* and *B*, as shown in the first diagram. When you consider only the outcomes shared by both *A* and *B*, you form the *intersection* of *A* and *B*, as shown in the second diagram. The union or intersection of two events is called a **compound event**.
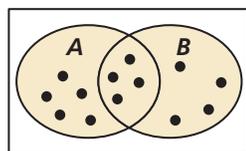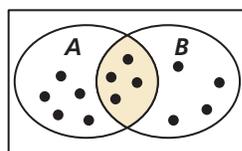
Union of *A* and *B*   Intersection of *A* and *B*   Intersection of *A* and *B*
                                                                is empty.

To find *P*(*A* or *B*) you must consider what outcomes, if any, are in the intersection of *A* and *B*. Two events are **overlapping** when they have one or more outcomes in common, as shown in the first two diagrams. Two events are **disjoint**, or **mutually exclusive**, when they have no outcomes in common, as shown in the third diagram.

STUDY TIP

If two events *A* and *B* are overlapping, then the outcomes in the intersection of *A* and *B* are counted *twice* when *P*(*A*) and *P*(*B*) are added. So, *P*(*A* and *B*) must be subtracted from the sum.

Core Concept

Probability of Compound Events

If *A* and *B* are any two events, then the probability of *A* or *B* is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

If *A* and *B* are disjoint events, then the probability of *A* or *B* is

$$P(A \text{ or } B) = P(A) + P(B).$$

EXAMPLE 1   Finding the Probability of Disjoint Events

A card is randomly selected from a standard deck of 52 playing cards. What is the probability that it is a 10 *or* a face card?

SOLUTION

Let event *A* be selecting a 10 and event *B* be selecting a face card. From the diagram, *A* has 4 outcomes and *B* has 12 outcomes. Because *A* and *B* are disjoint, the probability is
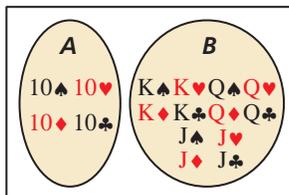
$$P(A \text{ or } B) = P(A) + P(B) \qquad \text{Write disjoint probability formula.}$$

$$= \frac{4}{52} + \frac{12}{52} \qquad \text{Substitute known probabilities.}$$

$$= \frac{16}{52} \qquad \text{Add.}$$

$$= \frac{4}{13} \qquad \text{Simplify.}$$

$$\approx 0.308. \qquad \text{Use a calculator.}$$

COMMON ERROR

When two events A and B overlap, as in Example 2, P(A or B) does not equal P(A) + P(B).

EXAMPLE 2 **Finding the Probability of Overlapping Events**

A card is randomly selected from a standard deck of 52 playing cards. What is the probability that it is a face card *or* a spade?

SOLUTION

Let event A be selecting a face card and event B be selecting a spade. From the diagram, A has 12 outcomes and B has 13 outcomes. Of these, 3 outcomes are common to A and B. So, the probability of selecting a face card or a spade is
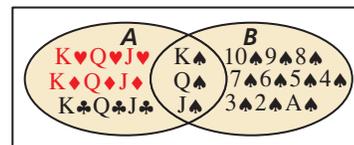
$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$ Write general formula.

$$= \frac{12}{52} + \frac{13}{52} - \frac{3}{52}$$ Substitute known probabilities.

$$= \frac{22}{52}$$ Add.

$$= \frac{11}{26}$$ Simplify.

$$\approx 0.423.$$ Use a calculator.

EXAMPLE 3 **Using a Formula to Find $P(A \text{ and } B)$**

Out of 200 students in a senior class, 113 students are either varsity athletes or on the honor roll. There are 74 seniors who are varsity athletes and 51 seniors who are on the honor roll. What is the probability that a randomly selected senior is both a varsity athlete *and* on the honor roll?

SOLUTION

Let event A be selecting a senior who is a varsity athlete and event B be selecting a senior on the honor roll. From the given information, you know that $P(A) = \frac{74}{200}$, $P(B) = \frac{51}{200}$, and $P(A \text{ or } B) = \frac{113}{200}$. The probability that a randomly selected senior is both a varsity athlete *and* on the honor roll is $P(A \text{ and } B)$.

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$ Write general formula.

$$\frac{113}{200} = \frac{74}{200} + \frac{51}{200} - P(A \text{ and } B)$$ Substitute known probabilities.

$$P(A \text{ and } B) = \frac{74}{200} + \frac{51}{200} - \frac{113}{200}$$ Solve for P(A and B).

$$P(A \text{ and } B) = \frac{12}{200}$$ Simplify.

$$P(A \text{ and } B) = \frac{3}{50}, \text{ or } 0.06$$ Simplify.

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A card is randomly selected from a standard deck of 52 playing cards. Find the probability of the event.

1. selecting an ace *or* an 8
2. selecting a 10 *or* a diamond

3. **WHAT IF?** In Example 3, suppose 32 seniors are in the band and 64 seniors are in the band or on the honor roll. What is the probability that a randomly selected senior is both in the band and on the honor roll?

Using More Than One Probability Rule

In the first four sections of this chapter, you have learned several probability rules. The solution to some real-life problems may require the use of two or more of these probability rules, as shown in the next example.

EXAMPLE 4   **Solving a Real-Life Problem**

The American Diabetes Association estimates that 8.3% of people in the United States have diabetes. Suppose that a medical lab has developed a simple diagnostic test for diabetes that is 98% accurate for people who have the disease and 95% accurate for people who do not have it. The medical lab gives the test to a randomly selected person. What is the probability that the diagnosis is correct?

SOLUTION

Let event A be "person has diabetes" and event B be "correct diagnosis." Notice that the probability of B depends on the occurrence of A, so the events are dependent. When A occurs, $P(B) = 0.98$. When A does not occur, $P(B) = 0.95$.

A probability tree diagram, where the probabilities are given along the branches, can help you see the different ways to obtain a correct diagnosis. Use the complements of events A and B to complete the diagram, where $\overline{A}$ is "person does not have diabetes" and $\overline{B}$ is "incorrect diagnosis." Notice that the probabilities for all branches from the same point must sum to 1.
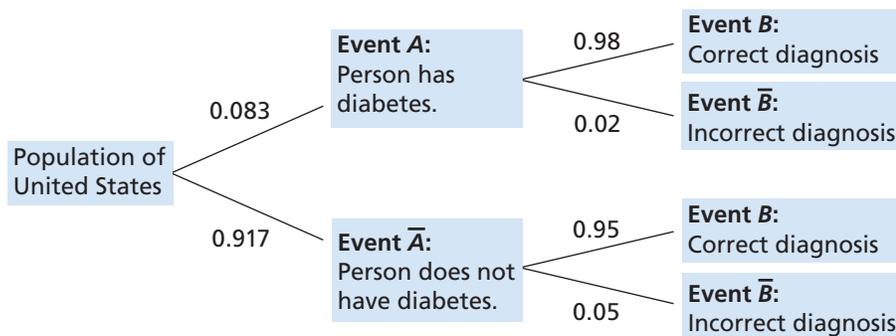
To find the probability that the diagnosis is correct, follow the branches leading to event B.

$$P(B) = P(A \text{ and } B) + P(\overline{A} \text{ and } B) \qquad \text{Use tree diagram.}$$

$$= P(A) \cdot P(B|A) + P(\overline{A}) \cdot P(B|\overline{A}) \qquad \text{Probability of dependent events}$$

$$= (0.083)(0.98) + (0.917)(0.95) \qquad \text{Substitute.}$$

$$\approx 0.952 \qquad \text{Use a calculator.}$$

▶   The probability that the diagnosis is correct is about 0.952, or 95.2%.

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4. In Example 4, what is the probability that the diagnosis is *incorrect*?

5. A high school basketball team leads at halftime in 60% of the games in a season. The team wins 80% of the time when they have the halftime lead, but only 10% of the time when they do not. What is the probability that the team wins a particular game during the season?

Vocabulary and Core Concept Check

1. **WRITING** Are the events A and $\overline{A}$ disjoint? Explain. Then give an example of a real-life event and its complement.

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

How many outcomes are in the intersection of A and B?

