3.7 Comparing Linear, Exponential, and Quadratic Functions

Essential Question How can you compare the growth rates of linear, exponential, and quadratic functions?

EXPLORATION 1 **Comparing Speeds**

Work with a partner. Three cars start traveling at the same time. The distance traveled in t minutes is y miles. Complete each table and sketch all three graphs in the same coordinate plane. Compare the speeds of the three cars. Which car has a constant speed? Which car is accelerating the most? Explain your reasoning.

t	$y = t$
0	
0.2	
0.4	
0.6	
0.8	
1.0	

t	$y = 2^t - 1$
0	
0.2	
0.4	
0.6	
0.8	
1.0	

t	$y = t^2$
0	
0.2	
0.4	
0.6	
0.8	
1.0	

COMPARING PREDICTIONS

To be proficient in math, you need to visualize the results of varying assumptions, explore consequences, and compare predictions with data.

EXPLORATION 2 **Comparing Speeds**

Work with a partner. Analyze the speeds of the three cars over the given time periods. The distance traveled in t minutes is y miles. Which car eventually overtakes the others?

t	$y = t$
1.0	
1.5	
2.0	
2.5	
3.0	
3.5	
4.0	
4.5	
5.0	

t	$y = 2^t - 1$
1.0	
1.5	
2.0	
2.5	
3.0	
3.5	
4.0	
4.5	
5.0	

t	$y = t^2$
1.0	
1.5	
2.0	
2.5	
3.0	
3.5	
4.0	
4.5	
5.0	

Communicate Your Answer

3. How can you compare the growth rates of linear, exponential, and quadratic functions?

4. Which function has a growth rate that is eventually much greater than the growth rates of the other two functions? Explain your reasoning.

3.7 Lesson

Core Vocabulary

Previous
average rate of change
slope

What You Will Learn

- Choose functions to model data.
- Write functions to model data.
- Compare functions using average rates of change.

Choosing Functions to Model Data

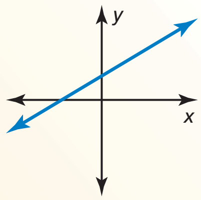
So far, you have studied linear functions, exponential functions, and quadratic functions. You can use these functions to model data.

Core Concept

Linear, Exponential, and Quadratic Functions

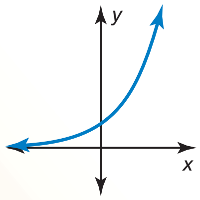
Linear Function

$y = mx + b$



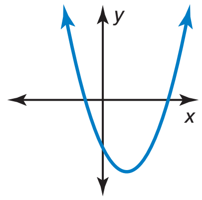
Exponential Function

$y = ab^x$



Quadratic Function

$y = ax^2 + bx + c$



EXAMPLE 1 **Using Graphs to Identify Functions**

Plot the points. Tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function.

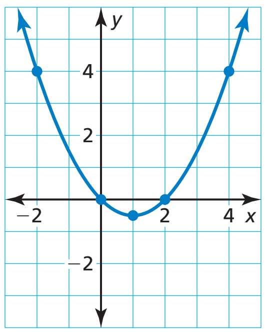
a. $(4, 4), (2, 0), (0, 0), \left(1, -\frac{1}{2}\right), (-2, 4)$

b. $(0, 1), (2, 4), (4, 7), (-2, -2), (-4, -5)$

c. $(0, 2), (2, 8), (1, 4), (-1, 1), \left(-2, \frac{1}{2}\right)$

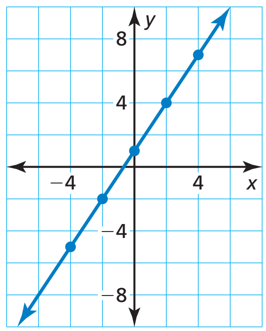
SOLUTION

a.



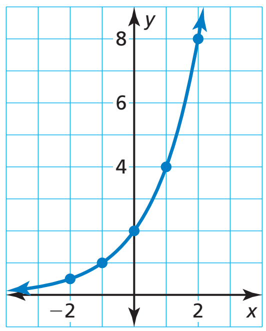
▶ quadratic

b.



