# 4.7 Solving Quadratic Equations with Complex Solutions

**Essential Question** How can you determine whether a quadratic equation has real solutions or imaginary solutions?

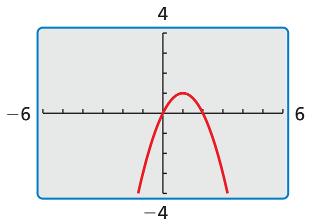
## EXPLORATION 1 Using Graphs to Solve Quadratic Equations

**Work with a partner.** Use the discriminant of $f(x) = 0$ and the sign of the leading coefficient of $f(x)$ to match each quadratic function with its graph. Explain your reasoning. Then find the real solution(s) (if any) of each quadratic equation $f(x) = 0$.

**a.** $f(x) = x^2 - 2x$ **b.** $f(x) = x^2 - 2x + 1$ **c.** $f(x) = x^2 - 2x + 2$

**d.** $f(x) = -x^2 + 2x$ **e.** $f(x) = -x^2 + 2x - 1$ **f.** $f(x) = -x^2 + 2x - 2$

**A.**

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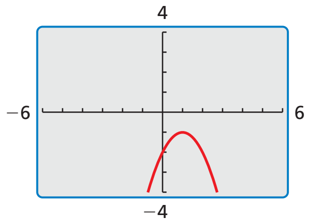


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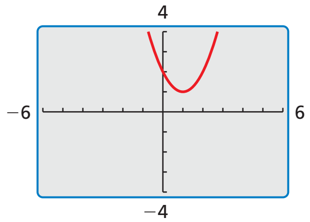


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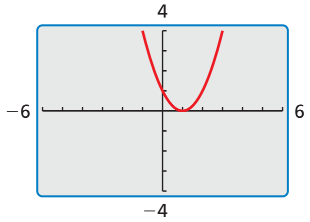
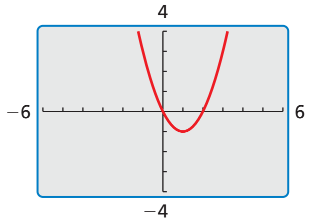
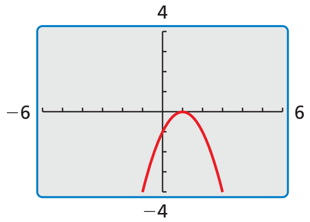



**E.**




**F.**




## EXPLORATION 2 Finding Imaginary Solutions

**Work with a partner.** What do you know about the discriminants of quadratic equations that have no real solutions? Use the Quadratic Formula and what you learned about the imaginary unit $i$ to find the *imaginary* solutions of each equation in Exploration 1 that has no real solutions. Use substitution to check your answers.

### MAKING SENSE OF PROBLEMS

To be proficient in math, you need to make conjectures about the form and meaning of solutions.

## Communicate Your Answer

**3.** How can you determine whether a quadratic equation has real solutions or imaginary solutions?

**4.** Describe the number and type of solutions of $x^2 + 2x + 3 = 0$. How do you know? What are the solutions?

# 4.7 Lesson

## What You Will Learn

- Solve quadratic equations and find zeros of quadratic functions.
- Use the discriminant.

## Finding Solutions and Zeros

Previously, you learned that you can use the discriminant of a quadratic equation to determine whether the equation has two real solutions, one real solution, or no real solutions. When the discriminant is negative, you can use the imaginary unit $i$ to write two *imaginary* solutions of the equation. So, all quadratic equations have complex number solutions.

You have solved quadratic equations with real solutions. Now you will solve quadratic equations with *imaginary* solutions.

**STUDY TIP**

In general, every polynomial equation has complex number solutions. This is implied by the *Fundamental Theorem of Algebra*. You will learn more about this theorem in a future course.

### EXAMPLE 1 Solving Quadratic Equations

Solve each equation.

