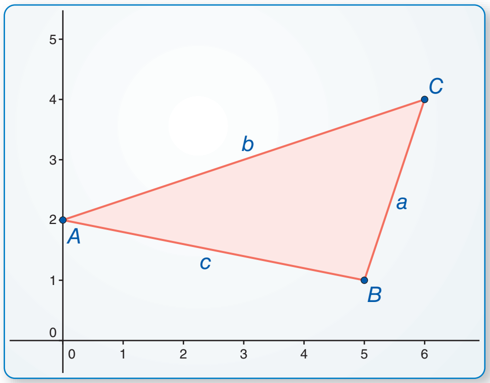
9.7 Law of Sines and Law of Cosines

Essential Question What are the Law of Sines and the Law of Cosines?

EXPLORATION 1 Discovering the Law of Sines

Work with a partner.

a. Copy and complete the table for the triangle shown. What can you conclude?



Sample
Segments
$a = 3.16$
$b = 6.32$
$c = 5.10$
Angles
$m\angle A = 29.74°$
$m\angle B = 97.13°$
$m\angle C = 53.13°$

$m\angle A$	a	$\frac{\sin A}{a}$	$m\angle B$	b	$\frac{\sin B}{b}$	$m\angle C$	c	$\frac{\sin C}{c}$

USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technology to compare predictions with data.

b. Use dynamic geometry software to draw two other triangles. Copy and complete the table in part (a) for each triangle. Use your results to write a conjecture about the relationship between the sines of the angles and the lengths of the sides of a triangle.

EXPLORATION 2 Discovering the Law of Cosines

Work with a partner.

a. Copy and complete the table for the triangle in Exploration 1(a). What can you conclude?

c	c^2	a	a^2	b	b^2	$m\angle C$	$a^2 + b^2 - 2ab \cos C$

b. Use dynamic geometry software to draw two other triangles. Copy and complete the table in part (a) for each triangle. Use your results to write a conjecture about what you observe in the completed tables.

Communicate Your Answer

3. What are the Law of Sines and the Law of Cosines?

4. When would you use the Law of Sines to solve a triangle? When would you use the Law of Cosines to solve a triangle?

9.7 Lesson

Core Vocabulary

Law of Sines, *p. 509*
Law of Cosines, *p. 511*

What You Will Learn

- Find areas of triangles.
- Use the Law of Sines to solve triangles.
- Use the Law of Cosines to solve triangles.

Finding Areas of Triangles

So far, you have used trigonometric ratios to solve right triangles. In this lesson, you will learn how to solve any triangle. When the triangle is obtuse, you may need to find a trigonometric ratio for an obtuse angle.

EXAMPLE 1 **Finding Trigonometric Ratios for Obtuse Angles**

Use a calculator to find each trigonometric ratio. Round your answer to four decimal places.

a. $\tan 150^\circ$ **b.** $\sin 120^\circ$ **c.** $\cos 95^\circ$

SOLUTION

a. $\tan 150^\circ \approx -0.5774$ **b.** $\sin 120^\circ \approx 0.8660$ **c.** $\cos 95^\circ \approx -0.0872$

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

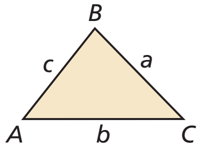
Use a calculator to find the trigonometric ratio. Round your answer to four decimal places.

1. $\tan 110^\circ$ **2.** $\sin 97^\circ$ **3.** $\cos 165^\circ$

Core Concept

Area of a Triangle

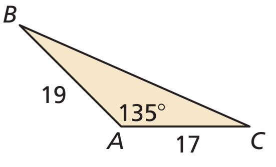
The area of any triangle is given by one-half the product of the lengths of two sides times the sine of their included angle. For $\triangle ABC$ shown, there are three ways to calculate the area.



$\text{Area} = \frac{1}{2}bc \sin A$ $\text{Area} = \frac{1}{2}ac \sin B$ $\text{Area} = \frac{1}{2}ab \sin C$

EXAMPLE 2 **Finding the Area of a Triangle**

Find the area of the triangle. Round your answer to the nearest tenth.



SOLUTION

$$\text{Area} = \frac{1}{2}bc \sin A = \frac{1}{2}(17)(19) \sin 135^\circ \approx 114.2$$

- The area of the triangle is about 114.2 square units.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Find the area of $\triangle ABC$ with the given side lengths and included angle. Round your answer to the nearest tenth.

