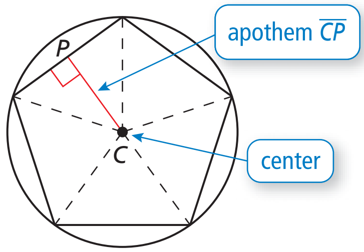
11.3 Areas of Polygons

Essential Question

How can you find the area of a regular polygon?

The **center of a regular polygon** is the center of its circumscribed circle.

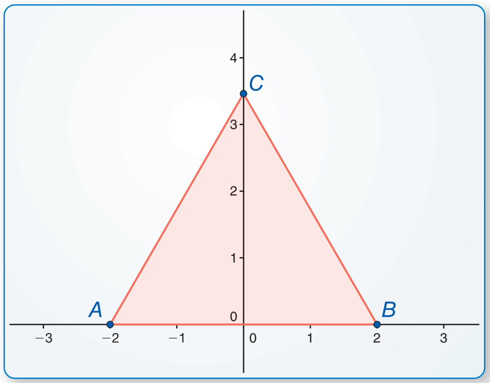
The distance from the center to any side of a regular polygon is called the **apothem of a regular polygon**.



EXPLORATION 1 Finding the Area of a Regular Polygon

Work with a partner. Use dynamic geometry software to construct each regular polygon with side lengths of 4, as shown. Find the apothem and use it to find the area of the polygon. Describe the steps that you used.

a.



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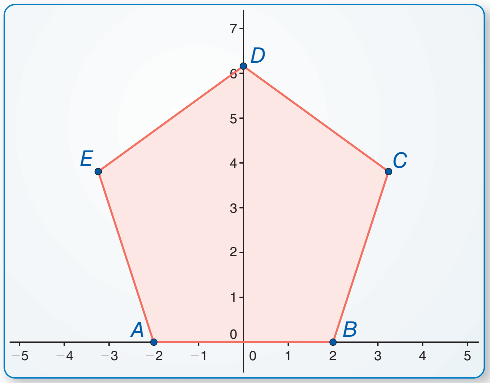
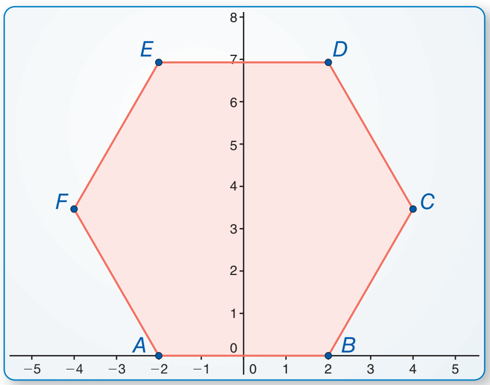


c.



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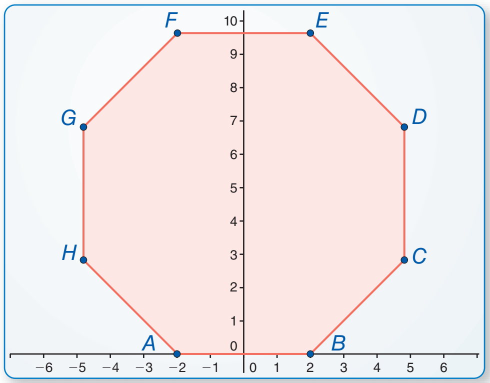


EXPLORATION 2 Writing a Formula for Area

Work with a partner. Generalize the steps you used in Exploration 1 to develop a formula for the area of a regular polygon.

REASONING ABSTRACTLY

To be proficient in math, you need to know and flexibly use different properties of operations and objects.

Communicate Your Answer

3. How can you find the area of a regular polygon?

4. Regular pentagon *ABCDE* has side lengths of 6 meters and an apothem of approximately 4.13 meters. Find the area of *ABCDE*.

