

4.2 Solving Quadratic Equations by Graphing

Essential Question How can you use a graph to solve a quadratic equation in one variable?

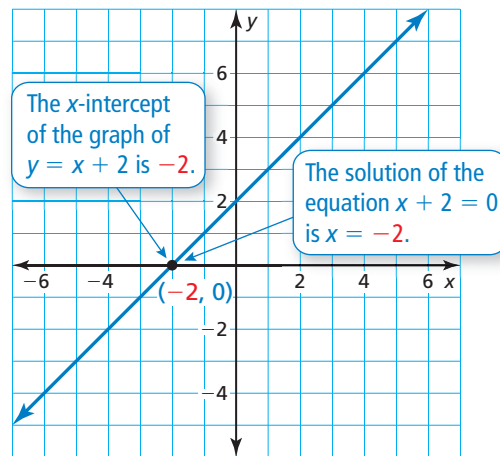
Based on the definition of an x -intercept of a graph, it follows that the x -intercept of the graph of the linear equation

$$y = ax + b \quad \text{2 variables}$$

is the same value as the solution of

$$ax + b = 0. \quad \text{1 variable}$$

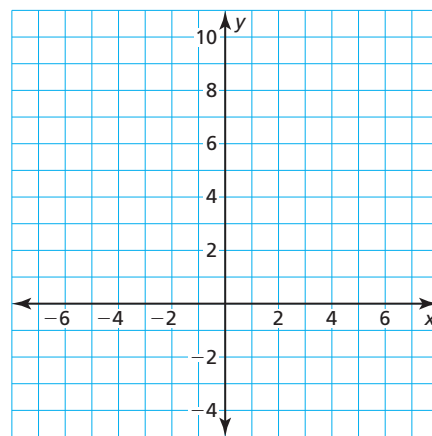
You can use similar reasoning to solve *quadratic equations*.



EXPLORATION 1 Solving a Quadratic Equation by Graphing

Work with a partner.

- Sketch the graph of $y = x^2 - 2x$.
- What is the definition of an x -intercept of a graph? How many x -intercepts does this graph have? What are they?
- What is the definition of a solution of an equation in x ? How many solutions does the equation $x^2 - 2x = 0$ have? What are they?
- Explain how you can verify the solutions you found in part (c).



EXPLORATION 2 Solving Quadratic Equations by Graphing

Work with a partner. Solve each equation by graphing.

- | | |
|-----------------------|------------------------|
| a. $x^2 - 4 = 0$ | b. $x^2 + 3x = 0$ |
| c. $-x^2 + 2x = 0$ | d. $x^2 - 2x + 1 = 0$ |
| e. $x^2 - 3x + 5 = 0$ | f. $-x^2 + 3x - 6 = 0$ |

MAKING SENSE OF PROBLEMS

To be proficient in math, you need to check your answers to problems using a different method and continually ask yourself, "Does this make sense?"

Communicate Your Answer

- How can you use a graph to solve a quadratic equation in one variable?
- After you find a solution graphically, how can you check your result algebraically? Check your solutions for parts (a)–(d) in Exploration 2 algebraically.
- How can you determine graphically that a quadratic equation has no solution?

4.2 Lesson

Core Vocabulary

quadratic equation, p. 202

Previous

x-intercept

root

zero of a function

What You Will Learn

- ▶ Solve quadratic equations by graphing.
- ▶ Use graphs to find and approximate the zeros of functions.
- ▶ Solve real-life problems using graphs of quadratic functions.

Solving Quadratic Equations by Graphing

A **quadratic equation** is a nonlinear equation that can be written in the standard form $ax^2 + bx + c = 0$, where $a \neq 0$.

In Chapter 2, you solved quadratic equations by factoring. You can also solve quadratic equations by graphing.

Core Concept

Solving Quadratic Equations by Graphing

Step 1 Write the equation in standard form, $ax^2 + bx + c = 0$.

Step 2 Graph the related function $y = ax^2 + bx + c$.

Step 3 Find the x -intercepts, if any.

The solutions, or *roots*, of $ax^2 + bx + c = 0$ are the x -intercepts of the graph.

EXAMPLE 1

Solving a Quadratic Equation: Two Real Solutions

Solve $x^2 + 2x = 3$ by graphing.

SOLUTION

Step 1 Write the equation in standard form.

$$x^2 + 2x = 3$$

Write original equation.