**a.** $x^2 + 9 = 0$ **b.** $x^2 + 4x + 5 = 0$ **c.** $5x^2 - 4x + 1 = 0$

**SOLUTION**

**a.** The equation does not have an $x$-term. So, solve using square roots.

$x^2 + 9 = 0$ Write the equation.

$x^2 = -9$ Subtract 9 from each side.

$x = \pm\sqrt{-9}$ Take the square root of each side.

$x = \pm 3i$ Write in terms of $i$.

**LOOKING FOR STRUCTURE**

You can use the pattern $(a + bi)(a - bi) = a^2 + b^2$ to rewrite $x^2 + 9 = 0$ as $(x + 3i)(x - 3i) = 0$. So, $x = \pm 3i$.

**b.** The coefficient of the $x^2$-term is 1, and the coefficient of the $x$-term is an even number. So, solve by completing the square.

$$x^2 + 4x + 5 = 0$$
$$x^2 + 4x = -5$$
$$x^2 + 4x + 4 = -5 + 4$$
$$(x + 2)^2 = -1$$
$$x + 2 = \pm\sqrt{-1}$$
$$x = -2 \pm \sqrt{-1}$$
$$x = -2 \pm i$$

**Check** You can check imaginary solutions algebraically. The check for one of the imaginary solutions, $-2 + i$, is shown.

$$(-2 + i)^2 + 4(-2 + i) + 5 \stackrel{?}{=} 0$$
$$3 - 4i - 8 + 4i + 5 \stackrel{?}{=} 0$$
$$0 = 0 \checkmark$$

**c.** The equation is not factorable, and completing the square would result in fractions. So, solve using the Quadratic Formula.

$x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(5)(1)}}{2(5)}$ Substitute 5 for $a$, $-4$ for $b$, and 1 for $c$.

$x = \dfrac{4 \pm \sqrt{-4}}{10}$ Simplify.

$x = \dfrac{4 \pm 2i}{10}$ Write in terms of $i$.

$x = \dfrac{2 \pm i}{5}$ Simplify.

## EXAMPLE 2 Finding Zeros of a Quadratic Function

Find the zeros of $f(x) = 4x^2 + 20$.

### SOLUTION

| | |
|---|---|
| $4x^2 + 20 = 0$ | Set $f(x)$ equal to 0. |
| $4x^2 = -20$ | Subtract 20 from each side. |
| $x^2 = -5$ | Divide each side by 4. |
| $x = \pm\sqrt{-5}$ | Take the square root of each side. |
| $x = \pm i\sqrt{5}$ | Write in terms of $i$. |

▶ So, the zeros of $f$ are $i\sqrt{5}$ and $-i\sqrt{5}$.

**Check**

$$f(i\sqrt{5}) = 4(i\sqrt{5})^2 + 20 = 4(-5) + 20 = 0 \checkmark$$

$$f(-i\sqrt{5}) = 4(-i\sqrt{5})^2 + 20 = 4(-5) + 20 = 0 \checkmark$$

### Monitoring Progress

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**Solve the equation using any method. Explain your choice of method.**

**1.** $-x^2 - 25 = 0$ **2.** $x^2 - 4x + 8 = 0$ **3.** $8x^2 + 5 = 12x$

**Find the zeros of the function.**

**4.** $f(x) = -2x^2 - 18$ **5.** $f(x) = 9x^2 + 1$ **6.** $f(x) = x^2 - 6x + 10$

## Using the Discriminant

## EXAMPLE 3 Writing an Equation

Find a possible pair of integer values for $a$ and $c$ so that the equation $ax^2 - 4x + c = 0$ has two imaginary solutions. Then write the equation.