How many outcomes are shared by both A and B?

How many outcomes are in the union of A and B?

How many outcomes in B are also in A?
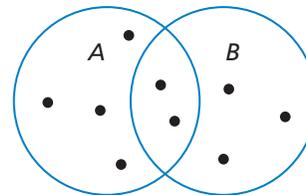
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In Exercises 3–6, events A and B are disjoint. Find $P(A \text{ or } B)$.

3. $P(A) = 0.3, P(B) = 0.1$
4. $P(A) = 0.55, P(B) = 0.2$

5. $P(A) = \frac{1}{3}, P(B) = \frac{1}{4}$
6. $P(A) = \frac{2}{3}, P(B) = \frac{1}{5}$

7. **PROBLEM SOLVING** Your dart is equally likely to hit any point inside the board shown. You throw a dart and pop a balloon. What is the probability that the balloon is red or blue? *(See Example 1.)*
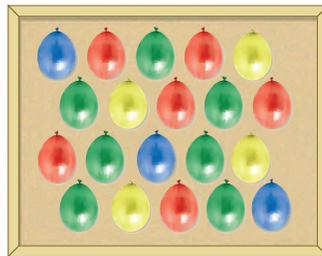
8. **PROBLEM SOLVING** You and your friend are among several candidates running for class president. You estimate that there is a 45% chance you will win and a 25% chance your friend will win. What is the probability that you or your friend win the election?

9. **PROBLEM SOLVING** You are performing an experiment to determine how well plants grow under different light sources. Of the 30 plants in the experiment, 12 receive visible light, 15 receive ultraviolet light, and 6 receive both visible and ultraviolet light. What is the probability that a plant in the experiment receives visible or ultraviolet light? *(See Example 2.)*

10. **PROBLEM SOLVING** Of 162 students honored at an academic awards banquet, 48 won awards for mathematics and 78 won awards for English. There are 14 students who won awards for both mathematics and English. A newspaper chooses a student at random for an interview. What is the probability that the student interviewed won an award for English or mathematics?

ERROR ANALYSIS In Exercises 11 and 12, describe and correct the error in finding the probability of randomly drawing the given card from a standard deck of 52 playing cards.

11.

$P(\text{heart or face card})$
$= P(\text{heart}) + P(\text{face card})$
$= \frac{13}{52} + \frac{12}{52} = \frac{25}{52}$

12.

$P(\text{club or 9})$
$= P(\text{club}) + P(9) + P(\text{club and 9})$
$= \frac{13}{52} + \frac{4}{52} + \frac{1}{52} = \frac{9}{26}$

In Exercises 13 and 14, you roll a six-sided die. Find $P(A \text{ or } B)$.

13. Event A: Roll a 6.
Event B: Roll a prime number.

14. Event A: Roll an odd number.
Event B: Roll a number less than 5.

15. DRAWING CONCLUSIONS A group of 40 trees in a forest are not growing properly. A botanist determines that 34 of the trees have a disease or are being damaged by insects, with 18 trees having a disease and 20 being damaged by insects. What is the probability that a randomly selected tree has both a disease and is being damaged by insects? *(See Example 3.)*

16. DRAWING CONCLUSIONS A company paid overtime wages or hired temporary help during 9 months of the year. Overtime wages were paid during 7 months, and temporary help was hired during 4 months. At the end of the year, an auditor examines the accounting records and randomly selects one month to check the payroll. What is the probability that the auditor will select a month in which the company paid overtime wages and hired temporary help?

17. DRAWING CONCLUSIONS A company is focus testing a new type of fruit drink. The focus group is 47% male. Of the responses, 40% of the males and 54% of the females said they would buy the fruit drink. What is the probability that a randomly selected person would buy the fruit drink? *(See Example 4.)*

18. DRAWING CONCLUSIONS The Redbirds trail the Bluebirds by one goal with 1 minute left in the hockey game. The Redbirds' coach must decide whether to remove the goalie and add a frontline player. The probabilities of each team scoring are shown in the table.

	Goalie	No goalie
Redbirds score	0.1	0.3
Bluebirds score	0.1	0.6

a. Find the probability that the Redbirds score and the Bluebirds do not score when the coach leaves the goalie in.

b. Find the probability that the Redbirds score and the Bluebirds do not score when the coach takes the goalie out.

c. Based on parts (a) and (b), what should the coach do?

19. PROBLEM SOLVING You can win concert tickets from a radio station if you are the first person to call when the song of the day is played, or if you are the first person to correctly answer the trivia question. The song of the day is announced at a random time between 7:00 and 7:30 A.M. The trivia question is asked at a random time between 7:15 and 7:45 A.M. You begin listening to the radio station at 7:20. Find the probability that you miss the announcement of the song of the day or the trivia question.

20. HOW DO YOU SEE IT?
Are events A and B disjoint events? Explain your reasoning.
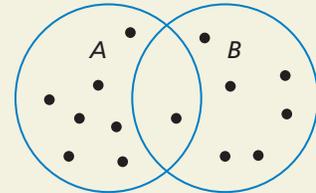
21. PROBLEM SOLVING You take a bus from your neighborhood to your school. The express bus arrives at your neighborhood at a random time between 7:30 and 7:36 A.M. The local bus arrives at your neighborhood at a random time between 7:30 and 7:40 A.M. You arrive at the bus stop at 7:33 A.M. Find the probability that you missed both the express bus and the local bus.

22. THOUGHT PROVOKING Write a general rule for finding $P(A \text{ or } B \text{ or } C)$ for (a) disjoint and (b) overlapping events A, B, and C.

23. MAKING AN ARGUMENT A bag contains 40 cards numbered 1 through 40 that are either red or blue. A card is drawn at random and placed back in the bag. This is done four times. Two red cards are drawn, numbered 31 and 19, and two blue cards are drawn, numbered 22 and 7. Your friend concludes that red cards and even numbers must be mutually exclusive. Is your friend correct? Explain.

Maintaining Mathematical Proficiency
Reviewing what you learned in previous grades and lessons

Graph the function. Compare the graph to the graph of $f(x) = x^2$. *(Section 3.1)*

24. $r(x) = 3x^2$

25. $g(x) = \frac{3}{4}x^2$

26. $h(x) = -5x^2$

5.5　Permutations and Combinations

Essential Question　How can a tree diagram help you visualize the number of ways in which two or more events can occur?

EXPLORATION 1　**Reading a Tree Diagram**

Work with a partner. Two coins are flipped and the spinner is spun. The tree diagram shows the possible outcomes.

Coin is flipped.

Coin is flipped.

Spinner is spun.
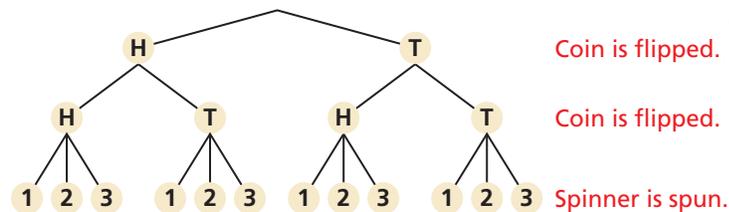
a. How many outcomes are possible?

b. List the possible outcomes.

EXPLORATION 2　**Reading a Tree Diagram**

Work with a partner. Consider the tree diagram below.
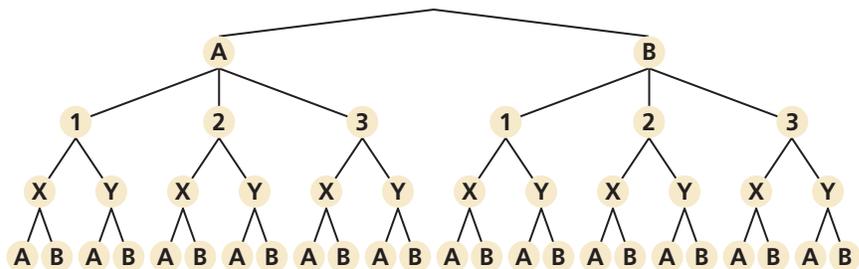
a. How many events are shown?　　**b.** What outcomes are possible for each event?

c. How many outcomes are possible?　**d.** List the possible outcomes.