11.3 Lesson

Core Vocabulary

center of a regular polygon, *p. 611*
radius of a regular polygon, *p. 611*
apothem of a regular polygon, *p. 611*
central angle of a regular polygon, *p. 611*

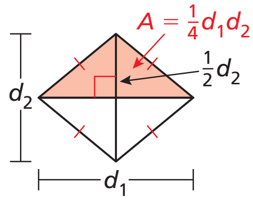
Previous
rhombus
kite

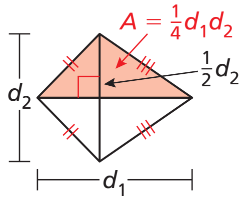
What You Will Learn

- Find areas of rhombuses and kites.
- Find angle measures in regular polygons.
- Find areas of regular polygons.

Finding Areas of Rhombuses and Kites

You can divide a rhombus or kite with diagonals d_1 and d_2 into two congruent triangles with base d_1, height $\frac{1}{2}d_2$, and area $\frac{1}{2}d_1\left(\frac{1}{2}d_2\right) = \frac{1}{4}d_1d_2$. So, the area of a rhombus or kite is $2\left(\frac{1}{4}d_1d_2\right) = \frac{1}{2}d_1d_2$.

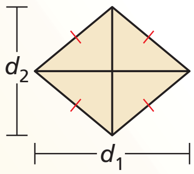


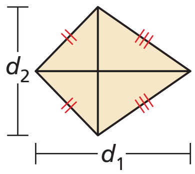


Core Concept

Area of a Rhombus or Kite

The area of a rhombus or kite with diagonals d_1 and d_2 is $\frac{1}{2}d_1d_2$.

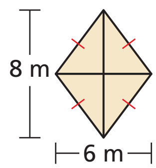




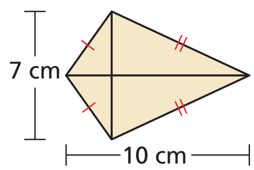
EXAMPLE 1 Finding the Area of a Rhombus or Kite

Find the area of each rhombus or kite.

a.



b.



SOLUTION

a. $A = \frac{1}{2}d_1d_2$

$= \frac{1}{2}(6)(8)$

$= 24$

▶ So, the area is 24 square meters.

b. $A = \frac{1}{2}d_1d_2$

$= \frac{1}{2}(10)(7)$

$= 35$

▶ So, the area is 35 square centimeters.

Monitoring Progress

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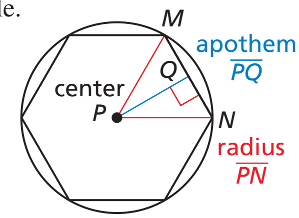
1. Find the area of a rhombus with diagonals $d_1 = 4$ feet and $d_2 = 5$ feet.
2. Find the area of a kite with diagonals $d_1 = 12$ inches and $d_2 = 9$ inches.

Finding Angle Measures in Regular Polygons

The diagram shows a regular polygon inscribed in a circle. The **center of a regular polygon** and the **radius of a regular polygon** are the center and the radius of its circumscribed circle.



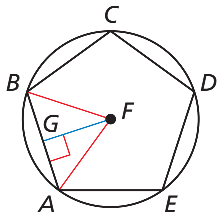
$\angle MPN$ is a central angle.

The distance from the center to any side of a regular polygon is called the **apothem of a regular polygon**. The apothem is the height to the base of an isosceles triangle that has two radii as legs. The word "apothem" refers to a segment as well as a length. For a given regular polygon, think of *an* apothem as a segment and *the* apothem as a length.

A **central angle of a regular polygon** is an angle formed by two radii drawn to consecutive vertices of the polygon. To find the measure of each central angle, divide 360° by the number of sides.

EXAMPLE 2 Finding Angle Measures in a Regular Polygon

In the diagram, *ABCDE* is a regular pentagon inscribed in $\odot F$. Find each angle measure.



a. $m\angle AFB$ **b.** $m\angle AFG$ **c.** $m\angle GAF$

ANALYZING RELATIONSHIPS

$\overline{FG}$ is an altitude of an isosceles triangle, so it is also a median and angle bisector of the isosceles triangle.

