6.3 Comparing Linear and Exponential Functions

Essential Question

How can you compare the growth rates of linear and exponential functions?

EXPLORATION 1 Comparing Values

Work with a partner. An art collector buys two paintings. The value of each painting after t years is y dollars. Complete each table. Compare the values of the two paintings. Which painting's value has a constant growth rate? Which painting's value has an increasing growth rate? Explain your reasoning.

t	$y = 19t + 5$
0	
1	
2	
3	
4	

t	$y = 3^t$
0	
1	
2	
3	
4	

COMPARING PREDICTIONS

To be proficient in math, you need to visualize the results of varying assumptions, explore consequences, and compare predictions with data.

EXPLORATION 2 Comparing Values

Work with a partner. Analyze the values of the two paintings over the given time periods. The value of each painting after t years is y dollars. Which painting's value eventually overtakes the other?

t	$y = 19t + 5$
4	
5	
6	
7	
8	
9	

t	$y = 3^t$
4	
5	
6	
7	
8	
9	

EXPLORATION 3 Comparing Graphs

Work with a partner. Use the tables in Explorations 1 and 2 to graph $y = 19t + 5$ and $y = 3^t$ in the same coordinate plane. Compare the graphs of the functions.

Communicate Your Answer

4. How can you compare the growth rates of linear and exponential functions?

5. Which function has a growth rate that is eventually much greater than the growth rate of the other function? Explain your reasoning.

6.3 Lesson

Core Vocabulary

average rate of change, *p. 291*

Previous

slope

What You Will Learn

- Choose functions to model data.
- Compare functions using average rates of change.
- Solve real-life problems involving different function types.

Choosing Functions to Model Data

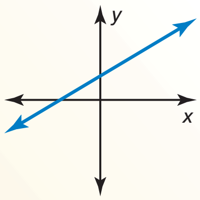
So far, you have studied linear functions and exponential functions. You can use these functions to model data.

Core Concept

Linear and Exponential Functions

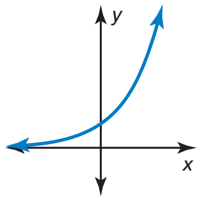
Linear Function

$y = mx + b$



Exponential Function

$y = ab^x$



EXAMPLE 1 Using Graphs to Identify Functions

Plot the points. Tell whether the points appear to represent a *linear function,* an *exponential function*, or *neither*.

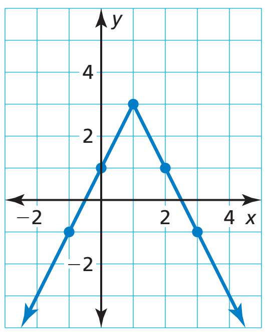
a. (3, −1), (2, 1), (0, 1), (1, 3), (−1, −1)

b. (0, 1), (2, −2), (4, −5), (−2, 4), (−4, 7)

c. (0, 2), (1, 1), $\left(2, \frac{1}{2}\right)$, (−2, 8), (−1, 4)

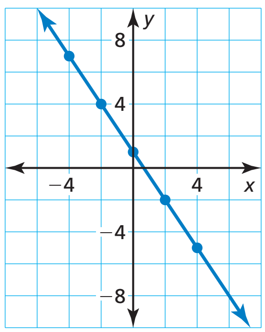
SOLUTION

a.



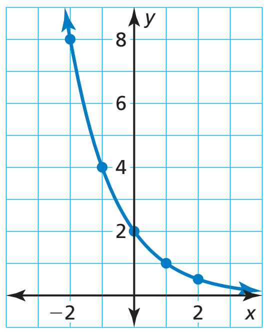
▶ neither

b.



▶ linear

c.



▶ exponential

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Plot the points. Tell whether the points appear to represent a *linear function*, an *exponential function*, or *neither*.

1. (−3, −4), (0, 2), (−2, 2), (1, −4), (−1, 4)

2. (1, 1), (2, 3), (3, 9), $\left(0, \frac{1}{3}\right)$, $\left(-1, \frac{1}{9}\right)$

3. (−2, −7), (0, −1), (2, 5), (−1, −4), (1, 2)

Core Concept

Differences and Ratios of Functions

You can use patterns between consecutive data pairs to determine which type of function models the data.

- **Linear Function** The differences of consecutive y-values are constant.
- **Exponential Function** Consecutive y-values have a common *ratio*.