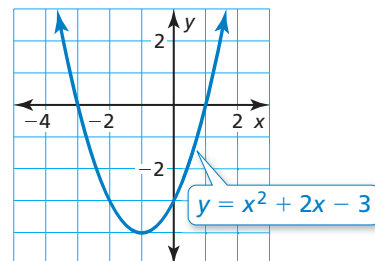
$$x^2 + 2x - 3 = 0$$

Subtract 3 from each side.

Step 2 Graph the related function
 $y = x^2 + 2x - 3$.

Step 3 Find the x -intercepts.
The x -intercepts are -3 and 1 .

- ▶ So, the solutions are $x = -3$
and $x = 1$.



Check

$$x^2 + 2x = 3$$

Original equation

$$x^2 + 2x = 3$$

$$(-3)^2 + 2(-3) \stackrel{?}{=} 3$$

Substitute.

$$1^2 + 2(1) \stackrel{?}{=} 3$$

$$3 = 3 \quad \checkmark$$

Simplify.

$$3 = 3 \quad \checkmark$$

Monitoring Progress



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Solve the equation by graphing. Check your solutions.

1. $x^2 - x - 2 = 0$

2. $x^2 + 7x = -10$

3. $x^2 + x = 12$

EXAMPLE 2 Solving a Quadratic Equation: One Real Solution

Solve $x^2 - 8x = -16$ by graphing.

SOLUTION

Step 1 Write the equation in standard form.

$$x^2 - 8x = -16$$

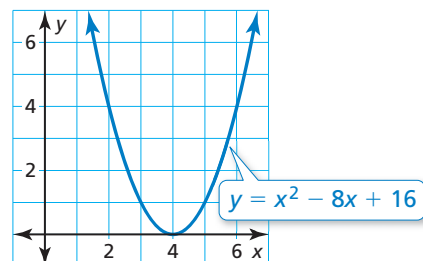
Write original equation.

$$x^2 - 8x + 16 = 0$$

Add 16 to each side.

Step 2 Graph the related function $y = x^2 - 8x + 16$.

Step 3 Find the x -intercept. The only x -intercept is at the vertex, $(4, 0)$.



► So, the solution is $x = 4$.

ANOTHER WAY

You can also solve the equation in Example 2 by factoring.

$$x^2 - 8x + 16 = 0$$

$$(x - 4)(x - 4) = 0$$

So, $x = 4$.



EXAMPLE 3 Solving a Quadratic Equation: No Real Solutions

Solve $-x^2 = 2x + 4$ by graphing.

SOLUTION

Method 1 Write the equation in standard form, $x^2 + 2x + 4 = 0$. Then graph the related function $y = x^2 + 2x + 4$, as shown at the left.

► There are no x -intercepts. So, $-x^2 = 2x + 4$ has no real solutions.

Method 2 Graph each side of the equation.

$$y = -x^2$$

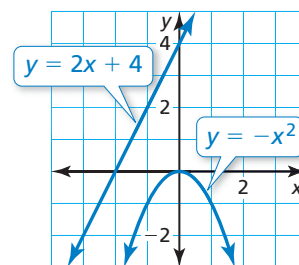
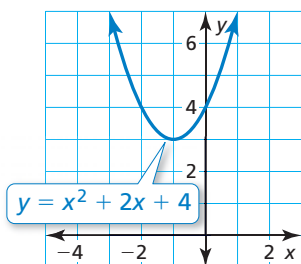
Left side

$$y = 2x + 4$$

Right side

► The graphs do not intersect.

So, $-x^2 = 2x + 4$ has no real solutions.



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Solve the equation by graphing.

4. $x^2 + 36 = 12x$

5. $x^2 + 4x = 0$

6. $x^2 + 10x = -25$

7. $x^2 = 3x - 3$

8. $x^2 + 7x = -6$

9. $2x + 5 = -x^2$

Concept Summary

Number of Solutions of a Quadratic Equation

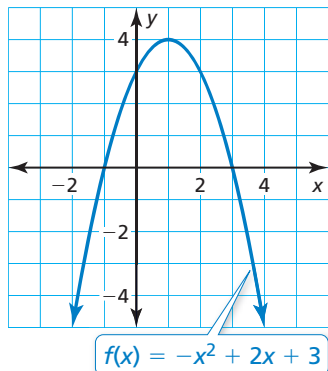
A quadratic equation has:

- two real solutions when the graph of its related function has two x -intercepts.
- one real solution when the graph of its related function has one x -intercept.
- no real solutions when the graph of its related function has no x -intercepts.