▶ linear

c.



▶ exponential

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Plot the points. Tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function.

1. $(-1, 5), (2, -1), (0, -1), (3, 5), (1, -3)$

2. $(-1, 2), (-2, 8), (-3, 32), \left(0, \frac{1}{2}\right), \left(1, \frac{1}{8}\right)$

3. $(-3, 5), (0, -1), (2, -5), (-4, 7), (1, -3)$

Core Concept

Differences and Ratios of Functions

You can use patterns between consecutive data pairs to determine which type of function models the data. The differences of consecutive y-values are called *first differences*. The differences of consecutive first differences are called *second differences*.

- **Linear Function** The first differences are constant.
- **Exponential Function** Consecutive y-values have a common *ratio*.
- **Quadratic Function** The second differences are constant.

In all cases, the differences of consecutive x-values need to be constant.

STUDY TIP

The first differences for exponential and quadratic functions are *not* constant.

EXAMPLE 2 Using Differences or Ratios to Identify Functions

Tell whether each table of values represents a *linear*, an *exponential*, or a *quadratic* function.

a.

x	−3	−2	−1	0	1
y	11	8	5	2	−1

b.

x	−2	−1	0	1	2
y	1	2	4	8	16

c.

x	−2	−1	0	1	2
y	−1	−2	−1	2	7

STUDY TIP

First determine that the differences of consecutive x-values are constant. Then check whether the first differences are constant or consecutive y-values have a common ratio. If neither of these is true, check whether the second differences are constant.

SOLUTION

a.

+1 +1 +1 +1

x	−3	−2	−1	0	1
y	11	8	5	2	−1

−3 −3 −3 −3

The first differences are constant. So, the table represents a linear function.

b.

+1 +1 +1 +1

x	−2	−1	0	1	2
y	1	2	4	8	16

×2 ×2 ×2 ×2

Consecutive y-values have a common ratio. So, the table represents an exponential function.

c.

+1 +1 +1 +1

x	−2	−1	0	1	2
y	−1	−2	−1	2	7

first differences → −1 +1 +3 +5

second differences → +2 +2 +2

The second differences are constant. So, the table represents a quadratic function.

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4. Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function.

x	−1	0	1	2	3
y	1	3	9	27	81

Writing Functions to Model Data

EXAMPLE 3 Writing a Function to Model Data

x	0	1	2	3	4
f(x)	12	0	−4	0	12

Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. Then write the function.

SOLUTION

Step 1 Determine which type of function the table of values represents.

The second differences are constant. So, the table represents a quadratic function.

+1 +1 +1 +1

x	0	1	2	3	4
f(x)	12	0	−4	0	12

first differences → −12 −4 +4 +12

second differences → +8 +8 +8

Step 2 Write an equation of the quadratic function. Using the table, notice that the x-intercepts are 1 and 3. So, use intercept form to write a function.

$f(x) = a(x - 1)(x - 3)$ Substitute for p and q in intercept form.

Use another point from the table, such as (0, 12), to find a.

$12 = a(0 - 1)(0 - 3)$ Substitute 0 for x and 12 for $f(x)$.

$4 = a$ Solve for a.

Use the value of a to write the function.

$f(x) = 4(x - 1)(x - 3)$ Substitute 4 for a.

$= 4x^2 - 16x + 12$ Use the FOIL Method and combine like terms.

STUDY TIP

To check your function in Example 3, substitute the other points from the table to verify that they satisfy the function.

▶ So, the quadratic function is $f(x) = 4x^2 - 16x + 12$.

Previously, you wrote recursive rules for linear and exponential functions. You can use the pattern in the first differences to write recursive rules for quadratic functions.

EXAMPLE 4 Writing a Recursive Rule

Write a recursive rule for the quadratic function f in Example 3.

SOLUTION

An expression for the nth term of the sequence of first differences is $-12 + 8(n - 1)$, or $8n - 20$. Notice that $f(n) - f(n - 1) = 8n - 20$.