4. $m\angle B = 60^\circ, a = 19, c = 14$ **5.** $m\angle C = 29^\circ, a = 38, b = 31$

Using the Law of Sines

The trigonometric ratios in the previous sections can only be used to solve right triangles. You will learn two laws that can be used to solve any triangle.

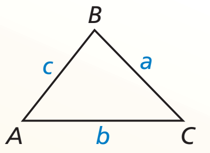
You can use the **Law of Sines** to solve triangles when two angles and the length of any side are known (AAS or ASA cases), or when the lengths of two sides and an angle opposite one of the two sides are known (SSA case).

Theorem

Theorem 9.9 Law of Sines

The Law of Sines can be written in either of the following forms for $\triangle ABC$ with sides of length a, b, and c.

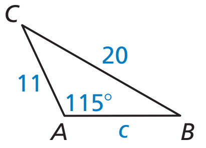
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Proof Ex. 51, p. 516

EXAMPLE 3 Using the Law of Sines (SSA Case)

Solve the triangle. Round decimal answers to the nearest tenth.



SOLUTION

Use the Law of Sines to find $m\angle B$.

$\dfrac{\sin B}{b} = \dfrac{\sin A}{a}$	Law of Sines
$\dfrac{\sin B}{11} = \dfrac{\sin 115^\circ}{20}$	Substitute.
$\sin B = \dfrac{11 \sin 115^\circ}{20}$	Multiply each side by 11.
$m\angle B \approx 29.9^\circ$	Use a calculator.

By the Triangle Sum Theorem (Theorem 5.1), $m\angle C \approx 180^\circ - 115^\circ - 29.9^\circ = 35.1^\circ$.

Use the Law of Sines again to find the remaining side length c of the triangle.

$\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$	Law of Sines
$\dfrac{c}{\sin 35.1^\circ} = \dfrac{20}{\sin 115^\circ}$	Substitute.
$c = \dfrac{20 \sin 35.1^\circ}{\sin 115^\circ}$	Multiply each side by sin 35.1°.
$c \approx 12.7$	Use a calculator.

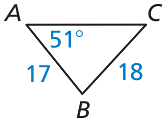
▶ In $\triangle ABC$, $m\angle B \approx 29.9^\circ$, $m\angle C \approx 35.1^\circ$, and $c \approx 12.7$.

Monitoring Progress

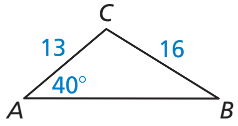
Help in English and Spanish at *BigIdeasMath.com*

Solve the triangle. Round decimal answers to the nearest tenth.

6.

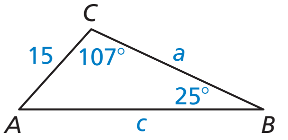


7.



EXAMPLE 4 Using the Law of Sines (AAS Case)

Solve the triangle. Round decimal answers to the nearest tenth.



SOLUTION

By the Triangle Sum Theorem (Theorem 5.1), $m\angle A = 180° - 107° - 25° = 48°$.

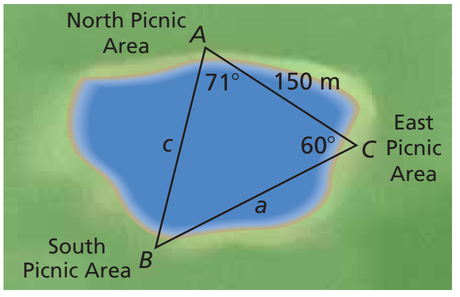
By the Law of Sines, you can write $\frac{a}{\sin 48°} = \frac{15}{\sin 25°} = \frac{c}{\sin 107°}$.

$\frac{a}{\sin 48°} = \frac{15}{\sin 25°}$	Write two equations, each with one variable.	$\frac{c}{\sin 107°} = \frac{15}{\sin 25°}$
$a = \frac{15 \sin 48°}{\sin 25°}$	Solve for each variable.	$c = \frac{15 \sin 107°}{\sin 25°}$
$a \approx 26.4$	Use a calculator.	$c \approx 33.9$

▶ In $\triangle ABC$, $m\angle A = 48°$, $a \approx 26.4$, and $c \approx 33.9$.

EXAMPLE 5 Using the Law of Sines (ASA Case)

A surveyor makes the measurements shown to determine the length of a bridge to be built across a small lake from the North Picnic Area to the South Picnic Area. Find the length of the bridge.