### SOLUTION

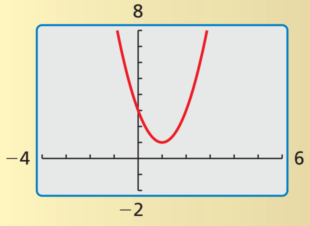
For the equation to have two imaginary solutions, the discriminant must be less than zero.

| | |
|---|---|
| $b^2 - 4ac < 0$ | Write the discriminant. |
| $(-4)^2 - 4ac < 0$ | Substitute $-4$ for $b$. |
| $16 - 4ac < 0$ | Evaluate the power. |
| $-4ac < -16$ | Subtract 16 from each side. |
| $ac > 4$ | Divide each side by $-4$. Reverse inequality symbol. |

Because $ac > 4$, choose two integers whose product is greater than 4, such as $a = 2$ and $c = 3$.

▶ So, one possible equation is $2x^2 - 4x + 3 = 0$.

**Check** The graph of $y = 2x^2 - 4x + 3$ does not have any $x$-intercepts. ✓




**ANOTHER WAY**

Another possible equation in Example 3 is $3x^2 - 4x + 2 = 0$. You can obtain this equation by letting $a = 3$ and $c = 2$.

### Monitoring Progress

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**7.** Find a possible pair of integer values for $a$ and $c$ so that the equation $ax^2 + 3x + c = 0$ has two imaginary solutions. Then write the equation.

The function $h = -16t^2 + s_0$ is used to model the height of a *dropped* object, where $h$ is the height (in feet), $t$ is the time in motion (in seconds), and $s_0$ is the initial height (in feet). For an object that is *launched* or *thrown*, an extra term $v_0t$ must be added to the model to account for the object's initial vertical velocity $v_0$ (in feet per second).

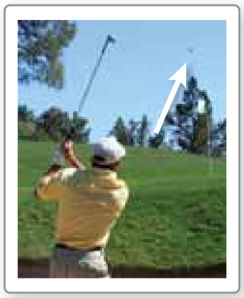
$h = -16t^2 + s_0$ Object is dropped.

$h = -16t^2 + v_0t + s_0$ Object is launched or thrown.

**STUDY TIP**

These models assume that the force of air resistance on the object is negligible. Also, these models apply only to objects on Earth. For planets with stronger or weaker gravitational forces, different models are used.

As shown below, the value of $v_0$ can be positive, negative, or zero depending on whether the object is launched upward, downward, or parallel to the ground.



$V_0 > 0$

$V_0 < 0$

$V_0 = 0$

## EXAMPLE 4 Modeling a Launched Object

A juggler tosses a ball into the air. The ball leaves the juggler's hand 4 feet above the ground and has an initial vertical velocity of 30 feet per second. Does the ball reach a height of 25 feet? 10 feet? Explain your reasoning.

### SOLUTION

Because the ball is *thrown*, use the model $h = -16t^2 + v_0t + s_0$ to write a function that represents the height of the ball.

$h = -16t^2 + v_0t + s_0$ Write the height model.

$h = -16t^2 + 30t + 4$ Substitute 30 for $v_0$ and 4 for $s_0$.

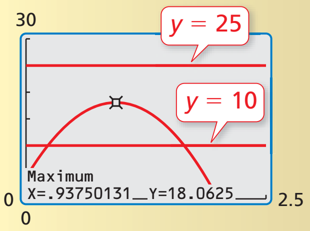
To determine whether the ball reaches each height, substitute each height for $h$ to create two equations. Then solve each equation using the Quadratic Formula.

| | |
|---|---|
| $25 = -16t^2 + 30t + 4$ | $10 = -16t^2 + 30t + 4$ |
| $0 = -16t^2 + 30t - 21$ | $0 = -16t^2 + 30t - 6$ |
| $t = \dfrac{-30 \pm \sqrt{30^2 - 4(-16)(-21)}}{2(-16)}$ | $t = \dfrac{-30 \pm \sqrt{30^2 - 4(-16)(-6)}}{2(-16)}$ |
| $t = \dfrac{-30 \pm \sqrt{-444}}{-32}$ | $t = \dfrac{-30 \pm \sqrt{516}}{-32}$ |

When $h = 25$, the equation has two imaginary solutions because the discriminant is negative. When $h = 10$, the equation has two real solutions, $t \approx 0.23$ and $t \approx 1.65$.