▶ So, a recursive rule for the quadratic function is $f(0) = 12$, $f(n) = f(n - 1) + 8n - 20$.

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5. Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. Then write the function.

x	−1	0	1	2	3
y	16	8	4	2	1

6. Write a recursive rule for the function represented by the table of values.

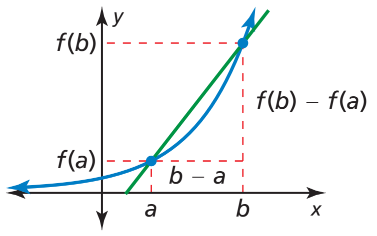
x	0	1	2	3	4
f(x)	−8	−1	10	25	44

Comparing Functions Using Average Rates of Change

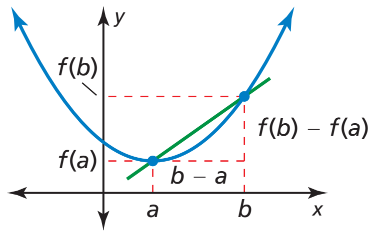
For nonlinear functions, the rate of change is not constant. You can compare two nonlinear functions over the same interval using their average rates of change. Recall that the average rate of change of a function $y = f(x)$ between $x = a$ and $x = b$ is the slope of the line through $(a, f(a))$ and $(b, f(b))$.

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

Exponential Function



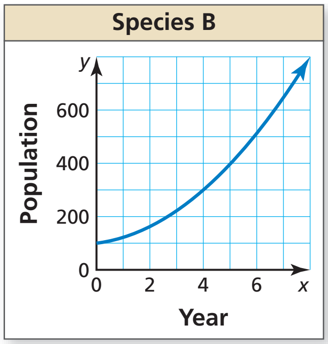
Quadratic Function



EXAMPLE 5 Using and Interpreting Average Rates of Change

The table and graph show the populations of two species of fish introduced into a lake. Compare the populations by calculating and interpreting the average rates of change from Year 0 to Year 4.

Species A	
Year, x	**Population, y**
0	100
2	264
4	428
6	592
8	756
10	920



SOLUTION

Calculate the average rates of change by using the points whose x-coordinates are 0 and 4.

Species A: Use (0, 100) and (4, 428).

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a} = \frac{428 - 100}{4 - 0} = 82$$

Species B: Use the graph to estimate the points when $x = 0$ and $x = 4$. Use (0, 100) and (4, 300).

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a} \approx \frac{300 - 100}{4 - 0} = 50$$

▶ From Year 0 to Year 4, the population of Species A increases at an average rate of 82 fish per year, and the population of Species B increases at an average rate of about 50 fish per year. So, the population of Species A is growing faster.

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7. Compare the populations by calculating and interpreting the average rates of change from Year 4 to Year 8.

Core Concept

Comparing Functions Using Average Rates of Change

- As a and b increase, the average rate of change between $x = a$ and $x = b$ of an increasing exponential function $y = f(x)$ will eventually exceed the average rate of change between $x = a$ and $x = b$ of an increasing quadratic function $y = g(x)$ or an increasing linear function $y = h(x)$. So, as x increases, $f(x)$ will eventually exceed $g(x)$ or $h(x)$.
- As a and b increase, the average rate of change between $x = a$ and $x = b$ of an increasing quadratic function $y = g(x)$ will eventually exceed the average rate of change between $x = a$ and $x = b$ of an increasing linear function $y = h(x)$. So, as x increases, $g(x)$ will eventually exceed $h(x)$.

STUDY TIP

You can explore these concepts using a graphing calculator.

EXAMPLE 6 Comparing Different Function Types

Let x represent the number of years since 1900. The function $C(x) = 12x^2 + 12x + 100$ represents the population of Cedar Ridge. In 1900, Pine Valley had a population of 50 people. Pine Valley's population increased by 9% each year.

a. From 1900 to 1950, which town's population had a greater average rate of change?

b. Which town will eventually have a greater population? Explain.