In each case, the differences of consecutive x-values need to be constant.

REMEMBER

Linear functions have a *constant rate of change*. So, for equally-spaced x-values, the differences of consecutive y-values are constant. Exponential functions do *not* have a constant rate of change.

EXAMPLE 2 Using Differences or Ratios to Identify Functions

Tell whether each table of values represents a *linear* or an *exponential* function. Then write the function.

a.

x	-2	-1	0	1	2
y	9	5	1	-3	-7

b.

x	-3	-2	-1	0	1
y	$\frac{1}{3}$	1	3	9	27

SOLUTION

a.

$+1$ $+1$ $+1$ $+1$

x	-2	-1	0	1	2
y	9	5	1	-3	-7

-4 -4 -4 -4

The differences of consecutive y-values are constant. The slope is $\frac{-4}{1} = -4$ and the y-intercept is 1. So, the table represents the linear function $y = -4x + 1$.

b.

$+1$ $+1$ $+1$ $+1$

x	-3	-2	-1	0	1
y	$\frac{1}{3}$	1	3	9	27

$\times 3$ $\times 3$ $\times 3$ $\times 3$

Consecutive y-values have a common ratio of 3 and the y-intercept is 9. So, the table represents the exponential function $y = 9(3)^x$.

STUDY TIP

First determine that the differences of consecutive x-values are constant. Then check whether the y-values have a constant difference or a common ratio.

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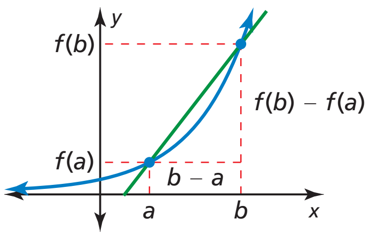
4. Tell whether the table of values represents a *linear* or an *exponential* function. Then write the function.

x	-1	0	1	2	3
y	32	16	8	4	2

Comparing Functions Using Average Rates of Change

For nonlinear functions, the rate of change is not constant. You can compare two functions over the same interval using their *average rates of change*. The **average rate of change** of a function $y = f(x)$ between $x = a$ and $x = b$ is the slope of the line through $(a, f(a))$ and $(b, f(b))$.

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$



STUDY TIP

You can explore this concept using a graphing calculator.

Core Concept

Comparing Functions Using Average Rates of Change

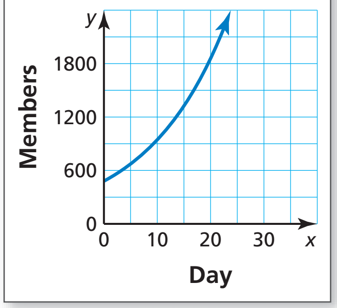
As a and b increase, the average rate of change between $x = a$ and $x = b$ of an increasing exponential function $y = f(x)$ will eventually exceed the average rate of change between $x = a$ and $x = b$ of an increasing linear function $y = g(x)$. So, as x increases, $f(x)$ will eventually exceed $g(x)$.

EXAMPLE 3 Using and Interpreting Average Rates of Change

Two social media websites open their memberships to the public. (a) Compare the websites by calculating and interpreting the average rates of change from Day 10 to Day 20. (b) Predict which website will have more members after 50 days. Explain.

Website A	
Day, x	**Members, y**
0	650
5	1025
10	1400
15	1775
20	2150
25	2525

Website B



SOLUTION

a. Calculate the average rates of change by using the points whose x-coordinates are 10 and 20.

Website A: Use (10, 1400) and (20, 2150).

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a} = \frac{2150 - 1400}{20 - 10} = \frac{750}{10} = 75$$

Website B: Use the graph to estimate the points when $x = 10$ and $x = 20$. Use (10, 950) and (20, 1850).

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a} \approx \frac{1850 - 950}{20 - 10} = \frac{900}{10} = 90$$

▶ From Day 10 to Day 20, Website A membership increases at an average rate of 75 people per day, and Website B membership increases at an average rate of about 90 people per day. So, Website B membership is growing faster.

b. Using the table, membership increases and the average rates of change are constant. So, Website A membership can be represented by an increasing linear function. Using the graph, membership increases and the average rates of change are increasing. It appears that Website B membership can be represented by an increasing exponential function.

After 25 days, the memberships of both websites are about equal and the average rate of change of Website B exceeds the average rate of change of Website A. So, Website B will have more members after 50 days.