EXPLORATION 3　**Writing a Conjecture**

Work with a partner.

a. Consider the following general problem: Event 1 can occur in m ways and event 2 can occur in n ways. Write a conjecture about the number of ways the two events can occur. Explain your reasoning.

b. Use the conjecture you wrote in part (a) to write a conjecture about the number of ways *more than* two events can occur. Explain your reasoning.

c. Use the results of Explorations 1(a) and 2(c) to verify your conjectures.

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to make conjectures and build a logical progression of statements to explore the truth of your conjectures.

Communicate Your Answer

4. How can a tree diagram help you visualize the number of ways in which two or more events can occur?

5. In Exploration 1, the spinner is spun a second time. How many outcomes are possible?

Core Vocabulary

permutation, *p. 312*
n factorial, *p. 312*
combination, *p. 314*

Previous
Fundamental Counting
  Principle

What You Will Learn

▶ Use the formula for the number of permutations.

▶ Use the formula for the number of combinations.

Permutations

A **permutation** is an arrangement of objects in which order is important. For instance, the 6 possible permutations of the letters A, B, and C are shown.

$$ABC \quad ACB \quad BAC \quad BCA \quad CAB \quad CBA$$

EXAMPLE 1   **Counting Permutations**

Consider the number of permutations of the letters in the word JULY. In how many ways can you arrange (a) all of the letters and (b) 2 of the letters?

SOLUTION

a. Use the Fundamental Counting Principle to find the number of permutations of the letters in the word JULY.

$$\text{Number of permutations} = \begin{pmatrix}\text{Choices for}\\\text{1st letter}\end{pmatrix}\begin{pmatrix}\text{Choices for}\\\text{2nd letter}\end{pmatrix}\begin{pmatrix}\text{Choices for}\\\text{3rd letter}\end{pmatrix}\begin{pmatrix}\text{Choices for}\\\text{4th letter}\end{pmatrix}$$

$$= 4 \cdot 3 \cdot 2 \cdot 1$$

$$= 24$$

▶ There are 24 ways you can arrange all of the letters in the word JULY.

b. When arranging 2 letters of the word JULY, you have 4 choices for the first letter and 3 choices for the second letter.

$$\text{Number of permutations} = \begin{pmatrix}\text{Choices for}\\\text{1st letter}\end{pmatrix}\begin{pmatrix}\text{Choices for}\\\text{2nd letter}\end{pmatrix}$$

$$= 4 \cdot 3$$

$$= 12$$

▶ There are 12 ways you can arrange 2 of the letters in the word JULY.

REMEMBER

Fundamental Counting Principle: If one event can occur in *m* ways and another event can occur in *n* ways, then the number of ways that both events can occur is *m* • *n*. The Fundamental Counting Principle can be extended to three or more events.

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1. In how many ways can you arrange the letters in the word HOUSE?

2. In how many ways can you arrange 3 of the letters in the word MARCH?

In Example 1(a), you evaluated the expression 4 • 3 • 2 • 1. This expression can be written as 4! and is read "4 *factorial*." For any positive integer *n*, the product of the integers from 1 to *n* is called ***n* factorial** and is written as

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1.$$

As a special case, the value of 0! is defined to be 1.

In Example 1(b), you found the permutations of 4 objects taken 2 at a time. You can find the number of permutations using the formulas on the next page.

Core Concept

Permutations

Formulas	Examples

The number of permutations of n objects is given by

$$_nP_n = n!.$$

The number of permutations of 4 objects is

$$_4P_4 = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24.$$

The number of permutations of n objects taken r at a time, where $r \leq n$, is given by

$$_nP_r = \frac{n!}{(n-r)!}.$$

The number of permutations of 4 objects taken 2 at a time is

$$_4P_2 = \frac{4!}{(4-2)!} = \frac{4 \cdot 3 \cdot 2!}{2!} = 12.$$

EXAMPLE 2 Using a Permutations Formula

Ten horses are running in a race. In how many different ways can the horses finish first, second, and third? (Assume there are no ties.)

SOLUTION

To find the number of permutations of 3 horses chosen from 10, find $_{10}P_3$.

$$_{10}P_3 = \frac{10!}{(10-3)!} \qquad \text{Permutations formula}$$

$$= \frac{10!}{7!} \qquad \text{Subtract.}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} \qquad \text{Expand factorial. Divide out common factor, } 7!.$$

$$= 720 \qquad \text{Simplify.}$$

▶ There are 720 ways for the horses to finish first, second, and third.

EXAMPLE 3 Finding a Probability Using Permutations

For a town parade, you will ride on a float with your soccer team. There are 12 floats in the parade, and their order is chosen at random. Find the probability that your float is first and the float with the school chorus is second.

SOLUTION

Step 1 Write the number of possible outcomes as the number of permutations of the 12 floats in the parade. This is $_{12}P_{12} = 12!$.

Step 2 Write the number of favorable outcomes as the number of permutations of the other floats, given that the soccer team is first and the chorus is second. This is $_{10}P_{10} = 10!$.

Step 3 Find the probability.

$$P(\text{soccer team is 1st, chorus is 2nd}) = \frac{10!}{12!} \qquad \text{Form a ratio of favorable to possible outcomes.}$$

$$= \frac{10!}{12 \cdot 11 \cdot 10!} \qquad \text{Expand factorial. Divide out common factor, } 10!.$$

$$= \frac{1}{132} \qquad \text{Simplify.}$$

3. WHAT IF? In Example 2, suppose there are 8 horses in the race. In how many different ways can the horses finish first, second, and third? (Assume there are no ties.)

4. WHAT IF? In Example 3, suppose there are 14 floats in the parade. Find the probability that the soccer team is first and the chorus is second.

Combinations

A **combination** is a selection of objects in which order is *not* important. For instance, in a drawing for 3 identical prizes, you would use combinations, because the order of the winners would not matter. If the prizes were different, then you would use permutations, because the order would matter.

EXAMPLE 4   **Counting Combinations**

Count the possible combinations of 2 letters chosen from the list A, B, C, D.

SOLUTION

List all of the permutations of 2 letters from the list A, B, C, D. Because order is not important in a combination, cross out any duplicate pairs.

AB	AC	AD	B̶A̶	BC	B̶D̶
C̶A̶	C̶B̶	CD	D̶A̶	D̶B̶	D̶C̶

> BD and DB are the same pair.

▶ There are 6 possible combinations of 2 letters from the list A, B, C, D.

5. Count the possible combinations of 3 letters chosen from the list A, B, C, D, E.

In Example 4, you found the number of combinations of objects by making an organized list. You can also find the number of combinations using the following formula.

USING A GRAPHING CALCULATOR

Most graphing calculators can calculate combinations.

```
4 nCr 2
              6
```
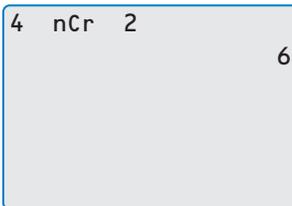
🔄 Core Concept

Combinations

Formula   The number of combinations of n objects taken r at a time, where $r \leq n$, is given by

$$_nC_r = \frac{n!}{(n-r)! \cdot r!}.$$

Example   The number of combinations of 4 objects taken 2 at a time is

$$_4C_2 = \frac{4!}{(4-2)! \cdot 2!} = \frac{4 \cdot 3 \cdot 2!}{2! \cdot (2 \cdot 1)} = 6.$$

Using the Combinations Formula

You order a sandwich at a restaurant. You can choose 2 side dishes from a list of 8. How many combinations of side dishes are possible?

SOLUTION

The order in which you choose the side dishes is not important. So, to find the number of combinations of 8 side dishes taken 2 at a time, find $_8C_2$.

Check

```
8  nCr  2
                  28
```

$$_8C_2 = \frac{8!}{(8-2)! \cdot 2!} \qquad \text{Combinations formula}$$

$$= \frac{8!}{6! \cdot 2!} \qquad \text{Subtract.}$$

$$= \frac{8 \cdot 7 \cdot \cancel{6!}}{\cancel{6!} \cdot (2 \cdot 1)} \qquad \text{Expand factorials. Divide out common factor, 6!.}$$

$$= 28 \qquad \text{Multiply.}$$

▶ There are 28 different combinations of side dishes you can order.