SOLUTION

a. $\angle AFB$ is a central angle, so $m\angle AFB = \frac{360°}{5} = 72°$.

b. $\overline{FG}$ is an apothem, which makes it an altitude of isosceles $\triangle AFB$.

So, $\overline{FG}$ bisects $\angle AFB$ and $m\angle AFG = \frac{1}{2}m\angle AFB = 36°$.

c. By the Triangle Sum Theorem (Theorem 5.1), the sum of the angle measures of right $\triangle GAF$ is 180°.

$$m\angle GAF = 180° - 90° - 36°$$
$$= 54°$$

So, $m\angle GAF = 54°$.

Monitoring Progress

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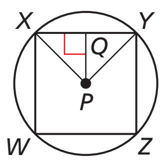
In the diagram, *WXYZ* is a square inscribed in $\odot P$.



3. Identify the center, a radius, an apothem, and a central angle of the polygon.

4. Find $m\angle XPY$, $m\angle XPQ$, and $m\angle PXQ$.

Finding Areas of Regular Polygons

You can find the area of any regular n-gon by dividing it into congruent triangles.

READING DIAGRAMS

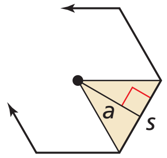
In this book, a point shown inside a regular polygon marks the center of the circle that can be circumscribed about the polygon.

$A =$ Area of one triangle • Number of triangles

$= \left(\frac{1}{2} \cdot s \cdot a\right) \cdot n$ — Base of triangle is s and height of triangle is a. Number of triangles is n.

$= \frac{1}{2} \cdot a \cdot (n \cdot s)$ — Commutative and Associative Properties of Multiplication

$= \frac{1}{2} a \cdot P$ — There are n congruent sides of length s, so perimeter P is $n \cdot s$.

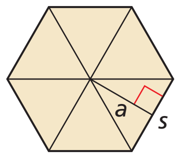


Core Concept

Area of a Regular Polygon

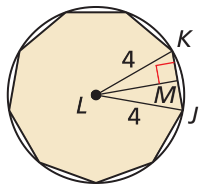
The area of a regular n-gon with side length s is one-half the product of the apothem a and the perimeter P.

$$A = \frac{1}{2}aP, \text{ or } A = \frac{1}{2}a \cdot ns$$



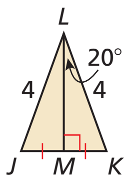
EXAMPLE 3 Finding the Area of a Regular Polygon

A regular nonagon is inscribed in a circle with a radius of 4 units. Find the area of the nonagon.



SOLUTION

The measure of central $\angle JLK$ is $\frac{360^\circ}{9}$, or 40°. Apothem $\overline{LM}$ bisects the central angle, so $m\angle KLM$ is 20°. To find the lengths of the legs, use trigonometric ratios for right $\triangle KLM$.



$$\sin 20^\circ = \frac{MK}{LK} \qquad \cos 20^\circ = \frac{LM}{LK}$$

$$\sin 20^\circ = \frac{MK}{4} \qquad \cos 20^\circ = \frac{LM}{4}$$

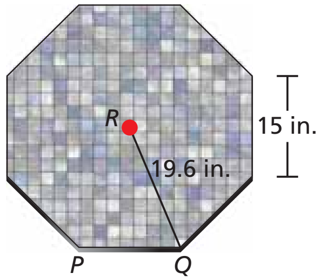
$$4 \sin 20^\circ = MK \qquad 4 \cos 20^\circ = LM$$

The regular nonagon has side length $s = 2(MK) = 2(4 \sin 20^\circ) = 8 \sin 20^\circ$, and apothem $a = LM = 4 \cos 20^\circ$.

▶ So, the area is $A = \frac{1}{2}a \cdot ns = \frac{1}{2}(4 \cos 20^\circ) \cdot (9)(8 \sin 20^\circ) \approx 46.3$ square units.