SOLUTION

a. Write the function that represents the population of Cedar Ridge and a function to model the population of Pine Valley.

Cedar Ridge: $C(x) = 12x^2 + 12x + 100$ Quadratic function

Pine Valley: $P(x) = 50(1.09)^x$ Exponential function

Find the average rate of change from 1900 to 1950 for the population of each town.

Cedar Ridge

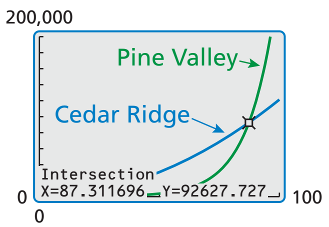
$$\frac{C(50) - C(0)}{50 - 0} = 612$$

Pine Valley

$$\frac{P(50) - P(0)}{50 - 0} \approx 73$$

▶ From 1900 to 1950, the average rate of change of Cedar Ridge's population was greater.

b. Because Pine Valley's population is given by an increasing exponential function and Cedar Ridge's population is given by an increasing quadratic function, Pine Valley will eventually have a greater population. Using a graphing calculator, you can see that Pine Valley's population caught up to and exceeded Cedar Ridge's population between $x = 87$ and $x = 88$, which corresponds to 1987.



X	Y1 (Pine Valley)	Y2 (Cedar Ridge)
87	90173	91972
87.1	90953	92182
87.2	91740	92392
87.3	92534	92603
87.4	93335	92814
87.5	94143	93025
87.6	94958	93236

X=87

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8. From 1950 to 2000, which town's population had a greater average rate of change?

3.7 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

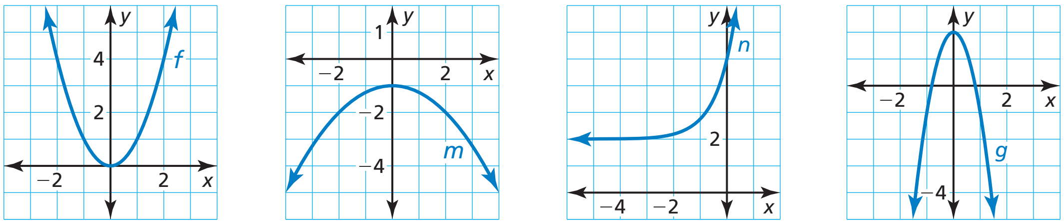
Vocabulary and Core Concept Check

1. WRITING Name three types of functions that you can use to model data. Describe the equation and graph of each type of function.

2. WRITING How can you decide whether to use a linear, an exponential, or a quadratic function to model a data set?

3. VOCABULARY Describe how to find the average rate of change of a function $y = f(x)$ between $x = a$ and $x = b$.

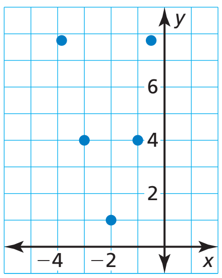
4. WHICH ONE DOESN'T BELONG? Which graph does *not* belong with the other three? Explain your reasoning.



Monitoring Progress and Modeling with Mathematics

In Exercises 5–8, tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function.

5.



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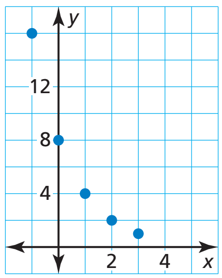

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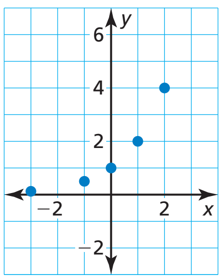

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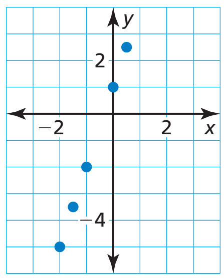


In Exercises 9–14, plot the points. Tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function. *(See Example 1.)*

9. $(-2, -1), (-1, 0), (1, 2), (2, 3), (0, 1)$

10. $\left(0, \frac{1}{4}\right), (1, 1), (2, 4), (3, 16), \left(-1, \frac{1}{16}\right)$