Finding a Probability Using Combinations

A yearbook editor has selected 14 photos, including one of you and one of your friend, to use in a collage for the yearbook. The photos are placed at random. There is room for 2 photos at the top of the page. What is the probability that your photo and your friend's photo are the 2 placed at the top of the page?

SOLUTION

Step 1 Write the number of possible outcomes as the number of combinations of 14 photos taken 2 at a time, or $_{14}C_2$, because the order in which the photos are chosen is not important.

$$_{14}C_2 = \frac{14!}{(14-2)! \cdot 2!} \qquad \text{Combinations formula}$$

$$= \frac{14!}{12! \cdot 2!} \qquad \text{Subtract.}$$

$$= \frac{14 \cdot 13 \cdot \cancel{12!}}{\cancel{12!} \cdot (2 \cdot 1)} \qquad \text{Expand factorials. Divide out common factor, 12!.}$$

$$= 91 \qquad \text{Multiply.}$$

Step 2 Find the number of favorable outcomes. Only one of the possible combinations includes your photo and your friend's photo.

Step 3 Find the probability.

$$P(\text{your photo and your friend's photos are chosen}) = \frac{1}{91}$$

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6. **WHAT IF?** In Example 5, suppose you can choose 3 side dishes out of the list of 8 side dishes. How many combinations are possible?

7. **WHAT IF?** In Example 6, suppose there are 20 photos in the collage. Find the probability that your photo and your friend's photo are the 2 placed at the top of the page.

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** An arrangement of objects in which order is important is called a(n) _____.

2. **WHICH ONE DOESN'T BELONG?** Which expression does *not* belong with the other three? Explain your reasoning.

$$\frac{7!}{2! \cdot 5!}$$

$$_7C_5$$

$$_7C_2$$

$$\frac{7!}{(7-2)!}$$

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, find the number of ways you can arrange (a) all of the letters and (b) 2 of the letters in the given word. *(See Example 1.)*

3. AT

4. TRY

5. ROCK

6. WATER

7. FAMILY

8. FLOWERS

In Exercises 9–16, evaluate the expression.

9. $_5P_2$

10. $_7P_3$

11. $_9P_1$

12. $_6P_5$

13. $_8P_6$

14. $_{12}P_0$

15. $_{30}P_2$

16. $_{25}P_5$

17. **PROBLEM SOLVING** Eleven students are competing in an art contest. In how many different ways can the students finish first, second, and third? *(See Example 2.)*

18. **PROBLEM SOLVING** Six friends go to a movie theater. In how many different ways can they sit together in a row of 6 empty seats?

19. **PROBLEM SOLVING** You and your friend are 2 of 8 servers working a shift in a restaurant. At the beginning of the shift, the manager randomly assigns one section to each server. Find the probability that you are assigned Section 1 and your friend is assigned Section 2. *(See Example 3.)*

20. **PROBLEM SOLVING** You make 6 posters to hold up at a basketball game. Each poster has a letter of the word TIGERS. You and 5 friends sit next to each other in a row. The posters are distributed at random. Find the probability that TIGERS is spelled correctly when you hold up the posters.

In Exercises 21–24, count the possible combinations of *r* letters chosen from the given list. *(See Example 4.)*

21. A, B, C, D; $r = 3$

22. L, M, N, O; $r = 2$

23. U, V, W, X, Y, Z; $r = 3$

24. D, E, F, G, H; $r = 4$

In Exercises 25–32, evaluate the expression.

25. $_5C_1$

26. $_8C_5$

27. $_9C_9$

28. $_8C_6$

29. $_{12}C_3$

30. $_{11}C_4$

31. $_{15}C_8$

32. $_{20}C_5$

33. **PROBLEM SOLVING** Each year, 64 golfers participate in a golf tournament. The golfers play in groups of 4. How many groups of 4 golfers are possible? *(See Example 5.)*

34. PROBLEM SOLVING You want to purchase vegetable dip for a party. A grocery store sells 7 different flavors of vegetable dip. You have enough money to purchase 2 flavors. How many combinations of 2 flavors of vegetable dip are possible?

ERROR ANALYSIS In Exercises 35 and 36, describe and correct the error in evaluating the expression.

35.

✗ $$_{11}P_7 = \frac{11!}{(11-7)} = \frac{11!}{4} = 9,979,200$$

36.

✗ $$_9C_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = 3024$$

REASONING In Exercises 37–40, tell whether the question can be answered using *permutations* or *combinations*. Explain your reasoning. Then answer the question.

37. To complete an exam, you must answer 8 questions from a list of 10 questions. In how many ways can you complete the exam?

38. Ten students are auditioning for 3 different roles in a play. In how many ways can the 3 roles be filled?

39. Fifty-two athletes are competing in a bicycle race. In how many orders can the bicyclists finish first, second, and third? (Assume there are no ties.)

40. An employee at a pet store needs to catch 5 tetras in an aquarium containing 27 tetras. In how many groupings can the employee capture 5 tetras?

41. CRITICAL THINKING Compare the quantities $_{50}C_9$ and $_{50}C_{41}$ without performing any calculations. Explain your reasoning.

42. CRITICAL THINKING Show that each identity is true for any whole numbers r and n, where $0 \le r \le n$.

 a. $_nC_n = 1$ 　　　　　**b.** $_nC_r = {}_nC_{n-r}$

 c. $_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$

43. REASONING Complete the table for each given value of r. Then write an inequality relating $_nP_r$ and $_nC_r$. Explain your reasoning.

	$r = 0$	$r = 1$	$r = 2$	$r = 3$
$_3P_r$				
$_3C_r$				

44. REASONING Write an equation that relates $_nP_r$ and $_nC_r$. Then use your equation to find and interpret the value of $\dfrac{_{182}P_4}{_{182}C_4}$.

45. PROBLEM SOLVING You and your friend are in the studio audience on a television game show. From an audience of 300 people, 2 people are randomly selected as contestants. What is the probability that you and your friend are chosen? *(See Example 6.)*

46. PROBLEM SOLVING You work 5 evenings each week at a bookstore. Your supervisor assigns you 5 evenings at random from the 7 possibilities. What is the probability that your schedule does not include working on the weekend?

REASONING In Exercises 47 and 48, find the probability of winning a lottery using the given rules. Assume that lottery numbers are selected at random.

47. You must correctly select 6 numbers, each an integer from 0 to 49. The order is not important.

48. You must correctly select 4 numbers, each an integer from 0 to 9. The order is important.

49. MATHEMATICAL CONNECTIONS
A polygon is convex when no line that contains a side of the polygon contains a point in the interior of the polygon. Consider a convex polygon with n sides.

 a. Use the combinations formula to write an expression for the number of diagonals in an n-sided polygon.

 b. Use your result from part (a) to write a formula for the number of diagonals of an n-sided convex polygon.

50. PROBLEM SOLVING You are ordering a burrito with 2 main ingredients and 3 toppings. The menu below shows the possible choices. How many different burritos are possible?

MAIN INGREDIENTS
chicken　black beans
steak　red beans
pork　rice

TOPPINGS
peppers　salsa
onions　pico de gallo
guacamole　extra cheese
jalapeños　black olives

51. PROBLEM SOLVING You want to purchase 2 different types of contemporary music CDs and 1 classical music CD from the music collection shown. How many different sets of music types can you choose for your purchase?

Contemporary	Classical
Blues	Opera
Country	Concerto
Jazz	Symphony
Rap	
Rock & Roll	

52. HOW DO YOU SEE IT? A bag contains one green marble, one red marble, and one blue marble. The diagram shows the possible outcomes of randomly drawing three marbles from the bag without replacement.

1st Draw 2nd Draw 3rd Draw
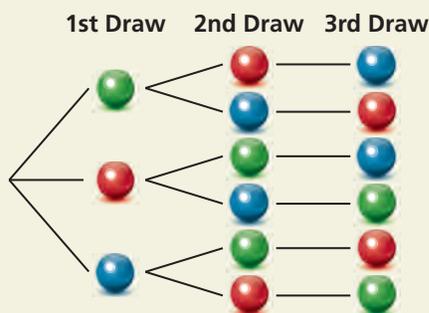
a. How many combinations of three marbles can be drawn from the bag? Explain.

b. How many permutations of three marbles can be drawn from the bag? Explain.