SOLUTION

In the diagram, c represents the distance from the North Picnic Area to the South Picnic Area, so c represents the length of the bridge.

By the Triangle Sum Theorem (Theorem 5.1), $m\angle B = 180° - 71° - 60° = 49°$.

By the Law of Sines, you can write $\frac{a}{\sin 71°} = \frac{150}{\sin 49°} = \frac{c}{\sin 60°}$.

$\frac{c}{\sin 60°} = \frac{150}{\sin 49°}$ Write an equation involving c.

$c = \frac{150 \sin 60°}{\sin 49°}$ Multiply each side by sin 60°.

$c \approx 172.1$ Use a calculator.

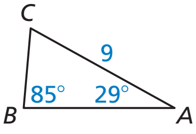
▶ The length of the bridge will be about 172.1 meters.

Monitoring Progress

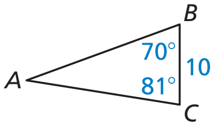
Help in English and Spanish at *BigIdeasMath.com*

Solve the triangle. Round decimal answers to the nearest tenth.

8.



9.



10. WHAT IF? In Example 5, what would be the length of a bridge from the South Picnic Area to the East Picnic Area?

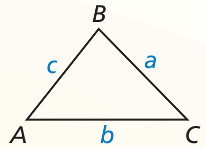
Using the Law of Cosines

You can use the **Law of Cosines** to solve triangles when two sides and the included angle are known (SAS case), or when all three sides are known (SSS case).

Theorem

Theorem 9.10 Law of Cosines

If $\triangle ABC$ has sides of length a, b, and c, as shown, then the following are true.



$$a^2 = b^2 + c^2 - 2bc \cos A$$

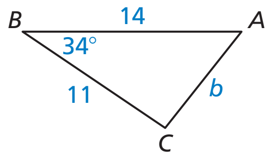
$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Proof Ex. 52, p. 516

EXAMPLE 6 Using the Law of Cosines (SAS Case)

Solve the triangle. Round decimal answers to the nearest tenth.



SOLUTION

Use the Law of Cosines to find side length b.

$b^2 = a^2 + c^2 - 2ac \cos B$	Law of Cosines
$b^2 = 11^2 + 14^2 - 2(11)(14) \cos 34°$	Substitute.
$b^2 = 317 - 308 \cos 34°$	Simplify.
$b = \sqrt{317 - 308 \cos 34°}$	Find the positive square root.
$b \approx 7.9$	Use a calculator.

ANOTHER WAY

When you know all three sides and one angle, you can use the Law of Cosines or the Law of Sines to find the measure of a second angle.

Use the Law of Sines to find $m\angle A$.

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$	Law of Sines
$\dfrac{\sin A}{11} = \dfrac{\sin 34°}{\sqrt{317 - 308 \cos 34°}}$	Substitute.
$\sin A = \dfrac{11 \sin 34°}{\sqrt{317 - 308 \cos 34°}}$	Multiply each side by 11.
$m\angle A \approx 51.6°$	Use a calculator.

By the Triangle Sum Theorem (Theorem 5.1), $m\angle C \approx 180° - 34° - 51.6° = 94.4°$.

▶ In $\triangle ABC$, $b \approx 7.9$, $m\angle A \approx 51.6°$, and $m\angle C \approx 94.4°$.

COMMON ERROR

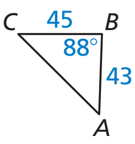
In Example 6, the smaller remaining angle is found first because the inverse sine feature of a calculator only gives angle measures from 0° to 90°. So, when an angle is obtuse, like $\angle C$ because $14^2 > (7.85)^2 + 11^2$, you will not get the obtuse measure.

Monitoring Progress

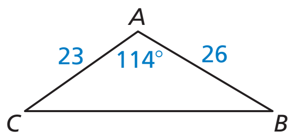
Help in English and Spanish at *BigIdeasMath.com*

Solve the triangle. Round decimal answers to the nearest tenth.

11.

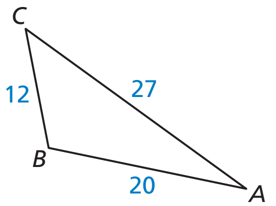


12.



EXAMPLE 7 Using the Law of Cosines (SSS Case)

Solve the triangle. Round decimal answers to the nearest tenth.



