9.5 The Sine and Cosine Ratios

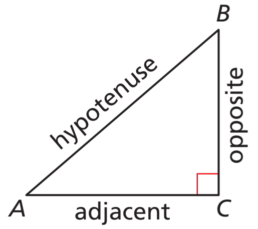
Essential Question

How is a right triangle used to find the sine and cosine of an acute angle? Is there a unique right triangle that must be used?

Let $\triangle ABC$ be a right triangle with acute $\angle A$. The *sine* of $\angle A$ and *cosine* of $\angle A$ (written as sin A and cos A, respectively) are defined as follows.

$$\sin A = \frac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}} = \frac{BC}{AB}$$

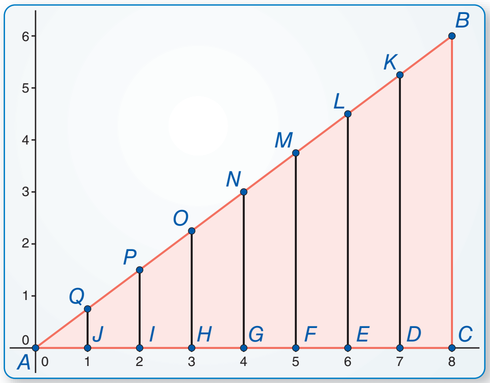
$$\cos A = \frac{\text{length of leg adjacent to } \angle A}{\text{length of hypotenuse}} = \frac{AC}{AB}$$



EXPLORATION 1 Calculating Sine and Cosine Ratios

Work with a partner. Use dynamic geometry software.

a. Construct $\triangle ABC$, as shown. Construct segments perpendicular to $\overline{AC}$ to form right triangles that share vertex A and are similar to $\triangle ABC$ with vertices, as shown.



Sample
Points
$A(0, 0)$
$B(8, 6)$
$C(8, 0)$
Angle
$m\angle BAC = 36.87°$

b. Calculate each given ratio to complete the table for the decimal values of sin A and cos A for each right triangle. What can you conclude?

Sine ratio	$\frac{BC}{AB}$	$\frac{KD}{AK}$	$\frac{LE}{AL}$	$\frac{MF}{AM}$	$\frac{NG}{AN}$	$\frac{OH}{AO}$	$\frac{PI}{AP}$	$\frac{QJ}{AQ}$
sin *A*								
Cosine ratio	$\frac{AC}{AB}$	$\frac{AD}{AK}$	$\frac{AE}{AL}$	$\frac{AF}{AM}$	$\frac{AG}{AN}$	$\frac{AH}{AO}$	$\frac{AI}{AP}$	$\frac{AJ}{AQ}$
cos *A*								

LOOKING FOR STRUCTURE

To be proficient in math, you need to look closely to discern a pattern or structure.

Communicate Your Answer

2. How is a right triangle used to find the sine and cosine of an acute angle? Is there a unique right triangle that must be used?

3. In Exploration 1, what is the relationship between $\angle A$ and $\angle B$ in terms of their measures? Find sin B and cos B. How are these two values related to sin A and cos A? Explain why these relationships exist.

9.5 Lesson

Core Vocabulary

sine, *p. 494*
cosine, *p. 494*
angle of depression, *p. 497*

What You Will Learn

- Use the sine and cosine ratios.
- Find the sine and cosine of angle measures in special right triangles.
- Solve real-life problems involving sine and cosine ratios.

Using the Sine and Cosine Ratios

The **sine** and **cosine** ratios are trigonometric ratios for acute angles that involve the lengths of a leg and the hypotenuse of a right triangle.

READING

Remember the following abbreviations.

sine $\rightarrow$ sin
cosine $\rightarrow$ cos
hypotenuse $\rightarrow$ hyp.

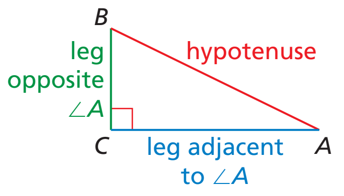
Core Concept

Sine and Cosine Ratios

Let $\triangle ABC$ be a right triangle with acute $\angle A$. The sine of $\angle A$ and cosine of $\angle A$ (written as $\sin A$ and $\cos A$) are defined as follows.

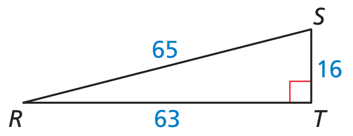
$$\sin A = \frac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}} = \frac{BC}{AB}$$

$$\cos A = \frac{\text{length of leg adjacent to } \angle A}{\text{length of hypotenuse}} = \frac{AC}{AB}$$



EXAMPLE 1 **Finding Sine and Cosine Ratios**

Find $\sin S$, $\sin R$, $\cos S$, and $\cos R$. Write each answer as a fraction and as a decimal rounded to four places.