53. PROBLEM SOLVING Every student in your history class is required to present a project in front of the class. Each day, 4 students make their presentations in an order chosen at random by the teacher. You make your presentation on the first day.

a. What is the probability that you are chosen to be the first or second presenter on the first day?

b. What is the probability that you are chosen to be the second or third presenter on the first day? Compare your answer with that in part (a).

54. PROBLEM SOLVING The organizer of a cast party for a drama club asks each of the 6 cast members to bring 1 food item from a list of 10 items. Assuming each member randomly chooses a food item to bring, what is the probability that at least 2 of the 6 cast members bring the same item?

55. PROBLEM SOLVING You are one of 10 students performing in a school talent show. The order of the performances is determined at random. The first 5 performers go on stage before the intermission.

a. What is the probability that you are the last performer before the intermission and your rival performs immediately before you?

b. What is the probability that you are *not* the first performer?

56. THOUGHT PROVOKING How many integers, greater than 999 but not greater than 4000, can be formed with the digits 0, 1, 2, 3, and 4? Repetition of digits is allowed.

57. PROBLEM SOLVING There are 30 students in your class. Your science teacher chooses 5 students at random to complete a group project. Find the probability that you and your 2 best friends in the science class are chosen to work in the group. Explain how you found your answer.

58. PROBLEM SOLVING Follow the steps below to explore a famous probability problem called the *birthday problem*. (Assume there are 365 equally likely birthdays possible.)

a. What is the probability that at least 2 people share the same birthday in a group of 6 randomly chosen people? in a group of 10 randomly chosen people?

b. Generalize the results from part (a) by writing a formula for the probability $P(n)$ that at least 2 people in a group of n people share the same birthday. (*Hint*: Use $_nP_r$ notation in your formula.)

c. Enter the formula from part (b) into a graphing calculator. Use the *table* feature to make a table of values. For what group size does the probability that at least 2 people share the same birthday first exceed 50%?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

59. A bag contains 12 white marbles and 3 black marbles. You pick 1 marble at random. What is the probability that you pick a black marble? *(Section 5.1)*

60. The table shows the results of flipping two coins 12 times. For what outcome is the experimental probability the same as the theoretical probability? *(Section 5.1)*

HH	HT	TH	TT
2	6	3	1

5.6 Binomial Distributions

Essential Question How can you determine the frequency of each outcome of an event?
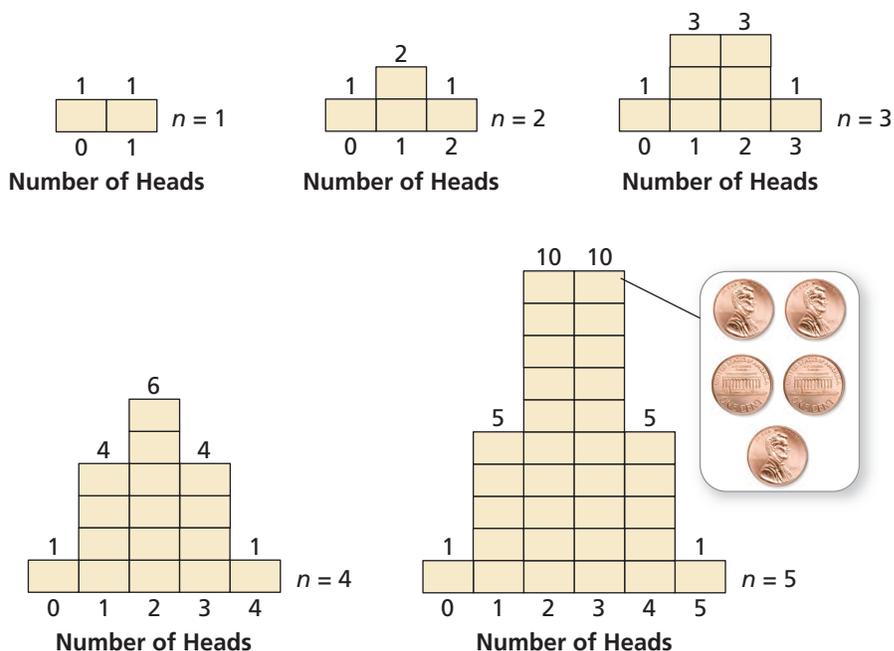
EXPLORATION 1 Analyzing Histograms

Work with a partner. The histograms show the results when n coins are flipped.

STUDY TIP

When 4 coins are flipped ($n = 4$), the possible outcomes are

TTTT TTTH TTHT TTHH

THTT THTH THHT THHH

HTTT HTTH HTHT HTHH

HHTT HHTH HHHT HHHH.

The histogram shows the numbers of outcomes having 0, 1, 2, 3, and 4 heads.

a. In how many ways can 3 heads occur when 5 coins are flipped?

b. Draw a histogram that shows the numbers of heads that can occur when 6 coins are flipped.

c. In how many ways can 3 heads occur when 6 coins are flipped?

EXPLORATION 2 Determining the Number of Occurrences

Work with a partner.

LOOKING FOR A PATTERN

To be proficient in math, you need to look closely to discern a pattern or structure.

a. Complete the table showing the numbers of ways in which 2 heads can occur when n coins are flipped.

n	3	4	5	6	7
Occurrences of 2 heads					

b. Determine the pattern shown in the table. Use your result to find the number of ways in which 2 heads can occur when 8 coins are flipped.

Communicate Your Answer

3. How can you determine the frequency of each outcome of an event?

4. How can you use a histogram to find the probability of an event?

What You Will Learn

▶ Construct and interpret probability distributions.

▶ Construct and interpret binomial distributions.

Probability Distributions

A **random variable** is a variable whose value is determined by the outcomes of a probability experiment. For example, when you roll a six-sided die, you can define a random variable x that represents the number showing on the die. So, the possible values of x are 1, 2, 3, 4, 5, and 6. For every random variable, a *probability distribution* can be defined.

🄖 Core Concept

Probability Distributions

A **probability distribution** is a function that gives the probability of each possible value of a random variable. The sum of all the probabilities in a probability distribution must equal 1.

Probability Distribution for Rolling a Six-Sided Die						
x	1	2	3	4	5	6
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

EXAMPLE 1 **Constructing a Probability Distribution**

Let x be a random variable that represents the sum when two six-sided dice are rolled. Make a table and draw a histogram showing the probability distribution for x.

SOLUTION

Step 1 Make a table. The possible values of x are the integers from 2 to 12. The table shows how many outcomes of rolling two dice produce each value of x. Divide the number of outcomes for x by 36 to find $P(x)$.

STUDY TIP

Recall that there are 36 possible outcomes when rolling two six-sided dice. These are listed in Example 3 on page 282.

x (sum)	2	3	4	5	6	7	8	9	10	11	12
Outcomes	1	2	3	4	5	6	5	4	3	2	1
$P(x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

Step 2 Draw a histogram where the intervals are given by x and the frequencies are given by $P(x)$.

EXAMPLE 2   **Interpreting a Probability Distribution**

Use the probability distribution in Example 1 to answer each question.

a. What is the most likely sum when rolling two six-sided dice?

b. What is the probability that the sum of the two dice is at least 10?

SOLUTION

a. The most likely sum when rolling two six-sided dice is the value of x for which $P(x)$ is greatest. This probability is greatest for $x = 7$. So, when rolling the two dice, the most likely sum is 7.

b. The probability that the sum of the two dice is at least 10 is

$$P(x \geq 10) = P(x = 10) + P(x = 11) + P(x = 12)$$

$$= \tfrac{3}{36} + \tfrac{2}{36} + \tfrac{1}{36}$$

$$= \tfrac{6}{36}$$

$$= \tfrac{1}{6}$$

$$\approx 0.167.$$

▶   The probability is about 16.7%.

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An octahedral die has eight sides numbered 1 through 8. Let x be a random variable that represents the sum when two such dice are rolled.