11. $(0, -3), (1, 0), (2, 9), (-2, 9), (-1, 0)$

12. $(-1, -3), (-3, 5), (0, -1), (1, 5), (2, 15)$

13. $(-4, -4), (-2, -3.4), (0, -3), (2, -2.6), (4, -2)$

14. $(0, 8), (-4, 0.25), (-3, 0.4), (-2, 1), (-1, 3)$

In Exercises 15–18, tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. *(See Example 2.)*

15.

x	−2	−1	0	1	2
y	0	0.5	1	1.5	2

16.

x	−1	0	1	2	3
y	0.2	1	5	25	125

17.

x	2	3	4	5	6
y	2	6	18	54	162

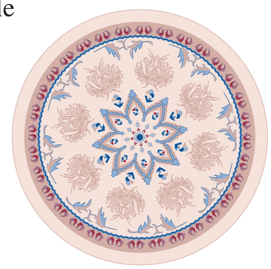
18.

x	−3	−2	−1	0	1
y	2	4.5	8	12.5	18

19. **MODELING WITH MATHEMATICS** A student takes a subway to a public library. The table shows the distances d (in miles) the student travels in t minutes. Let the time t represent the independent variable. Tell whether the data can be modeled by a *linear*, an *exponential*, or a *quadratic* function. Explain.

Time, t	0.5	1	3	5
Distance, d	0.335	0.67	2.01	3.35

20. **MODELING WITH MATHEMATICS** A store sells custom circular rugs. The table shows the costs c (in dollars) of rugs that have diameters of d feet. Let the diameter d represent the independent variable. Tell whether the data can be modeled by a *linear*, an *exponential*, or a *quadratic* function. Explain.



Diameter, d	3	4	5	6
Cost, c	63.90	113.60	177.50	255.60

In Exercises 21–26, tell whether the data represent a *linear*, an *exponential*, or a *quadratic* function. Then write the function. *(See Example 3.)*

21. $(-2, 8), (-1, 0), (0, -4), (1, -4), (2, 0), (3, 8)$

22. $(-3, 8), (-2, 4), (-1, 2), (0, 1), (1, 0.5)$

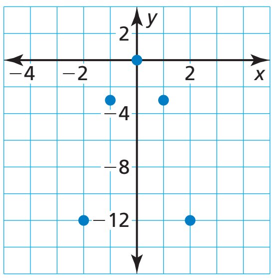
23.

x	-2	-1	0	1	2
y	4	1	-2	-5	-8

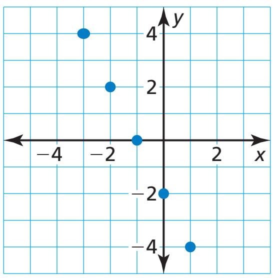
24.

x	-1	0	1	2	3
y	2.5	5	10	20	40

25.



26.



In Exercises 27 and 28, write a recursive rule for the function represented by the table of values. *(See Example 4.)*

27.

x	0	1	2	3	4
$f(x)$	2	1	2	5	10

28.

x	0	1	2	3	4
$f(x)$	3	6	12	24	48

29. **ERROR ANALYSIS** Describe and correct the error in determining whether the table represents a linear, an exponential, or a quadratic function.

✗

$+1 \quad +1 \quad +1 \quad +1$

x	1	2	3	4	5
y	3	9	27	81	243

$\times 3 \quad \times 3 \quad \times 3 \quad \times 3$

Consecutive y-values change by a constant amount. So, the table represents a linear function.

30. **ERROR ANALYSIS** Describe and correct the error in writing the function represented by the table.

✗

$+1 \quad +1 \quad +1 \quad +1$

x	-3	-2	-1	0	1
y	4	0	-2	-2	0

first differences → $-4 \quad -2 \quad +0 \quad +2$

second differences → $+2 \quad +2 \quad +2$

The table represents a quadratic function.

$$f(x) = a(x - 2)(x - 1)$$

$$4 = a(-3 - 2)(-3 - 1)$$

$$\frac{1}{5} = a$$

$$f(x) = \frac{1}{5}(x - 2)(x - 1)$$

$$= \frac{1}{5}x^2 - \frac{3}{5}x + \frac{2}{5}$$

So, the function is $f(x) = \frac{1}{5}x^2 - \frac{3}{5}x + \frac{2}{5}$.