Finding Zeros of Functions

Recall that a zero of a function is an x -intercept of the graph of the function.

EXAMPLE 4 Finding the Zeros of a Function



The graph of $f(x) = -x^2 + 2x + 3$ is shown. Find the zeros of f .

SOLUTION

The x -intercepts are -1 and 3 .

► So, the zeros of f are -1 and 3 .

Check

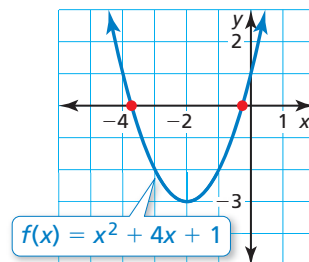
$$f(-1) = -(-1)^2 + 2(-1) + 3 = 0 \quad \checkmark$$

$$f(3) = -3^2 + 2(3) + 3 = 0 \quad \checkmark$$

The zeros of a function are not necessarily integers. To approximate zeros, analyze the signs of function values. When two function values have different signs, a zero lies between the x -values that correspond to the function values.

EXAMPLE 5 Approximating the Zeros of a Function

The graph of $f(x) = x^2 + 4x + 1$ is shown. Approximate the zeros of f to the nearest tenth.



SOLUTION

There are two x -intercepts: one between -4 and -3 , and another between -1 and 0 .

Make tables using x -values between -4 and -3 , and between -1 and 0 . Use an increment of 0.1 . Look for a change in the signs of the function values.

x	-3.9	-3.8	-3.7	-3.6	-3.5	-3.4	-3.3	-3.2	-3.1
$f(x)$	0.61	0.24	-0.11	-0.44	-0.75	-1.04	-1.31	-1.56	-1.79

change in signs

x	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
$f(x)$	-1.79	-1.56	-1.31	-1.04	-0.75	-0.44	-0.11	0.24	0.61

change in signs

The function values that are closest to 0 correspond to x -values that best approximate the zeros of the function.

► In each table, the function value closest to 0 is -0.11 . So, the zeros of f are about -3.7 and -0.3 .

ANOTHER WAY

You could approximate one zero using a table and then use the axis of symmetry to find the other zero.

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- Graph $f(x) = x^2 + x - 6$. Find the zeros of f .
- Graph $f(x) = -x^2 + 2x + 2$. Approximate the zeros of f to the nearest tenth.



Solving Real-Life Problems

EXAMPLE 6 Real-Life Application

A football player kicks a football 2 feet above the ground with an initial vertical velocity of 75 feet per second. The function $h = -16t^2 + 75t + 2$ represents the height h (in feet) of the football after t seconds. (a) Find the height of the football each second after it is kicked. (b) Use the results of part (a) to estimate when the height of the football is 50 feet. (c) Using a graph, after how many seconds is the football 50 feet above the ground?

SOLUTION

Seconds, t	Height, h
0	2
1	61
2	88
3	83
4	46
5	-23

- a. Make a table of values starting with $t = 0$ seconds using an increment of 1. Continue the table until a function value is negative.

▶ The height of the football is 61 feet after 1 second, 88 feet after 2 seconds, 83 feet after 3 seconds, and 46 feet after 4 seconds.

- b. From part (a), you can estimate that the height of the football is 50 feet between 0 and 1 second and between 3 and 4 seconds.

▶ Based on the function values, it is reasonable to estimate that the height of the football is 50 feet slightly less than 1 second and slightly less than 4 seconds after it is kicked.

- c. To determine when the football is 50 feet above the ground, find the t -values for which $h = 50$. So, solve the equation $-16t^2 + 75t + 2 = 50$ by graphing.

Step 1 Write the equation in standard form.

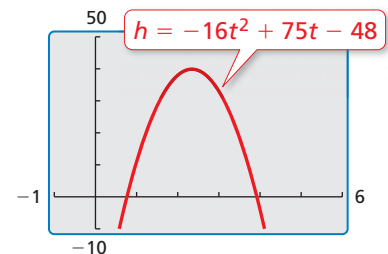
$$-16t^2 + 75t + 2 = 50$$

Write the equation.

$$-16t^2 + 75t - 48 = 0$$

Subtract 50 from each side.