1. Make a table and draw a histogram showing the probability distribution for x.

2. What is the most likely sum when rolling the two dice?

3. What is the probability that the sum of the two dice is at most 3?

Binomial Distributions

One type of probability distribution is a **binomial distribution**. A binomial distribution shows the probabilities of the outcomes of a *binomial experiment*.

🔄 Core Concept

Binomial Experiments

A **binomial experiment** meets the following conditions.

- There are n independent trials.

- Each trial has only two possible outcomes: success and failure.

- The probability of success is the same for each trial. This probability is denoted by p. The probability of failure is $1 - p$.

For a binomial experiment, the probability of exactly k successes in n trials is

$$P(k \text{ successes}) = {}_nC_k p^k (1 - p)^{n - k}.$$

EXAMPLE 3   Constructing a Binomial Distribution

According to a survey, about 33% of people ages 16 and older in the U.S. own an electronic book reading device, or e-reader. You ask 6 randomly chosen people (ages 16 and older) whether they own an e-reader. Draw a histogram of the binomial distribution for your survey.

SOLUTION

The probability that a randomly selected person has an e-reader is $p = 0.33$. Because you survey 6 people, $n = 6$.

$P(k = 0) = {_6}C_0(0.33)^0(0.67)^6 \approx 0.090$

$P(k = 1) = {_6}C_1(0.33)^1(0.67)^5 \approx 0.267$

$P(k = 2) = {_6}C_2(0.33)^2(0.67)^4 \approx 0.329$

$P(k = 3) = {_6}C_3(0.33)^3(0.67)^3 \approx 0.216$

$P(k = 4) = {_6}C_4(0.33)^4(0.67)^2 \approx 0.080$

$P(k = 5) = {_6}C_5(0.33)^5(0.67)^1 \approx 0.016$

$P(k = 6) = {_6}C_6(0.33)^6(0.67)^0 \approx 0.001$

Binomial Distribution for Your Survey

A histogram of the distribution is shown.
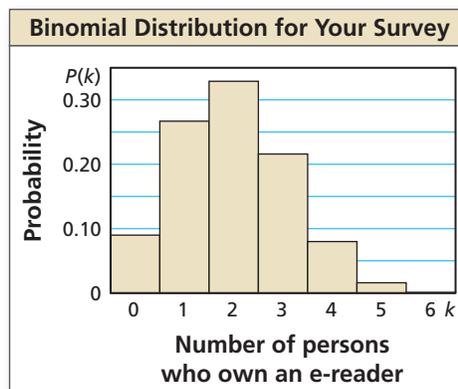
EXAMPLE 4   Interpreting a Binomial Distribution

Use the binomial distribution in Example 3 to answer each question.

a. What is the most likely outcome of the survey?

b. What is the probability that at most 2 people have an e-reader?

SOLUTION

a. The most likely outcome of the survey is the value of k for which $P(k)$ is greatest. This probability is greatest for $k = 2$. The most likely outcome is that 2 of the 6 people own an e-reader.

b. The probability that at most 2 people have an e-reader is

$P(k \le 2) = P(k = 0) + P(k = 1) + P(k = 2)$

$\approx 0.090 + 0.267 + 0.329$

$\approx 0.686.$

▶ The probability is about 68.6%.

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According to a survey, about 85% of people ages 18 and older in the U.S. use the Internet or e-mail. You ask 4 randomly chosen people (ages 18 and older) whether they use the Internet or e-mail.

4. Draw a histogram of the binomial distribution for your survey.

5. What is the most likely outcome of your survey?

6. What is the probability that at most 2 people you survey use the Internet or e-mail?

Vocabulary and Core Concept Check

1. **VOCABULARY** What is a random variable?

2. **WRITING** Give an example of a binomial experiment and describe how it meets the conditions of a binomial experiment.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, make a table and draw a histogram showing the probability distribution for the random variable. *(See Example 1.)*

3. $x =$ the number on a table tennis ball randomly chosen from a bag that contains 5 balls labeled "1," 3 balls labeled "2," and 2 balls labeled "3."

4. $c = 1$ when a randomly chosen card out of a standard deck of 52 playing cards is a heart and $c = 2$ otherwise.

5. $w = 1$ when a randomly chosen letter from the English alphabet is a vowel and $w = 2$ otherwise.

6. $n =$ the number of digits in a random integer from 0 through 999.

In Exercises 7 and 8, use the probability distribution to determine (a) the number that is most likely to be spun on a spinner, and (b) the probability of spinning an even number. *(See Example 2.)*

7.

Spinner Results

Number on spinner

8.

Spinner Results

Number on spinner
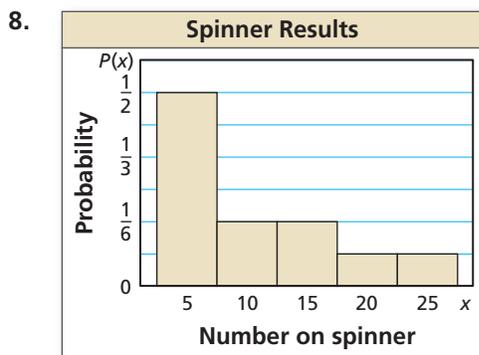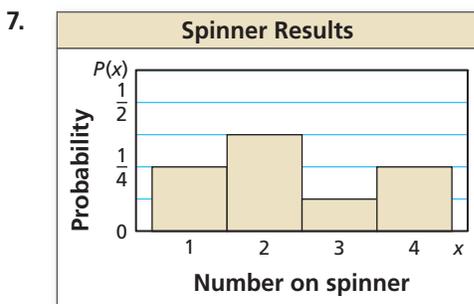
USING EQUATIONS In Exercises 9–12, calculate the probability of flipping a coin 20 times and getting the given number of heads.

9. 1

10. 4

11. 18

12. 20

13. **MODELING WITH MATHEMATICS** According to a survey, 27% of high school students in the United States buy a class ring. You ask 6 randomly chosen high school students whether they own a class ring. *(See Examples 3 and 4.)*

   a. Draw a histogram of the binomial distribution for your survey.

   b. What is the most likely outcome of your survey?

   c. What is the probability that at most 2 people have a class ring?

14. **MODELING WITH MATHEMATICS** According to a survey, 48% of adults in the United States believe that Unidentified Flying Objects (UFOs) are observing our planet. You ask 8 randomly chosen adults whether they believe UFOs are watching Earth.

   a. Draw a histogram of the binomial distribution for your survey.

   b. What is the most likely outcome of your survey?

   c. What is the probability that at most 3 people believe UFOs are watching Earth?

ERROR ANALYSIS In Exercises 15 and 16, describe and correct the error in calculating the probability of rolling a 1 exactly 3 times in 5 rolls of a six-sided die.
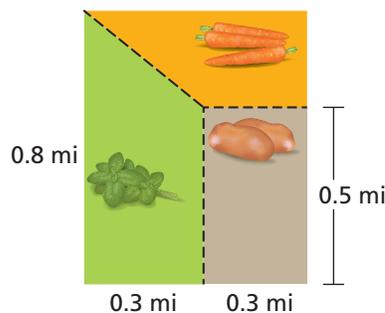
15.

✗ $P(k = 3) = {}_5C_3\left(\frac{1}{6}\right)^{5-3}\left(\frac{5}{6}\right)^3$

≈ 0.161

16.

✗ $P(k = 3) = \left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{5-3}$

≈ 0.003

17. MATHEMATICAL CONNECTIONS At most 7 gopher holes appear each week on the farm shown. Let x represent how many of the gopher holes appear in the carrot patch. Assume that a gopher hole has an equal chance of appearing at any point on the farm.

0.8 mi

0.5 mi

0.3 mi   0.3 mi

a. Find $P(x)$ for $x = 0, 1, 2, \ldots, 7$. (*Note*: The probability of a gopher hole appearing in the carrot patch is equal to the ratio of the area of the carrot patch to the area of the entire farm.)

b. Make a table showing the probability distribution for x.

c. Make a histogram showing the probability distribution for x.