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5. Compare the websites in Example 3 by calculating and interpreting the average rates of change from Day 0 to Day 10.

Solving Real-Life Problems

EXAMPLE 4 **Comparing Different Function Types**

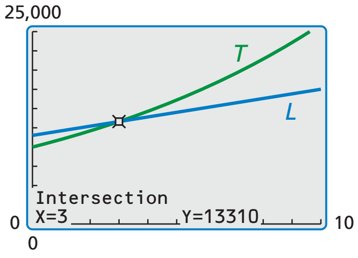
In 2000, Littleton had a population of 11,510 people. Littleton's population increased by 600 people each year. In 2000, Tinyville had a population of 10,000 people. Tinyville's population increased by 10% each year.

a. In what year were the populations equal?

b. Suppose Littleton's initial population doubled to 23,020 and maintained a constant rate of increase of 600 people each year. Did Tinyville's population still catch up to Littleton's population?

c. Suppose Littleton's rate of increase doubled to 1200 people each year, in addition to doubling the initial population. Did Tinyville's population still catch up to Littleton's population? Explain.

SOLUTION



a. Let x represent the number of years since 2000. Write a function to model the population of each town.

Littleton: $L(x) = 600x + 11{,}510$ — Linear function

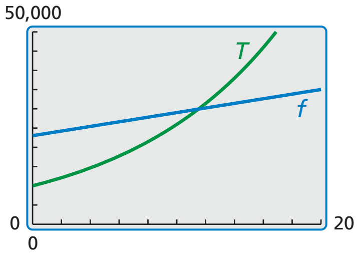
Tinyville: $T(x) = 10{,}000(1.1)^x$ — Exponential function

Use a graphing calculator to graph each function in the same viewing window. Use the *intersect* feature to find the value of x for which $L(x) = T(x)$. The graphs intersect when $x = 3$.

▶ So, the populations were equal in 2003.

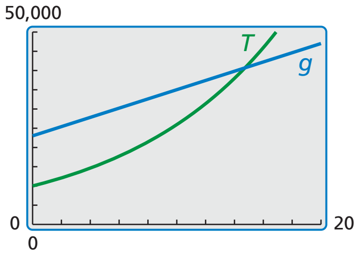
b. Littleton's new population function is $f(x) = 600x + 23{,}020$. Use a graphing calculator to graph f and T in the same viewing window.

▶ From the graph, you can see that Tinyville's population eventually caught up to and exceeded Littleton's population.



c. Littleton's new population function is $g(x) = 1200x + 23{,}020$. Use a graphing calculator to graph g and T in the same viewing window.

▶ From the graph, you can see that Tinyville's population eventually caught up to and exceeded Littleton's population. Because Littleton's population shows linear growth and Tinyville's population shows exponential growth, Tinyville's population eventually exceeded Littleton's regardless of Littleton's constant rate or initial value.



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6. WHAT IF? In 2000, Littleton had a population of 10,900 people and the population increased by 600 people each year. In what year were the populations equal?

6.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Name two types of functions that you can use to model data. Describe the equation and graph of each type of function.

2. **WRITING** How can you decide whether to use a linear or an exponential function to model a data set?

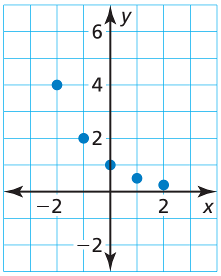
3. **VOCABULARY** Describe how to find the average rate of change of a function $y = f(x)$ between $x = a$ and $x = b$.

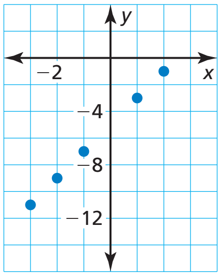
4. **DIFFERENT WORDS, SAME QUESTION** Let $f(x) = 4^x$. Which is different? Find "both" answers.

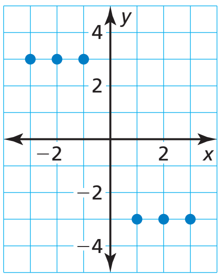
As the input increases by 1, by what factor does the output increase?	What is the average rate of change of f between $x = 1$ and $x = 2$?
What is $\dfrac{f(2) - f(1)}{2 - 1}$?	What is the slope of the line that passes through (1, 4) and (2, 16)?