SOLUTION

First, find the angle opposite the longest side, $\overline{AC}$. Use the Law of Cosines to find $m\angle B$.

$b^2 = a^2 + c^2 - 2ac \cos B$	Law of Cosines
$27^2 = 12^2 + 20^2 - 2(12)(20) \cos B$	Substitute.
$\dfrac{27^2 - 12^2 - 20^2}{-2(12)(20)} = \cos B$	Solve for cos *B*.
$m\angle B \approx 112.7^\circ$	Use a calculator.

Now, use the Law of Sines to find $m\angle A$.

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$	Law of Sines
$\dfrac{\sin A}{12} = \dfrac{\sin 112.7^\circ}{27}$	Substitute for *a*, *b*, and *B*.
$\sin A = \dfrac{12 \sin 112.7^\circ}{27}$	Multiply each side by 12.
$m\angle A \approx 24.2^\circ$	Use a calculator.

By the Triangle Sum Theorem (Theorem 5.1), $m\angle C \approx 180^\circ - 24.2^\circ - 112.7^\circ = 43.1^\circ$.

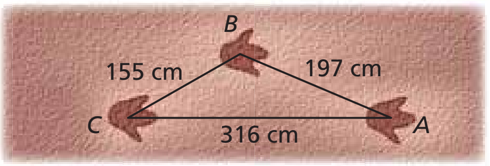
▶ In $\triangle ABC$, $m\angle A \approx 24.2^\circ$, $m\angle B \approx 112.7^\circ$, and $m\angle C \approx 43.1^\circ$.

COMMON ERROR

In Example 7, the largest angle is found first to make sure that the other two angles are acute. This way, when you use the Law of Sines to find another angle measure, you will know that it is between 0° and 90°.

EXAMPLE 8 Solving a Real-Life Problem

An organism's step angle is a measure of walking efficiency. The closer the step angle is to 180°, the more efficiently the organism walked. The diagram shows a set of footprints for a dinosaur. Find the step angle *B*.



SOLUTION

$b^2 = a^2 + c^2 - 2ac \cos B$	Law of Cosines
$316^2 = 155^2 + 197^2 - 2(155)(197) \cos B$	Substitute.
$\dfrac{316^2 - 155^2 - 197^2}{-2(155)(197)} = \cos B$	Solve for cos *B*.
$127.3^\circ \approx m\angle B$	Use a calculator.

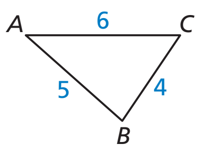
▶ The step angle *B* is about 127.3°.

Monitoring Progress

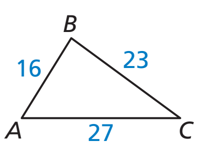
Help in English and Spanish at *BigIdeasMath.com*

Solve the triangle. Round decimal answers to the nearest tenth.

13.



14.



9.7 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** What type of triangle would you use the Law of Sines or the Law of Cosines to solve?

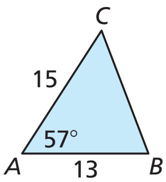
2. **VOCABULARY** What information do you need to use the Law of Sines?

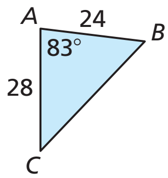
Monitoring Progress and Modeling with Mathematics

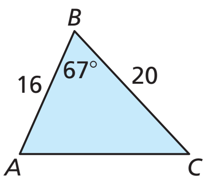
In Exercises 3–8, use a calculator to find the trigonometric ratio. Round your answer to four decimal places. *(See Example 1.)*

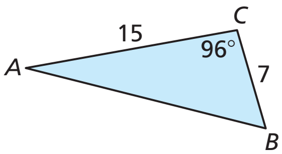
3. sin 127°
4. sin 98°
5. cos 139°
6. cos 108°
7. tan 165°
8. tan 116°

In Exercises 9–12, find the area of the triangle. Round your answer to the nearest tenth. *(See Example 2.)*

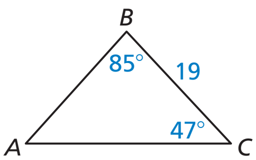
9.


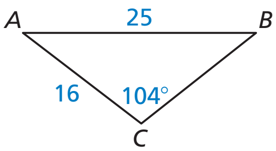
10.


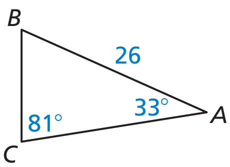
11.