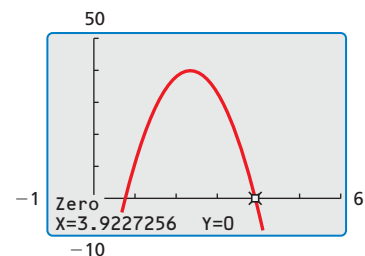
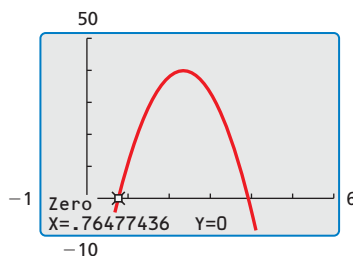
- Step 2** Use a graphing calculator to graph the related function $h = -16t^2 + 75t - 48$.



REMEMBER

Equations have *solutions*, or *roots*. Graphs have *x-intercepts*. Functions have *zeros*.

- Step 3** Use the *zero* feature to find the zeros of the function.



- ▶ The football is 50 feet above the ground after about 0.8 second and about 3.9 seconds, which supports the estimates in part (b).

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12. **WHAT IF?** After how many seconds is the football 65 feet above the ground?

Vocabulary and Core Concept Check

- VOCABULARY** What is a quadratic equation?
- WHICH ONE DOESN'T BELONG?** Which equation does *not* belong with the other three? Explain your reasoning.

$$x^2 + 5x = 20$$

$$x^2 + x - 4 = 0$$

$$x^2 - 6 = 4x$$

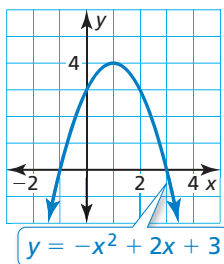
$$7x + 12 = x^2$$

- WRITING** How can you use a graph to find the number of solutions of a quadratic equation?
- WRITING** How are solutions, roots, x -intercepts, and zeros related?

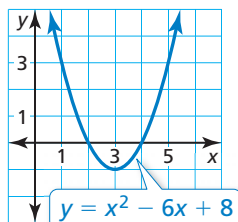
Monitoring Progress and Modeling with Mathematics

In Exercises 5–8, use the graph to solve the equation.

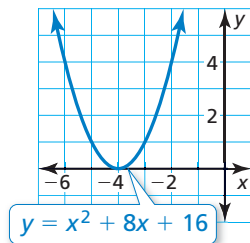
5. $-x^2 + 2x + 3 = 0$



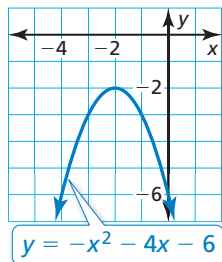
6. $x^2 - 6x + 8 = 0$



7. $x^2 + 8x + 16 = 0$



8. $-x^2 - 4x - 6 = 0$



19. $x^2 = -1 - 2x$

20. $x^2 = -x - 3$

21. $4x - 12 = -x^2$

22. $5x - 6 = x^2$

23. $x^2 - 2 = -x$

24. $16 + x^2 = -8x$

25. **ERROR ANALYSIS** Describe and correct the error in solving $x^2 + 3x = 18$ by graphing.

X

The solutions of the equation $x^2 + 3x = 18$ are $x = -3$ and $x = 0$.

In Exercises 9–12, write the equation in standard form.

9. $4x^2 = 12$

10. $-x^2 = 15$

11. $2x - x^2 = 1$

12. $5 + x = 3x^2$

In Exercises 13–24, solve the equation by graphing.
(See Examples 1, 2, and 3.)

13. $x^2 - 5x = 0$

14. $x^2 - 4x + 4 = 0$

15. $x^2 - 2x + 5 = 0$

16. $x^2 - 6x - 7 = 0$

17. $x^2 = 6x - 9$

18. $-x^2 = 8x + 20$

26. **ERROR ANALYSIS** Describe and correct the error in solving $x^2 + 6x + 9 = 0$ by graphing.

X

The solution of the equation $x^2 + 6x + 9 = 0$ is $x = 9$.

27. **MODELING WITH MATHEMATICS** The height y (in yards) of a flop shot in golf can be modeled by $y = -x^2 + 5x$, where x is the horizontal distance (in yards).