31. **REASONING** The table shows the numbers of people attending the first five football games at a high school.

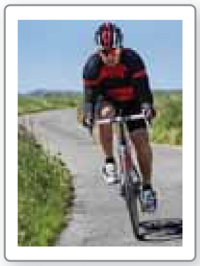
Game, g	1	2	3	4	5
People, p	252	325	270	249	310

a. Plot the points. Let the game g represent the independent variable.

b. Can a linear, an exponential, or a quadratic function represent this situation? Explain.

32. MODELING WITH MATHEMATICS The table shows the breathing rates y (in liters of air per minute) of a cyclist traveling at different speeds x (in miles per hour).

Speed, x	20	21	22	23	24
Breathing rate, y	51.4	57.1	63.3	70.3	78.0



a. Plot the points. Let the speed x represent the independent variable. Then determine the type of function that best represents this situation.

b. Write a function that models the data.

c. Find the breathing rate of a cyclist traveling 18 miles per hour. Round your answer to the nearest tenth.

33. ANALYZING RATES OF CHANGE The function $f(t) = -16t^2 + 48t + 3$ represents the height (in feet) of a volleyball t seconds after it is hit into the air.

a. Copy and complete the table.

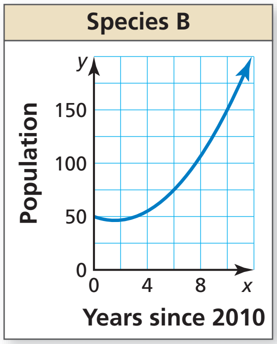
t	0	0.5	1	1.5	2	2.5	3
$f(t)$							

b. Plot the ordered pairs and draw a smooth curve through the points.

c. Describe where the function is increasing and decreasing.

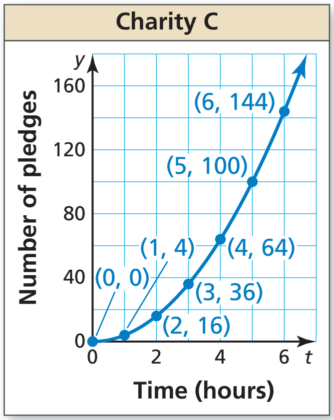
d. Find the average rate of change for each 0.5-second interval in the table. What do you notice about the average rates of change when the function is increasing? decreasing?

34. ANALYZING RELATIONSHIPS The population of Species A in 2010 was 100. The population of Species A increased by 6% every year. Let x represent the number of years since 2010. The graph shows the population of Species B. Compare the populations of the species by calculating and interpreting the average rates of change from 2010 to 2016. *(See Example 5.)*



35. ANALYZING RELATIONSHIPS Three charities are holding telethons. Charity A begins with one pledge, and the number of pledges triples each hour. The table shows the numbers of pledges received by Charity B. The graph shows the numbers of pledges received by Charity C.

Time (hours), t	**Number of pledges, y**
0	12
1	24
2	36
3	48
4	60
5	72
6	84



a. What type of function represents the numbers of pledges received by Charity A? B? C?

b. Find the average rates of change of each function for each 1-hour interval from $t = 0$ to $t = 6$.

c. For which function does the average rate of change increase most quickly? What does this tell you about the numbers of pledges received by the three charities?

36. COMPARING FUNCTIONS Let x represent the number of years since 1900. The function

$H(x) = 10x^2 + 10x + 500$

represents the population of Oak Hill. In 1900, Poplar Grove had a population of 200 people. Poplar Grove's population increased by 8% each year. *(See Example 6.)*

a. From 1900 to 1950, which town's population had a greater average rate of change?

b. Which town will eventually have a greater population? Explain.