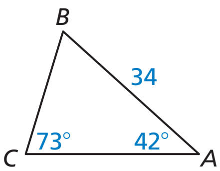
12.


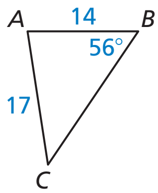
In Exercises 13–18, solve the triangle. Round decimal answers to the nearest tenth. *(See Examples 3, 4, and 5.)*

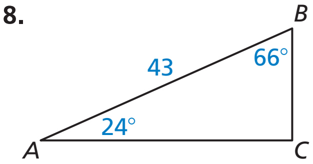
13.


14.


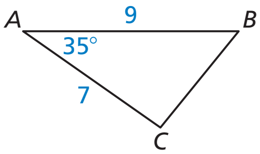
15.


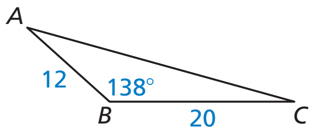
16.


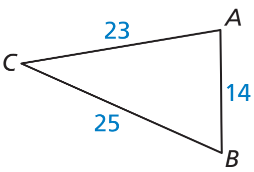
17.


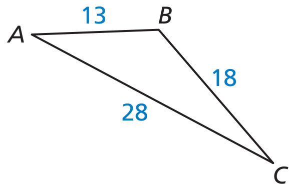
18.


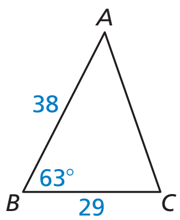
In Exercises 19–24, solve the triangle. Round decimal answers to the nearest tenth. *(See Examples 6 and 7.)*

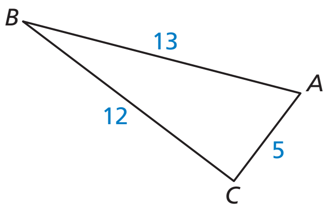
19.


20.


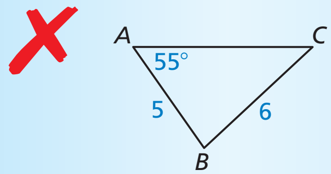
21.


22.


23.


24.


25. **ERROR ANALYSIS** Describe and correct the error in finding $m\angle C$.



$$\frac{\sin C}{6} = \frac{\sin 55°}{5}$$

$$\sin C = \frac{6 \sin 55°}{5}$$

$$m\angle C \approx 79.4°$$

26. **ERROR ANALYSIS** Describe and correct the error in finding $m\angle A$ in $\triangle ABC$ when $a = 19$, $b = 21$, and $c = 11$.

$$\cos A = \frac{19^2 - 21^2 - 11^2}{-2(19)(21)}$$

$$m\angle A \approx 75.4°$$

COMPARING METHODS **In Exercises 27–32, tell whether you would use the Law of Sines, the Law of Cosines, or the Pythagorean Theorem (Theorem 9.1) and trigonometric ratios to solve the triangle with the given information. Explain your reasoning. Then solve the triangle.**

27. $m\angle A = 72°$, $m\angle B = 44°$, $b = 14$

28. $m\angle B = 98°$, $m\angle C = 37°$, $a = 18$

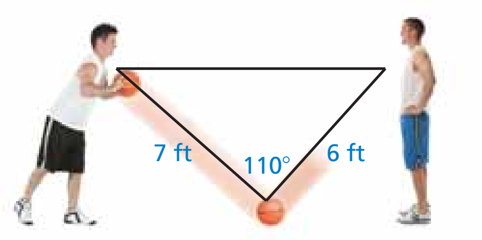
29. $m\angle C = 65°$, $a = 12$, $b = 21$

30. $m\angle B = 90°$, $a = 15$, $c = 6$

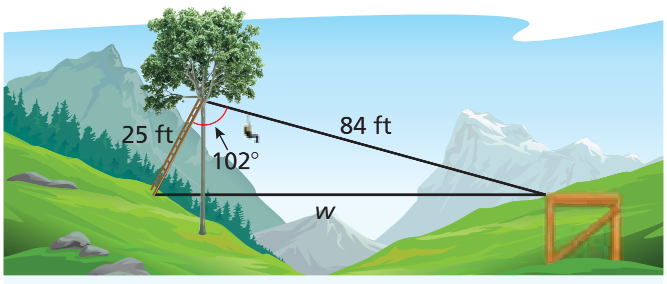
31. $m\angle C = 40°$, $b = 27$, $c = 36$