- Interpret the x -intercepts of the graph of the equation.
- How far away does the golf ball land?

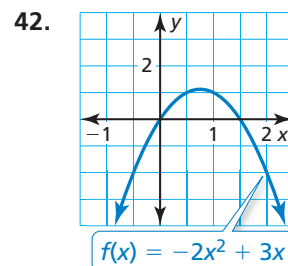
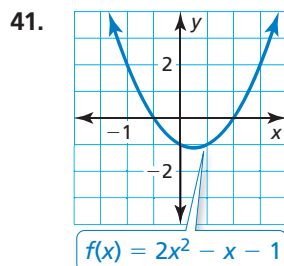
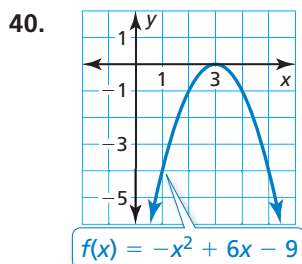
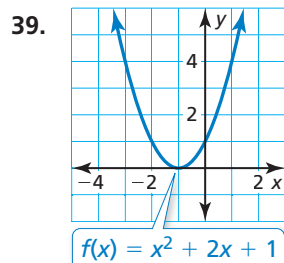
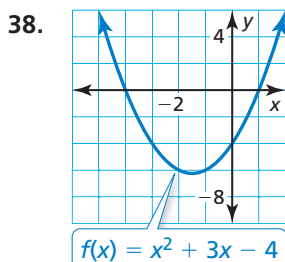
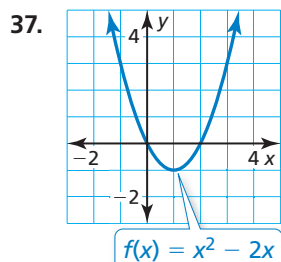
28. **MODELING WITH MATHEMATICS** The height h (in feet) of an underhand volleyball serve can be modeled by $h = -16t^2 + 30t + 4$, where t is the time (in seconds).

- Do both t -intercepts of the graph of the function have meaning in this situation? Explain.
- No one receives the serve. After how many seconds does the volleyball hit the ground?

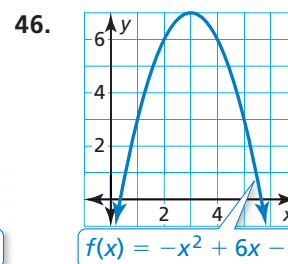
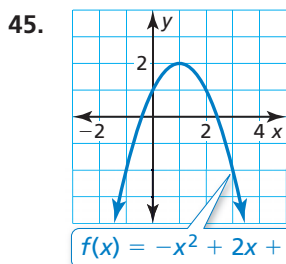
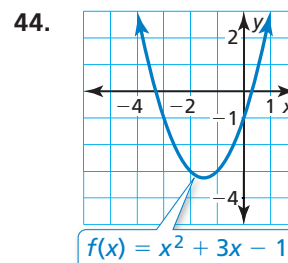
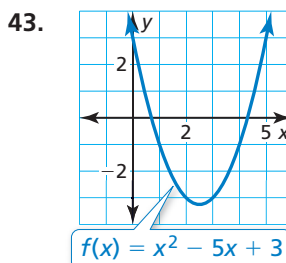
In Exercises 29–36, solve the equation by using Method 2 from Example 3.

- | | |
|-----------------------|----------------------|
| 29. $x^2 = 10 - 3x$ | 30. $2x - 3 = x^2$ |
| 31. $5x - 7 = x^2$ | 32. $x^2 = 6x - 5$ |
| 33. $x^2 + 12x = -20$ | 34. $x^2 + 8x = 9$ |
| 35. $-x^2 - 5 = -2x$ | 36. $-x^2 - 4 = -4x$ |

In Exercises 37–42, find the zero(s) of f . (See Example 4.)



In Exercises 43–46, approximate the zeros of f to the nearest tenth. (See Example 5.)



In Exercises 47–52, graph the function. Approximate the zeros of the function to the nearest tenth, if necessary.