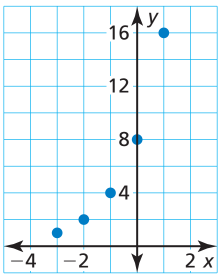
Monitoring Progress and Modeling with Mathematics

In Exercises 5–8, tell whether the points appear to represent a *linear function*, an *exponential function*, or *neither*.

5. 

6. 

7. 

8. 

In Exercises 9–14, plot the points. Tell whether the points appear to represent a *linear function*, an *exponential function*, or *neither*. *(See Example 1.)*

9. $(-1, -3), (0, -2), (1, -1), (2, 0), (3, 1)$

10. $\left(-1, \frac{1}{4}\right), \left(0, \frac{1}{2}\right), (1, 1), (2, 2), (3, 4)$

11. $(0, 0), (4, 6), (2, 3), (-2, 3), (-4, 6)$

12. $(-1, -1), (-2, -6), (0, -2), (3, 1), (2, -2)$

13. $(-3, 5.5), (-1, 0.5), (0, -2), (1, -4.5), (3, -9.5)$

14. $(0, 6), (1, 3), (2, 1.5), (3, 0.75), (-1, 12)$

In Exercises 15–18, tell whether the table of values represents a *linear* or an *exponential* function. Then write the function. *(See Example 2.)*

15.

x	−1	0	1	2	3
y	−1	−0.5	0	0.5	1

16.

x	−2	−1	0	1	2
y	$\frac{1}{5}$	1	5	25	125

17.

x	1	2	3	4	5
y	512	128	32	8	2

18.

x	−5	−4	−3	−2	−1
y	12	9	6	3	0

19. MODELING WITH MATHEMATICS You ride a bus to school. The table shows the distances d (in miles) the bus travels in t minutes. Let the time t represent the independent variable. Tell whether the data can be modeled by a *linear function*, an *exponential function*, or *neither.* Explain.

Time, t	1	2	3	4	5
Distance, d	0.7	1.4	2.1	2.1	2.8

20. MODELING WITH MATHEMATICS The table shows the size s (in hectares) of a glacier d decades after 1980.

Decades, d	0	1	2	3
Size, s	300	270	243	218.7

a. Plot the points. Let the number d of decades after 1980 represent the independent variable.

b. Tell whether the data can be modeled by a *linear* or an *exponential* function. Then write the function.

21. ERROR ANALYSIS Describe and correct the error in determining which type of function the table represents.

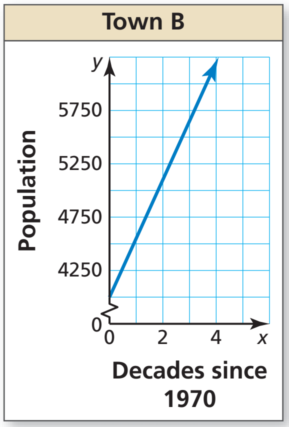
x	0	1	2	3	4
y	3	6	12	24	48

+1 +1 +1 +1

×2 ×2 ×2 ×2

Consecutive y-values change by a constant amount. So, the table represents a linear function.

22. ANALYZING RELATIONSHIPS The population of Town A in 1970 was 3000. The population of Town A increased by 20% every decade. Let x represent the number of decades since 1970. The graph shows the population of Town B. *(See Example 3.)*

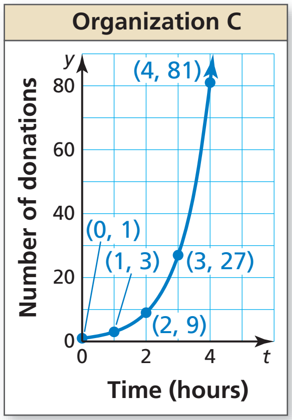


a. Compare the populations of the towns by calculating and interpreting the average rates of change from 1990 to 2010.

b. Predict which town will have a greater population after 2020. Explain.

23. ANALYZING RELATIONSHIPS Three organizations are collecting donations for a cause. Organization A begins with one donation, and the number of donations quadruples each hour. The table shows the numbers of donations collected by Organization B. The graph shows the numbers of donations collected by Organization C.

Time (hours), t	Number of donations, y
0	0
1	4
2	8
3	12
4	16
5	20
6	24



a. What type of function represents the numbers of donations collected by Organization A? B? C?

b. Find the average rates of change of each function for each 1-hour interval from $t = 0$ to $t = 6$.

c. For which function does the average rate of change increase most quickly? What does this tell you about the numbers of donations collected by the three organizations?