32. $a = 34$, $b = 19$, $c = 27$

33. **MODELING WITH MATHEMATICS** You and your friend are standing on the baseline of a basketball court. You bounce a basketball to your friend, as shown in the diagram. What is the distance between you and your friend? *(See Example 8.)*



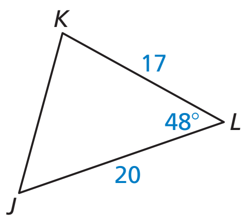
34. **MODELING WITH MATHEMATICS** A zip line is constructed across a valley, as shown in the diagram. What is the width w of the valley?



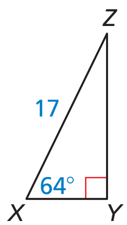
35. **MODELING WITH MATHEMATICS** You are on the observation deck of the Empire State Building looking at the Chrysler Building. When you turn 145° clockwise, you see the Statue of Liberty. You know that the Chrysler Building and the Empire State Building are about 0.6 mile apart and that the Chrysler Building and the Statue of Liberty are about 5.6 miles apart. Estimate the distance between the Empire State Building and the Statue of Liberty.

36. **MODELING WITH MATHEMATICS** The Leaning Tower of Pisa in Italy has a height of 183 feet and is 4° off vertical. Find the horizontal distance d that the top of the tower is off vertical.

37. **MAKING AN ARGUMENT** Your friend says that the Law of Sines can be used to find JK. Your cousin says that the Law of Cosines can be used to find JK. Who is correct? Explain your reasoning.



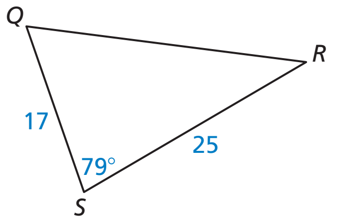
38. **REASONING** Use $\triangle XYZ$.



a. Can you use the Law of Sines to solve $\triangle XYZ$? Explain your reasoning.

b. Can you use another method to solve $\triangle XYZ$? Explain your reasoning.

39. **MAKING AN ARGUMENT** Your friend calculates the area of the triangle using the formula $A = \frac{1}{2}qr \sin S$ and says that the area is approximately 208.6 square units. Is your friend correct? Explain your reasoning.



40. **MODELING WITH MATHEMATICS** You are fertilizing a triangular garden. One side of the garden is 62 feet long, and another side is 54 feet long. The angle opposite the 62-foot side is 58°.

a. Draw a diagram to represent this situation.

b. Use the Law of Sines to solve the triangle from part (a).

c. One bag of fertilizer covers an area of 200 square feet. How many bags of fertilizer will you need to cover the entire garden?

41. **MODELING WITH MATHEMATICS** A golfer hits a drive 260 yards on a hole that is 400 yards long. The shot is 15° off target.

Not drawn to scale

a. What is the distance x from the golfer's ball to the hole?

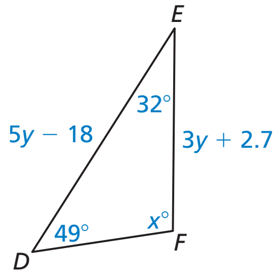
b. Assume the golfer is able to hit the ball precisely the distance found in part (a). What is the maximum angle θ (theta) by which the ball can be off target in order to land no more than 10 yards from the hole?

42. **COMPARING METHODS** A building is constructed on top of a cliff that is 300 meters high. A person standing on level ground below the cliff observes that the angle of elevation to the top of the building is 72° and the angle of elevation to the top of the cliff is 63°.

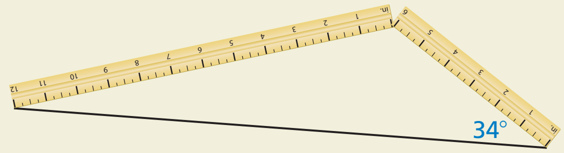
a. How far away is the person from the base of the cliff?

b. Describe two different methods you can use to find the height of the building. Use one of these methods to find the building's height.

43. **MATHEMATICAL CONNECTIONS** Find the values of x and y.



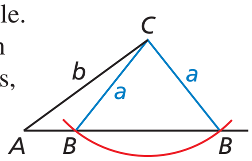
44. **HOW DO YOU SEE IT?** Would you use the Law of Sines or the Law of Cosines to solve the triangle?