18. HOW DO YOU SEE IT? Complete the probability distribution for the random variable x. What is the probability the value of x is greater than 2?

x	1	2	3	4
$P(x)$	0.1	0.3	0.4	

19. MAKING AN ARGUMENT The binomial distribution shows the results of a binomial experiment. Your friend claims that the probability p of a success must be greater than the probability $1 - p$ of a failure. Is your friend correct? Explain your reasoning.

Experiment Results
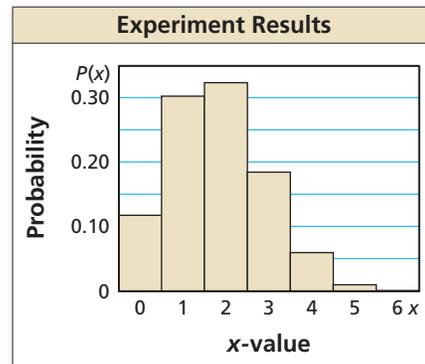
20. THOUGHT PROVOKING There are 100 coins in a bag. Only one of them has a date of 2010. You choose a coin at random, check the date, and then put the coin back in the bag. You repeat this 100 times. Are you certain of choosing the 2010 coin at least once? Explain your reasoning.

21. MODELING WITH MATHEMATICS Assume that having a male and having a female child are independent events, and that the probability of each is 0.5.

a. A couple has 4 male children. Evaluate the validity of this statement: "The first 4 kids were all boys, so the next one will probably be a girl."

b. What is the probability of having 4 male children and then a female child?

c. Let x be a random variable that represents the number of children a couple already has when they have their first female child. Draw a histogram of the distribution of $P(x)$ for $0 \le x \le 10$. Describe the shape of the histogram.

22. CRITICAL THINKING An entertainment system has n speakers. Each speaker will function properly with probability p, independent of whether the other speakers are functioning. The system will operate effectively when at least 50% of its speakers are functioning. For what values of p is a 5-speaker system more likely to operate than a 3-speaker system?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

List the possible outcomes for the situation. *(Section 5.1)*

23. guessing the gender of three children

24. picking one of two doors and one of three curtains

5.4–5.6   What Did You Learn?

Core Vocabulary

Core Concepts

Section 5.4

Section 5.5

Section 5.6

Mathematical Practices

1. How can you use diagrams to understand the situation in Exercise 22 on page 310?

2. Describe a relationship between the results in part (a) and part (b) in Exercise 52 on page 318.

3. Explain how you were able to break the situation into cases to evaluate the validity of the statement in part (a) of Exercise 21 on page 324.

Performance Task:

Risk Analysis

An arborist performs an extensive risk analysis to help identify and treat tree diseases. How can this process be applied to other things?

To explore the answer to this question and more, check out the Performance Task and Real-Life STEM video at *BigIdeasMath.com*.

5.1 Sample Spaces and Probability *(pp. 279–286)*

Each section of the spinner shown has the same area. The spinner was spun 30 times. The table shows the results. For which color is the experimental probability of stopping on the color the same as the theoretical probability?

Spinner Results	
green	4
orange	6
red	9
blue	8
yellow	3

The theoretical probability of stopping on each of the five colors is $\frac{1}{5}$. Use the outcomes in the table to find the experimental probabilities.

$P(\text{green}) = \frac{4}{30} = \frac{2}{15}$ $P(\text{orange}) = \frac{6}{30} = \frac{1}{5}$ $P(\text{red}) = \frac{9}{30} = \frac{3}{10}$ $P(\text{blue}) = \frac{8}{30} = \frac{4}{15}$ $P(\text{yellow}) = \frac{3}{30} = \frac{1}{10}$

▶ The experimental probability of stopping on orange is the same as the theoretical probability.

1. You record a person's gender and whether they respond *yes*, *no*, or *maybe* to a survey question. How many possible outcomes are in the sample space? List the possible outcomes.

2. A bag contains 9 tiles, one for each letter in the word HAPPINESS. You choose a tile at random. What is the probability that you choose a tile with the letter S? What is the probability that you choose a tile with a letter other than P?

3. Using the spinner above, what are the odds in favor of stopping on yellow? What are the odds against stopping on blue?

5.2 Independent and Dependent Events *(pp. 287–294)*

You randomly select 2 cards from a standard deck of 52 playing cards. What is the probability that both cards are jacks when (a) you replace the first card before selecting the second, and (b) you do not replace the first card. Compare the probabilities.

Let event *A* be "first card is a jack" and event *B* be "second card is a jack."

a. Because you replace the first card before you select the second card, the events are independent. So, the probability is

$$P(A \text{ and } B) = P(A) \cdot P(B) = \frac{4}{52} \cdot \frac{4}{52} = \frac{16}{2704} = \frac{1}{169} \approx 0.006.$$

b. Because you do not replace the first card before you select the second card, the events are dependent. So, the probability is

$$P(A \text{ and } B) = P(A) \cdot P(B|A) = \frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221} \approx 0.005.$$

▶ So, you are $\frac{1}{169} \div \frac{1}{221} \approx 1.3$ times more likely to select 2 jacks when you replace the first card before you select the second card.

Find the probability of randomly selecting the given marbles from a bag of 5 red, 8 green, and 3 blue marbles when (a) you replace the first marble before drawing the second, and (b) you do not replace the first marble. Compare the probabilities.

4. red, then green   5. blue, then red   6. green, then green

5.3 Two-Way Tables and Probability *(pp. 295–302)*

A survey asks residents of the east and west sides of a city whether they support the construction of a bridge. The results, given as joint relative frequencies, are shown in the two-way table. What is the probability that a randomly selected resident from the east side will support the project?

		Location	
		East Side	**West Side**
Response	**Yes**	0.47	0.36
	No	0.08	0.09

Find the joint and marginal relative frequencies. Then use these values to find the conditional probability.

$$P(\text{yes}\,|\,\text{east side}) = \frac{P(\text{east side and yes})}{P(\text{east side})} = \frac{0.47}{0.47 + 0.08} \approx 0.855$$

▶ So, the probability that a resident of the east side of the city will support the project is about 85.5%.

7. What is the probability that a randomly selected resident who does not support the project in the example above is from the west side?

8. After a conference, 220 men and 270 women respond to a survey. Of those, 200 men and 230 women say the conference was impactful. Organize these results in a two-way table. Then find and interpret the marginal frequencies.

5.4 Probability of Disjoint and Overlapping Events *(pp. 305–310)*

Let A and B be events such that $P(A) = \frac{2}{3}$, $P(B) = \frac{1}{2}$, and $P(A \text{ and } B) = \frac{1}{3}$. Find $P(A \text{ or } B)$.

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ Write general formula.

$\qquad\qquad = \frac{2}{3} + \frac{1}{2} - \frac{1}{3}$ Substitute known probabilities.

$\qquad\qquad = \frac{5}{6}$ Simplify.

$\qquad\qquad \approx 0.833$ Use a calculator.

9. Let A and B be events such that $P(A) = 0.32$, $P(B) = 0.48$, and $P(A \text{ and } B) = 0.12$. Find $P(A \text{ or } B)$.

10. Out of 100 employees at a company, 92 employees either work part time or work 5 days each week. There are 14 employees who work part time and 80 employees who work 5 days each week. What is the probability that a randomly selected employee works both part time and 5 days each week?

Permutations and Combinations *(pp. 311–318)*

A 5-digit code consists of 5 different integers from 0 to 9. How many different codes are possible?

To find the number of permutations of 5 integers chosen from 10, find $_{10}P_5$.

$$_{10}P_5 = \frac{10!}{(10-5)!}$$ Permutations formula

$$= \frac{10!}{5!}$$ Subtract.

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}}$$ Expand factorials. Divide out common factor, 5!.

$$= 30{,}240$$ Simplify.

▶ There are 30,240 possible codes.

Evaluate the expression.

11. $_7P_6$ **12.** $_{13}P_{10}$ **13.** $_6C_2$ **14.** $_8C_4$

15. In how many ways can you arrange (a) all of the letters and (b) 3 of the letters in the word UNCLE?

16. A random drawing will determine which 3 people in a group of 9 will win concert tickets. What is the probability that you and your 2 friends will win the tickets?