EXAMPLE 4 Finding the Area of a Regular Polygon

You are decorating the top of a table by covering it with small ceramic tiles. The tabletop is a regular octagon with 15-inch sides and a radius of about 19.6 inches. What is the area you are covering?



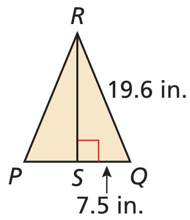
SOLUTION

Step 1 Find the perimeter P of the tabletop.
An octagon has 8 sides, so $P = 8(15) = 120$ inches.

Step 2 Find the apothem a. The apothem is height RS of $\triangle PQR$.

Because $\triangle PQR$ is isosceles, altitude $\overline{RS}$ bisects $\overline{QP}$.

So, $QS = \frac{1}{2}(QP) = \frac{1}{2}(15) = 7.5$ inches.



To find RS, use the Pythagorean Theorem (Theorem 9.1) for $\triangle RQS$.

$$a = RS = \sqrt{19.6^2 - 7.5^2} = \sqrt{327.91} \approx 18.108$$

Step 3 Find the area A of the tabletop.

$A = \frac{1}{2}aP$ Formula for area of a regular polygon

$= \frac{1}{2}(\sqrt{327.91})(120)$ Substitute.

≈ 1086.5 Simplify.

▶ The area you are covering with tiles is about 1086.5 square inches.

Monitoring Progress

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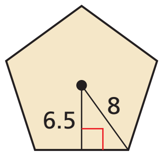
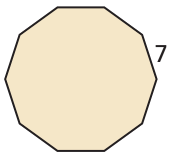
Find the area of the regular polygon.

5.



6.



11.3 Exercises

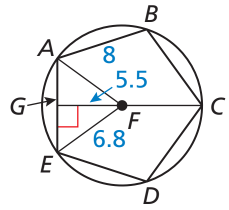
Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Explain how to find the measure of a central angle of a regular polygon.

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

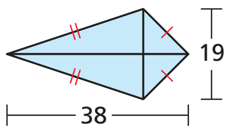
Find the radius of $\odot F$.	Find the apothem of polygon *ABCDE*.
Find *AF*.	Find the radius of polygon *ABCDE*.



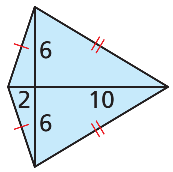
Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, find the area of the kite or rhombus. *(See Example 1.)*

3.



4.



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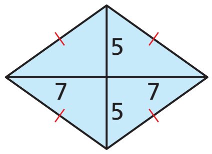

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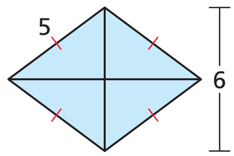
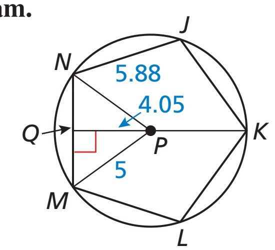


In Exercises 7–10, use the diagram.



7. Identify the center of polygon *JKLMN*.

8. Identify a central angle of polygon *JKLMN*.

9. What is the radius of polygon *JKLMN*?

10. What is the apothem of polygon *JKLMN*?

In Exercises 11–14, find the measure of a central angle of a regular polygon with the given number of sides. Round answers to the nearest tenth of a degree, if necessary.

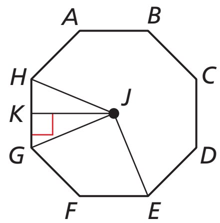
11. 10 sides

12. 18 sides

13. 24 sides

14. 7 sides

In Exercises 15–18, find the given angle measure for regular octagon *ABCDEFGH*. *(See Example 2.)*



15. $m\angle GJH$

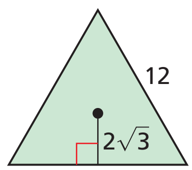
16. $m\angle GJK$

17. $m\angle KGJ$

18. $m\angle EJH$

In Exercises 19–24, find the area of the regular polygon. *(See Examples 3 and 4.)*

19.