37. COMPARING FUNCTIONS Let x represent the number of years since 2000. The function

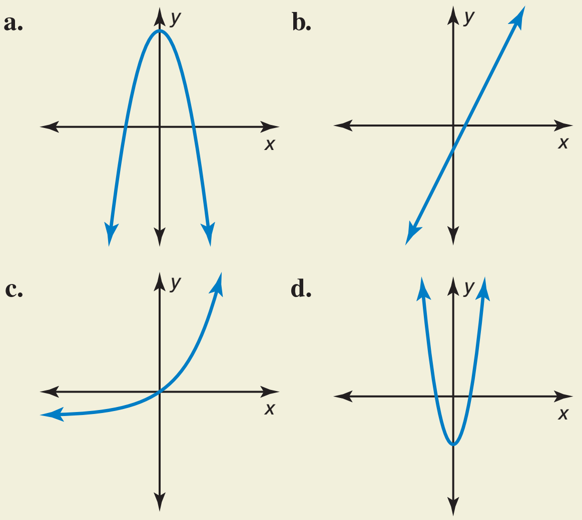
$R(x) = 0.01x^2 + 0.22x + 1.08$

represents the revenue (in millions of dollars) of Company A. In 2000, Company B had a revenue of \$2.12 million. Company B's revenue increased by \$0.32 million each year.

a. From 2000 to 2015, which company's revenue had a greater average rate of change?

b. Which company will eventually have a greater revenue? Explain.

38. **HOW DO YOU SEE IT?** Match each graph with its function. Explain your reasoning.



A. $y = 2x^2 - 4$ **B.** $y = 2^x - 1$

C. $y = 2x - 1$ **D.** $y = -2x^2 + 4$

39. **REASONING** Explain why the average rate of change of a linear function is constant and the average rate of change of a quadratic or exponential function is not constant.

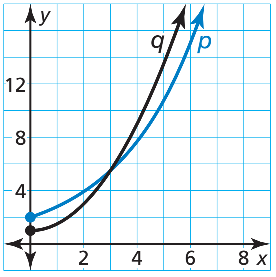
40. **CRITICAL THINKING** In the ordered pairs below, the y-values are given in terms of n. Tell whether the ordered pairs represent a *linear*, an *exponential*, or a *quadratic* function. Explain.

$(1, 3n - 1), (2, 10n + 2), (3, 26n),$
$(4, 51n - 7), (5, 85n - 19)$

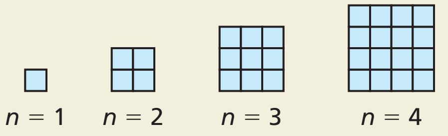
41. **USING STRUCTURE** Write a function that has constant second differences of 3.

42. **CRITICAL THINKING** Is the graph of a set of points enough to determine whether the points represent a linear, an exponential, or a quadratic function? Justify your answer.

43. **MAKING AN ARGUMENT** Function p is an exponential function and function q is a quadratic function. Your friend says that after about $x = 3$, function q will always have a greater y-value than function p. Is your friend correct? Explain.



44. **THOUGHT PROVOKING** Find three different patterns in the figure. Determine whether each pattern represents a *linear*, an *exponential*, or a *quadratic* function. Write a model for each pattern.



45. **USING TOOLS** The table shows the amount a (in billions of dollars) United States residents spent on pets or pet-related products and services each year for a 5-year period. Let the year x represent the independent variable. Using technology, find a function that models the data. How did you choose the model? Predict how much residents will spend on pets or pet-related products and services in Year 7.

Year, x	1	2	3	4	5
Amount, a	53.1	56.9	61.8	65.7	67.1

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the expression. *(Section 1.5)*

46. $\sqrt{121}$

47. $\sqrt[3]{125}$

48. $\sqrt[3]{512}$

49. $\sqrt[5]{243}$

Find the product. *(Section 2.3)*

50. $(x + 6)(x - 6)$

51. $(2y + 5)(2y - 5)$

52. $(4c - 3d)(4c + 3d)$

53. $(-3s + 8t)(-3s - 8t)$