45. **REWRITING A FORMULA** Simplify the Law of Cosines for when the given angle is a right angle.

46. **THOUGHT PROVOKING** Consider any triangle with side lengths of a, b, and c. Calculate the value of s, which is half the perimeter of the triangle. What measurement of the triangle is represented by $\sqrt{s(s-a)(s-b)(s-c)}$?

47. **ANALYZING RELATIONSHIPS** The *ambiguous case* of the Law of Sines occurs when you are given the measure of one acute angle, the length of one adjacent side, and the length of the side opposite that angle, which is less than the length of the adjacent side. This results in two possible triangles. Using the given information, find two possible solutions for $\triangle ABC$. Draw a diagram for each triangle. (*Hint*: The inverse sine function gives only acute angle measures, so consider the acute angle and its supplement for $\angle B$.)



a. $m\angle A = 40°$, $a = 13$, $b = 16$

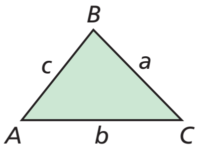
b. $m\angle A = 21°$, $a = 17$, $b = 32$

48. **ABSTRACT REASONING** Use the Law of Cosines to show that the measure of each angle of an equilateral triangle is 60°. Explain your reasoning.

49. **CRITICAL THINKING** An airplane flies 55° east of north from City A to City B, a distance of 470 miles. Another airplane flies 7° north of east from City A to City C, a distance of 890 miles. What is the distance between Cities B and C?

50. REWRITING A FORMULA Follow the steps to derive the formula for the area of a triangle,

Area $= \frac{1}{2}ab \sin C$.



a. Draw the altitude from vertex B to $\overline{AC}$. Label the altitude as h. Write a formula for the area of the triangle using h.

b. Write an equation for $\sin C$.

c. Use the results of parts (a) and (b) to write a formula for the area of a triangle that does not include h.

51. PROVING A THEOREM Follow the steps to use the formula for the area of a triangle to prove the Law of Sines (Theorem 9.9).

a. Use the derivation in Exercise 50 to explain how to derive the three related formulas for the area of a triangle.

$$\text{Area} = \frac{1}{2}bc \sin A,$$
$$\text{Area} = \frac{1}{2}ac \sin B,$$
$$\text{Area} = \frac{1}{2}ab \sin C$$

b. Why can you use the formulas in part (a) to write the following statement?

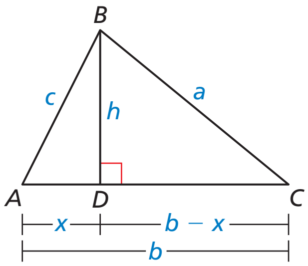
$$\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$$

c. Show how to rewrite the statement in part (b) to prove the Law of Sines. Justify each step.

52. PROVING A THEOREM Use the given information to complete the two-column proof of the Law of Cosines (Theorem 9.10).

Given $\overline{BD}$ is an altitude of $\triangle ABC$.

Prove $a^2 = b^2 + c^2 - 2bc \cos A$



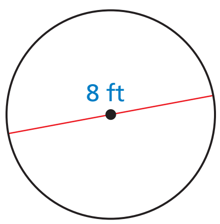
STATEMENTS	REASONS
1. $\overline{BD}$ is an altitude of $\triangle ABC$.	**1.** Given
2. $\triangle ADB$ and $\triangle CDB$ are right triangles.	**2.** ____________
3. $a^2 = (b - x)^2 + h^2$	**3.** ____________
4. ____________	**4.** Expand binomial.
5. $x^2 + h^2 = c^2$	**5.** ____________
6. ____________	**6.** Substitution Property of Equality
7. $\cos A = \frac{x}{c}$	**7.** ____________
8. $x = c \cos A$	**8.** ____________
9. $a^2 = b^2 + c^2 - 2bc \cos A$	**9.** ____________

Maintaining Mathematical Proficiency

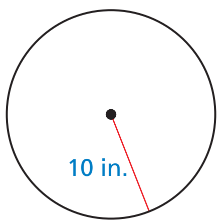
Reviewing what you learned in previous grades and lessons

Find the radius and diameter of the circle. *(Skills Review Handbook)*

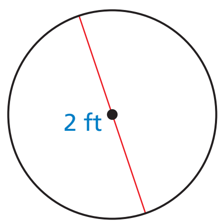
53.



54.



55.



56.