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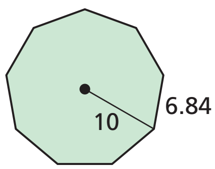

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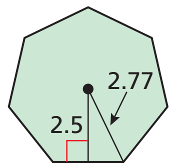

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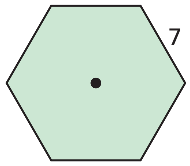
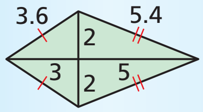


23. an octagon with a radius of 11 units

24. a pentagon with an apothem of 5 units

25. **ERROR ANALYSIS** Describe and correct the error in finding the area of the kite.

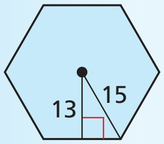


$A = \frac{1}{2}(3.6)(5.4)$

$= 9.72$

So, the area of the kite is 9.72 square units.

26. **ERROR ANALYSIS** Describe and correct the error in finding the area of the regular hexagon.



$s = \sqrt{15^2 - 13^2} \approx 7.5$

$A = \frac{1}{2}a \cdot ns$

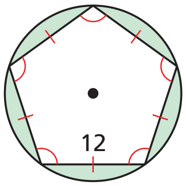
$\approx \frac{1}{2}(13)(6)(7.5)$

$= 292.5$

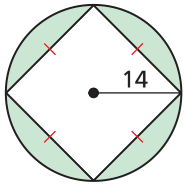
So, the area of the hexagon is about 292.5 square units.

In Exercises 27–30, find the area of the shaded region.

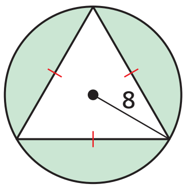
27.



28.



29.



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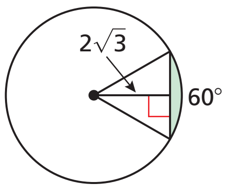
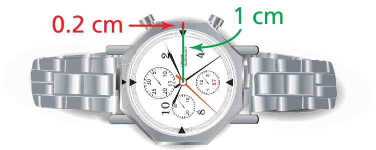


31. **MODELING WITH MATHEMATICS** Basaltic columns are geological formations that result from rapidly cooling lava. Giant's Causeway in Ireland contains many hexagonal basaltic columns. Suppose the top of one of the columns is in the shape of a regular hexagon with a radius of 8 inches. Find the area of the top of the column to the nearest square inch.

32. **MODELING WITH MATHEMATICS** A watch has a circular surface on a background that is a regular octagon. Find the area of the octagon. Then find the area of the silver border around the circular face.



CRITICAL THINKING In Exercises 33–35, tell whether the statement is *true* or *false*. Explain your reasoning.

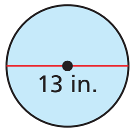
33. The area of a regular n-gon of a fixed radius r increases as n increases.

34. The apothem of a regular polygon is always less than the radius.

35. The radius of a regular polygon is always less than the side length.

36. **REASONING** Predict which figure has the greatest area and which has the least area. Explain your reasoning. Check by finding the area of each figure.

Ⓐ



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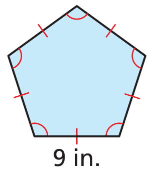

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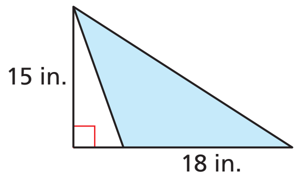


37. **USING EQUATIONS** Find the area of a regular pentagon inscribed in a circle whose equation is given by $(x - 4)^2 + (y + 2)^2 = 25$.

38. **REASONING** What happens to the area of a kite if you double the length of one of the diagonals? if you double the length of both diagonals? Justify your answer.

MATHEMATICAL CONNECTIONS In Exercises 39 and 40, write and solve an equation to find the indicated lengths. Round decimal answers to the nearest tenth.