▶ So, the ball reaches a height of 10 feet, but it does not reach a height of 25 feet.

**Check** The graph shows that the ball reaches a height of 10 feet but not 25 feet. ✓




## Monitoring Progress

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**8.** The ball leaves the juggler's hand with an initial vertical velocity of 40 feet per second. Does the ball reach a height of 30 feet? 20 feet? Explain.

# 4.7 Exercises



## Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** When the graph of a quadratic function $y = f(x)$ has no $x$-intercepts, the equation $f(x) = 0$ has two ____________ solutions.

2. **WRITING** Can a quadratic equation with real coefficients have one imaginary solution? Explain.

## Monitoring Progress and Modeling with Mathematics

**ANALYZING EQUATIONS In Exercises 3–6, use the discriminant to match the quadratic equation with the graph of the related function. Then describe the number and type of solutions of the equation.**

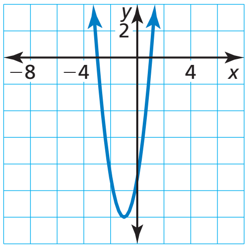
3. $x^2 - 6x + 25 = 0$

4. $2x^2 - 20x + 50 = 0$

5. $3x^2 + 6x - 9 = 0$

6. $5x^2 - 10x - 35 = 0$

**A.**

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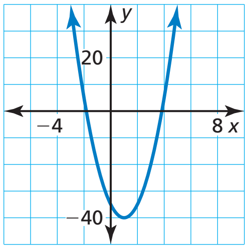
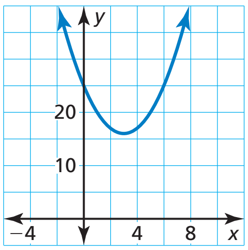
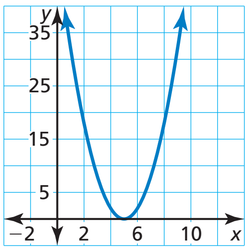



**C.**




**D.**




**In Exercises 7–20, solve the equation using any method. Explain your choice of method.** *(See Example 1.)*

7. $x^2 + 49 = 0$

8. $2x^2 - 7 = -3$

9. $x^2 - 4x + 3 = 0$

10. $3x^2 + 6x + 3 = 0$

11. $x^2 + 6x + 15 = 0$

12. $6x^2 - 2x + 1 = 0$

13. $9x^2 + 17 = 24x$

14. $-3x = 2x^2 - 4$

15. $-10x = -25 - x^2$

16. $-2x^2 - 5 = -2x$

17. $-4x^2 + 3x = -5$

18. $3x^2 + 87 = 30x$

19. $-z^2 = -12z + 6$

20. $-7w + 6 = -4w^2$

21. **ERROR ANALYSIS** Describe and correct the error in solving the equation.

$$x^2 + 10x + 74 = 0$$

$$x = \frac{-10 \pm \sqrt{10^2 - 4(1)(74)}}{2(1)}$$

$$= \frac{-10 \pm \sqrt{-196}}{2}$$

$$= \frac{-10 \pm 14}{2}$$

$$= -12 \text{ or } 2$$

22. **REASONING** Write a quadratic equation in the form $ax^2 + bx + c = 0$ that has the solutions $x = 1 \pm i$.