- | | |
|--------------------------------------|-----------------------------|
| 47. $f(x) = x^2 + 6x + 1$ | 48. $f(x) = x^2 - 3x + 2$ |
| 49. $y = -x^2 + 4x - 2$ | 50. $y = -x^2 + 9x - 6$ |
| 51. $f(x) = \frac{1}{2}x^2 + 2x - 5$ | 52. $f(x) = -3x^2 + 4x + 3$ |

53. **MODELING WITH MATHEMATICS** At a Civil War reenactment, a cannonball is fired into the air with an initial vertical velocity of 128 feet per second. The release point is 6 feet above the ground. The function $h = -16t^2 + 128t + 6$ represents the height h (in feet) of the cannonball after t seconds. (See Example 6.)

- Find the height of the cannonball each second after it is fired.
- Use the results of part (a) to estimate when the height of the cannonball is 150 feet.
- Using a graph, after how many seconds is the cannonball 150 feet above the ground?



- 54. MODELING WITH MATHEMATICS** You throw a softball straight up into the air with an initial vertical velocity of 40 feet per second. The release point is 5 feet above the ground. The function $h = -16t^2 + 40t + 5$ represents the height h (in feet) of the softball after t seconds.

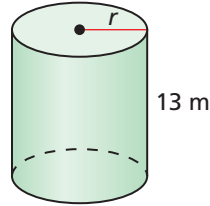
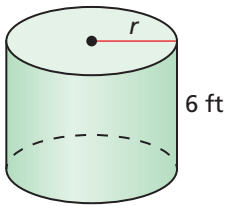
- Find the height of the softball each second after it is released.
- Use the results of part (a) to estimate when the height of the softball is 15 feet.
- Using a graph, after how many seconds is the softball 15 feet above the ground?



MATHEMATICAL CONNECTIONS In Exercises 55 and 56, use the given surface area S of the cylinder to find the radius r to the nearest tenth.

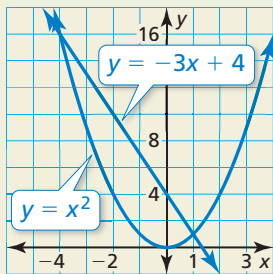
55. $S = 225 \text{ ft}^2$

56. $S = 750 \text{ m}^2$



- 57. WRITING** Explain how to approximate zeros of a function when the zeros are not integers.

- 58. HOW DO YOU SEE IT?** Consider the graph shown.

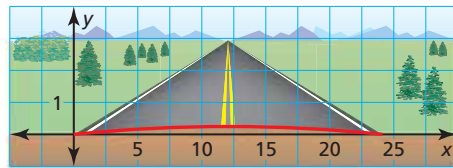


- How many solutions does the quadratic equation $x^2 = -3x + 4$ have? Explain.
- Without graphing, describe what you know about the graph of $y = x^2 + 3x - 4$.

- 59. COMPARING METHODS** Example 3 shows two methods for solving a quadratic equation. Which method do you prefer? Explain your reasoning.

- 60. THOUGHT PROVOKING** How many different parabolas have -2 and 2 as x -intercepts? Sketch examples of parabolas that have these two x -intercepts.

- 61. MODELING WITH MATHEMATICS** To keep water off a road, the surface of the road is shaped like a parabola. A cross section of the road is shown in the diagram. The surface of the road can be modeled by $y = -0.0017x^2 + 0.041x$, where x and y are measured in feet. Find the width of the road to the nearest tenth of a foot.



- 62. MAKING AN ARGUMENT** A stream of water from a fire hose can be modeled by $y = -0.003x^2 + 0.58x + 3$, where x and y are measured in feet. A firefighter is standing 57 feet from a building and is holding the hose 3 feet above the ground. The bottom of a window of the building is 26 feet above the ground. Your friend claims the stream of water will pass through the window. Is your friend correct? Explain.

REASONING In Exercises 63–65, determine whether the statement is *always*, *sometimes*, or *never* true. Justify your answer.

- The graph of $y = ax^2 + c$ has two x -intercepts when a is negative.
- The graph of $y = ax^2 + c$ has no x -intercepts when a and c have the same sign.
- The graph of $y = ax^2 + bx + c$ has more than two x -intercepts when $a \neq 0$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Show that an exponential model fits the data. Then write a recursive rule that models the data. (Section 1.6)

66.

n	0	1	2	3
$f(n)$	18	3	$\frac{1}{2}$	$\frac{1}{12}$

67.

n	0	1	2	3
$f(n)$	2	8	32	128