Binomial Distributions *(pp. 319–324)*

According to a survey, about 21% of adults in the U.S. visited an art museum last year. You ask 4 randomly chosen adults whether they visited an art museum last year. Draw a histogram of the binomial distribution for your survey.

The probability that a randomly selected person visited an art museum is $p = 0.21$. Because you survey 4 people, $n = 4$.
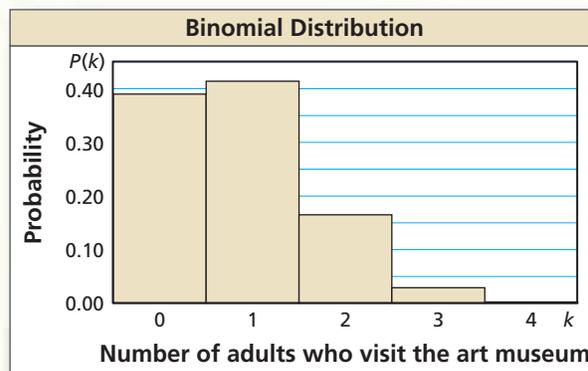
$$P(k = 0) = {_4}C_0(0.21)^0(0.79)^4 \approx 0.390$$

$$P(k = 1) = {_4}C_1(0.21)^1(0.79)^3 \approx 0.414$$

$$P(k = 2) = {_4}C_2(0.21)^2(0.79)^2 \approx 0.165$$

$$P(k = 3) = {_4}C_3(0.21)^3(0.79)^1 \approx 0.029$$

$$P(k = 4) = {_4}C_4(0.21)^4(0.79)^0 \approx 0.002$$

Binomial Distribution

(histogram: $P(k)$ vs. Number of adults who visit the art museum; bars at $k = 0$ through 4 with probabilities approximately 0.39, 0.41, 0.17, 0.03, 0.00; y-axis labeled Probability from 0.00 to 0.40)

17. Find the probability of flipping a coin 12 times and getting exactly 4 heads.

18. A basketball player makes a free throw 82.6% of the time. The player attempts 5 free throws. Draw a histogram of the binomial distribution of the number of successful free throws. What is the most likely outcome?

5 Chapter Test

You roll a six-sided die. Find the probability of the event described. Explain your reasoning.

1. You roll a number less than 5.

2. You roll a multiple of 3.

Evaluate the expression.

3. $_7P_2$

4. $_8P_3$

5. $_6C_3$

6. $_{12}C_7$

7. The odds in favor of an event are $2:9$. What are the odds against the event?

8. You find the probability $P(A$ or $B)$ by using the equation $P(A$ or $B) = P(A) + P(B) - P(A$ and $B)$. Describe why it is necessary to subtract $P(A$ and $B)$ when the events A and B are overlapping. Then describe why it is *not* necessary to subtract $P(A$ and $B)$ when the events A and B are disjoint.

9. Is it possible to use the formula $P(A$ and $B) = P(A) \cdot P(B|A)$ when events A and B are independent? Explain your reasoning.

10. According to a survey, about 58% of families sit down for a family dinner at least four times per week. You ask 5 randomly chosen families whether they have a family dinner at least four times per week.

    a. Draw a histogram of the binomial distribution for the survey.

    b. What is the most likely outcome of the survey?

    c. What is the probability that at least 3 families have a family dinner four times per week?

11. You are choosing a cell phone company to sign with for the next 2 years. The three plans you consider are equally priced. You ask several of your neighbors whether they are satisfied with their current cell phone company. The table shows the results. According to this survey, which company should you choose?

	Satisfied	Not Satisfied
Company A	IIII	II
Company B	IIII	III
Company C	JHT I	JHT

12. Out of 10,500 people that attend a math conference, 7315 people are either college professors or female. There are 4680 college professors and 5140 females. What is the probability that a randomly selected person is both a college professor and female?

13. Consider a bag that contains all the chess pieces in a set, as shown in the diagram.

	King	Queen	Bishop	Rook	Knight	Pawn
Black	1	1	2	2	2	8
White	1	1	2	2	2	8

    a. You choose one piece at random. Find the probability that you choose a black piece or a queen.

    b. You choose one piece at random, do not replace it, then choose a second piece at random. Find the probability that you choose a king, then a pawn.

14. Three volunteers are chosen at random from a group of 12 to help at a summer camp.

    a. What is the probability that you, your brother, and your friend are chosen?

    b. The first person chosen will be a counselor, the second will be a lifeguard, and the third will be a cook. What is the probability that you are the cook, your brother is the lifeguard, and your friend is the counselor?

1. According to a survey, 63% of Americans consider themselves sports fans. You randomly select 14 Americans to survey.

   a. Draw a histogram of the binomial distribution of your survey.

   b. What is the most likely number of Americans who consider themselves sports fans?

   c. What is the probability at least 7 Americans consider themselves sports fans?

2. Place each function into one of the three categories.

No zeros	One zero	Two zeros

$f(x) = 3x^2 + 4x + 2$
$f(x) = -x^2 + 2x$

$f(x) = 4x^2 - 8x + 4$
$f(x) = x^2 - 3x - 21$

$f(x) = 7x^2$
$f(x) = -6x^2 - 5$

3. You order a fruit smoothie made with 2 liquid ingredients and 3 fruit ingredients from the menu shown. How many different fruit smoothies can you order?

Liquids
- Water   • Tea
- Coconut Water
- Almond Milk
- Apple Juice
- Orange Juice

Fruits
- Orange   • Watermelon
- Banana   • Kiwi
- Pineapple   • Peach
- Cantaloupe • Blueberry
- Strawberry • Pomegranate

4. Select all the numbers that are in the range of the function shown.

$$y = \begin{cases} x^2 + 4x + 7, & \text{if } x \leq -1 \\ \frac{1}{2}x + 2, & \text{if } x > -1 \end{cases}$$
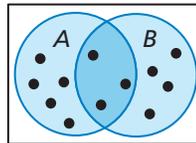
| 0 | $\frac{1}{2}$ | 1 | $1\frac{1}{2}$ | 2 | $2\frac{1}{2}$ | 3 | $3\frac{1}{2}$ | 4 |

5. Complete the equation so that the solutions of the system of equations are $(-2, 4)$ and $(1, -5)$.

$$y = \boxed{} \, x + \boxed{}$$
$$y = 2x^2 - x - 6$$

6. Use the diagram to explain why the equation is true.

$$P(A) + P(B) = P(A \text{ or } B) + P(A \text{ and } B)$$

7. Which polynomial represents the product of $2x - 4$ and $x^2 + 6x - 2$?

(**A**) $2x^3 + 8x^2 - 4x + 8$

(**B**) $2x^3 + 8x^2 - 28x + 8$

(**C**) $2x^3 + 8$

(**D**) $2x^3 - 24x - 2$

8. A survey asked male and female students about whether they prefer to take gym class or choir. The table shows the results of the survey.

		Class		
		Gym	**Choir**	**Total**
Gender	**Male**			50
	Female	23		
	Total		49	106

    a. Complete the two-way table.

    b. What is the probability that a randomly selected student is female and prefers choir?

    c. What is the probability that a randomly selected male student prefers gym class?

9. The owner of a lawn-mowing business has three mowers. As long as one of the mowers is working, the owner can stay productive. One of the mowers is unusable 10% of the time, one is unusable 8% of the time, and one is unusable 18% of the time.

    a. Find the probability that all three mowers are unusable on a given day.

    b. Find the probability that at least one of the mowers is unusable on a given day.

    c. Suppose the least-reliable mower stops working completely. How does this affect the probability that the lawn-mowing business can be productive on a given day?
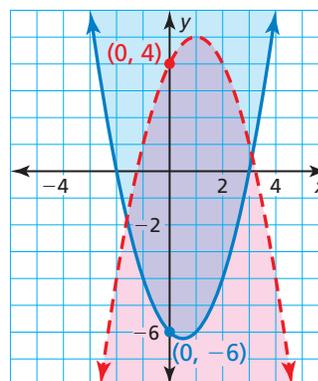
10. Write a system of quadratic inequalities whose solution is represented in the graph.