24. COMPARING FUNCTIONS The room expenses for two different resorts are shown. *(See Example 4.)*

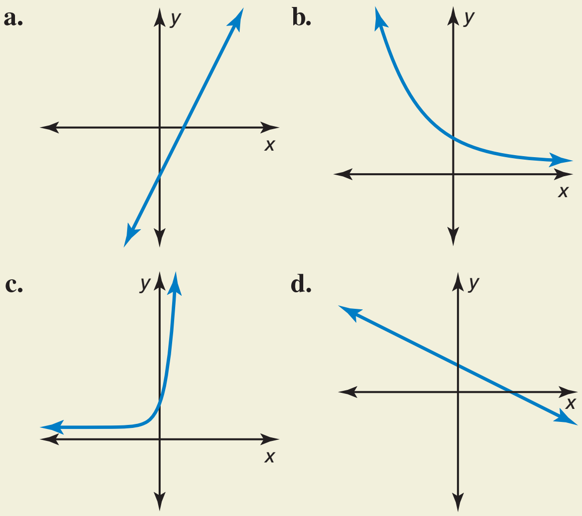
a. For what length of vacation does each resort cost about the same?

b. Suppose Blue Water Resort charges $1450 for the first three nights and $105 for each additional night. Would Sea Breeze Resort ever be more expensive than Blue Water Resort? Explain.

c. Suppose Sea Breeze Resort charges $1200 for the first three nights. The charge increases 10% for each additional night. Would Blue Water Resort ever be more expensive than Sea Breeze Resort? Explain.

25. **REASONING** Explain why the average rate of change of a linear function is constant and the average rate of change of an exponential function is not constant.

26. **HOW DO YOU SEE IT?** Match each graph with its function. Explain your reasoning.



A. $y = -\frac{1}{2}x + 1$ **B.** $y = 2(4)^x + 1$

C. $y = 2\left(\frac{3}{4}\right)^x + 1$ **D.** $y = 2x - 4$

27. **USING STRUCTURE** In the ordered pairs below, m is an integer and $n \neq 0$. Tell whether the ordered pairs represent a *linear* or an *exponential* function. Explain.

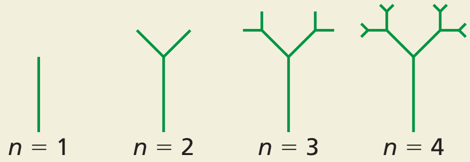
$(m, n), (m + 1, 2n), (m + 2, 4n),$
$(m + 3, 8n), (m + 4, 16n)$

28. **CRITICAL THINKING** Write and graph a linear function f and an exponential function g with the following characteristics.

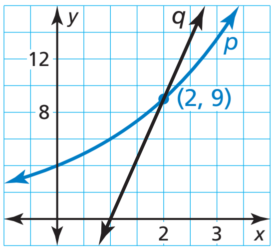
- $f(x) > g(x)$ when $0 < x < 4$
- $f(x) < g(x)$ when $x < 0$ and $x > 4$

29. **CRITICAL THINKING** Is the graph of a set of points enough to determine whether the points represent a linear function or an exponential function? Justify your answer.

30. **THOUGHT PROVOKING** Find four different patterns in the figure. Determine whether each pattern represents a *linear* or an *exponential* function. Write a model for each pattern.



31. **MAKING AN ARGUMENT** Function p is an exponential function and function q is a linear function. Your friend says that after $x = 2$, function q will always have a greater y-value than function p. Is your friend correct? Explain.



32. **USING TOOLS** The table shows the amount a (in milligrams) of venom remaining in an animal's body x days after being bitten by a venomous snake. Let the number x of days represent the independent variable. Using technology, find a function that models the data. How did you choose the model? About how much venom was initially injected into the animal's body? Explain your reasoning.

Day, x	1	2	3	4	5
Amount, a	10.2	5.3	2.5	1.2	0.7

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution. *(Section 1.3)*

33. $8x + 12 = 4x$

34. $5 - t = 7t + 21$

35. $6(r - 2) = 2r + 8$

36. $-6(s + 7) = 10(3 - s)$

Find the slope and the y-intercept of the graph of the linear equation. *(Section 3.5)*

37. $y = -6x + 7$

38. $y = \frac{1}{4}x + 7$

39. $3y = 6x - 12$

40. $2y + x = 8$