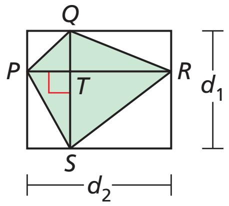
39. The area of a kite is 324 square inches. One diagonal is twice as long as the other diagonal. Find the length of each diagonal.

40. One diagonal of a rhombus is four times the length of the other diagonal. The area of the rhombus is 98 square feet. Find the length of each diagonal.

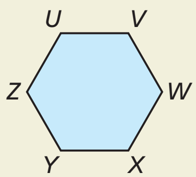
41. **REASONING** The perimeter of a regular nonagon, or 9-gon, is 18 inches. Is this enough information to find the area? If so, find the area and explain your reasoning. If not, explain why not.

42. MAKING AN ARGUMENT Your friend claims that it is possible to find the area of any rhombus if you only know the perimeter of the rhombus. Is your friend correct? Explain your reasoning.

43. PROOF Prove that the area of any quadrilateral with perpendicular diagonals is $A = \frac{1}{2}d_1d_2$, where d_1 and d_2 are the lengths of the diagonals.



44. HOW DO YOU SEE IT? Explain how to find the area of the regular hexagon by dividing the hexagon into equilateral triangles.



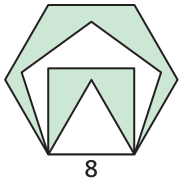
45. REWRITING A FORMULA Rewrite the formula for the area of a rhombus for the special case of a square with side length s. Show that this is the same as the formula for the area of a square, $A = s^2$.

46. REWRITING A FORMULA Use the formula for the area of a regular polygon to show that the area of an equilateral triangle can be found by using the formula $A = \frac{1}{4}s^2\sqrt{3}$, where s is the side length.

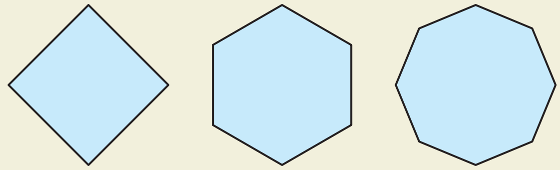
47. CRITICAL THINKING The area of a regular pentagon is 72 square centimeters. Find the length of one side.

48. CRITICAL THINKING The area of a dodecagon, or 12-gon, is 140 square inches. Find the apothem of the polygon.

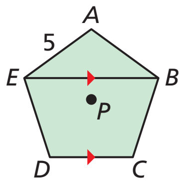
49. USING STRUCTURE In the figure, an equilateral triangle lies inside a square inside a regular pentagon inside a regular hexagon. Find the approximate area of the entire shaded region to the nearest whole number.



50. THOUGHT PROVOKING The area of a regular n-gon is given by $A = \frac{1}{2}aP$. As n approaches infinity, what does the n-gon approach? What does P approach? What does a approach? What can you conclude from your three answers? Explain your reasoning.



51. COMPARING METHODS Find the area of regular pentagon $ABCDE$ by using the formula $A = \frac{1}{2}aP$, or $A = \frac{1}{2}a \bullet ns$. Then find the area by adding the areas of smaller polygons. Check that both methods yield the same area. Which method do you prefer? Explain your reasoning.



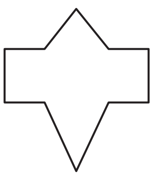
52. USING STRUCTURE Two regular polygons both have n sides. One of the polygons is inscribed in, and the other is circumscribed about, a circle of radius r. Find the area between the two polygons in terms of n and r.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Determine whether the figure has *line symmetry*, *rotational symmetry*, *both*, or *neither*. If the figure has line symmetry, determine the number of lines of symmetry. If the figure has rotational symmetry, describe any rotations that map the figure onto itself. *(Section 4.2 and Section 4.3)*

53.



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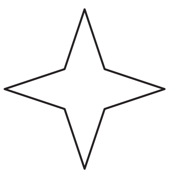

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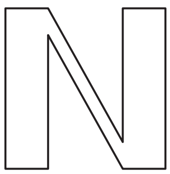

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