**In Exercises 23–28, find the zeros of the function.** *(See Example 2.)*

23. $f(x) = 5x^2 + 35$

24. $g(x) = -3x^2 + 24$

25. $h(x) = x^2 + 8x - 13$

26. $r(x) = 8x^2 + 4x + 5$

27. $m(x) = -5x^2 + 50x - 135$

28. $r(x) = 4x^2 + 9x + 3$

**OPEN-ENDED In Exercises 29–32, find a possible pair of integer values for $a$ and $c$ so that the quadratic equation has the given solution(s). Then write the equation.** *(See Example 3.)*

29. $ax^2 + 4x + c = 0$; two imaginary solutions

30. $ax^2 - 8x + c = 0$; two real solutions

31. $ax^2 + 10x = c$; one real solution

32. $-4x + c = -ax^2$; two imaginary solutions

**MODELING WITH MATHEMATICS In Exercises 33 and 34, write a function that represents the situation.**

33. A gannet is a bird that feeds on fish by diving into the water. A gannet spots a fish on the surface of the water and dives 100 feet to catch it. The bird plunges toward the water with an initial vertical velocity of −88 feet per second.

34. An archer is shooting at targets. The height of the arrow is 5 feet above the ground. Due to safety rules, the archer must aim the arrow parallel to the ground.

35. **MODELING WITH MATHEMATICS** A lacrosse player throws a ball in the air from an initial height of 7 feet. The ball has an initial vertical velocity of 35 feet per second. Does the ball reach a height of 30 feet? 26 feet? Explain your reasoning. *(See Example 4.)*

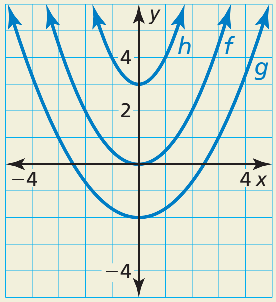
36. **PROBLEM SOLVING** A rocketry club is launching model rockets. The launching pad is 30 feet above the ground. Your model rocket has an initial vertical velocity of 105 feet per second. Your friend's model rocket has an initial vertical velocity of 100 feet per second.

   **a.** Does your rocket reach a height of 200 feet? Does your friend's rocket? Explain your reasoning.

   **b.** Which rocket is in the air longer? How much longer?

37. **CRITICAL THINKING** When a quadratic equation with real coefficients has imaginary solutions, why are the solutions complex conjugates? As part of your explanation, show that there is no such equation with solutions of $3i$ and $-2i$.

38. **HOW DO YOU SEE IT?** The graphs of three functions are shown. Which function(s) has real zeros? imaginary zeros? Explain your reasoning.




39. **USING STRUCTURE** Use the Quadratic Formula to write a quadratic equation that has the solutions $x = \dfrac{-8 \pm \sqrt{-176}}{-10}$.

40. **THOUGHT PROVOKING** Describe a real-life story that could be modeled by $h = -16t^2 + v_0t + s_0$. Write the height model for your story and determine how long your object is in the air.

41. **MODELING WITH MATHEMATICS** The Stratosphere Tower in Las Vegas is 921 feet tall and has a "needle" at its top that extends even higher into the air. A thrill ride called Big Shot catapults riders 160 feet up the needle and then lets them fall back to the launching pad.

   **a.** The height $h$ (in feet) of a rider on the Big Shot can be modeled by $h = -16t^2 + v_0t + 921$, where $t$ is the elapsed time (in seconds) after launch and $v_0$ is the initial vertical velocity (in feet per second). Find $v_0$ using the fact that the maximum value of $h$ is $921 + 160 = 1081$ feet.

   **b.** A brochure for the Big Shot states that the ride up the needle takes 2 seconds. Compare this time to the time given by the model $h = -16t^2 + v_0t + 921$, where $v_0$ is the value you found in part (a). Discuss the accuracy of the model.

## Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

**Solve the system of linear equations using any method. Explain why you chose the method.** *(Skills Review Handbook)*

42. $y = -x + 4$, $y = 2x - 8$

43. $x = 16 - 4y$, $3x + 4y = 8$

44. $2x - y = 7$, $2x + 7y = 31$

45. $3x - 2y = -20$, $x + 1.2y = 6.4$

**Find (a) the axis of symmetry and (b) the vertex of the graph of the function.** *(Section 3.3)*

46. $y = -x^2 + 2x + 1$

47. $y = 2x^2 - x + 3$

48. $f(x) = 0.5x^2 + 2x + 5$