SOLUTION

$$\sin S = \frac{\text{opp. } \angle S}{\text{hyp.}} = \frac{RT}{SR} = \frac{63}{65} \approx 0.9692 \qquad \sin R = \frac{\text{opp. } \angle R}{\text{hyp.}} = \frac{ST}{SR} = \frac{16}{65} \approx 0.2462$$

$$\cos S = \frac{\text{adj. to } \angle S}{\text{hyp.}} = \frac{ST}{SR} = \frac{16}{65} \approx 0.2462 \qquad \cos R = \frac{\text{adj. to } \angle R}{\text{hyp.}} = \frac{RT}{SR} = \frac{63}{65} \approx 0.9692$$

In Example 1, notice that $\sin S = \cos R$ and $\sin R = \cos S$. This is true because the side opposite $\angle S$ is adjacent to $\angle R$ and the side opposite $\angle R$ is adjacent to $\angle S$. The relationship between the sine and cosine of $\angle S$ and $\angle R$ is true for all complementary angles.

Core Concept

Sine and Cosine of Complementary Angles

The sine of an acute angle is equal to the cosine of its complement. The cosine of an acute angle is equal to the sine of its complement.

Let A and B be complementary angles. Then the following statements are true.

$\sin A = \cos(90° - A) = \cos B \qquad \sin B = \cos(90° - B) = \cos A$

$\cos A = \sin(90° - A) = \sin B \qquad \cos B = \sin(90° - B) = \sin A$

EXAMPLE 2 Rewriting Trigonometric Expressions

Write $\sin 56^\circ$ in terms of cosine.

SOLUTION

Use the fact that the sine of an acute angle is equal to the cosine of its complement.

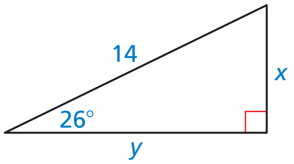
$$\sin 56^\circ = \cos(90^\circ - 56^\circ) = \cos 34^\circ$$

▶ The sine of 56° is the same as the cosine of 34°.

You can use the sine and cosine ratios to find unknown measures in right triangles.

EXAMPLE 3 Finding Leg Lengths

Find the values of x and y using sine and cosine. Round your answers to the nearest tenth.



SOLUTION

Step 1 Use a sine ratio to find the value of x.

$$\sin 26^\circ = \frac{\text{opp.}}{\text{hyp.}}$$ Write ratio for sine of 26°.

$$\sin 26^\circ = \frac{x}{14}$$ Substitute.

$$14 \cdot \sin 26^\circ = x$$ Multiply each side by 14.

$$6.1 \approx x$$ Use a calculator.

▶ The value of x is about 6.1.

Step 2 Use a cosine ratio to find the value of y.

$$\cos 26^\circ = \frac{\text{adj.}}{\text{hyp.}}$$ Write ratio for cosine of 26°.

$$\cos 26^\circ = \frac{y}{14}$$ Substitute.

$$14 \cdot \cos 26^\circ = y$$ Multiply each side by 14.

$$12.6 \approx y$$ Use a calculator.

▶ The value of y is about 12.6.

Monitoring Progress

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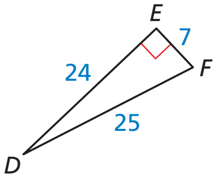
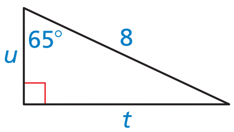
1. Find $\sin D$, $\sin F$, $\cos D$, and $\cos F$. Write each answer as a fraction and as a decimal rounded to four places.



2. Write $\cos 23^\circ$ in terms of sine.

3. Find the values of u and t using sine and cosine. Round your answers to the nearest tenth.



Finding Sine and Cosine in Special Right Triangles

EXAMPLE 4 Finding the Sine and Cosine of 45°

Find the sine and cosine of a 45° angle.

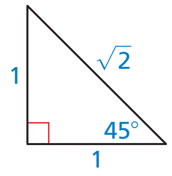
SOLUTION

Begin by sketching a 45°-45°-90° triangle. Because all such triangles are similar, you can simplify your calculations by choosing 1 as the length of each leg. Using the 45°-45°-90° Triangle Theorem (Theorem 9.4), the length of the hypotenuse is $\sqrt{2}$.

STUDY TIP

Notice that

$$\begin{aligned}\sin 45° &= \cos(90 - 45)° \\ &= \cos 45°.\end{aligned}$$



$$\sin 45° = \frac{\text{opp.}}{\text{hyp.}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.7071$$

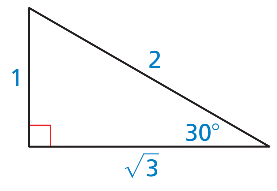
$$\cos 45° = \frac{\text{adj.}}{\text{hyp.}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.7071$$

EXAMPLE 5 Finding the Sine and Cosine of 30°

Find the sine and cosine of a 30° angle.

SOLUTION

Begin by sketching a 30°-60°-90° triangle. Because all such triangles are similar, you can simplify your calculations by choosing 1 as the length of the shorter leg. Using the 30°-60°-90° Triangle Theorem (Theorem 9.5), the length of the longer leg is $\sqrt{3}$ and the length of the hypotenuse is 2.



$$\sin 30° = \frac{\text{opp.}}{\text{hyp.}} = \frac{1}{2} = 0.5000$$

$$\cos 30° = \frac{\text{adj.}}{\text{hyp.}} = \frac{\sqrt{3}}{2} \approx 0.8660$$

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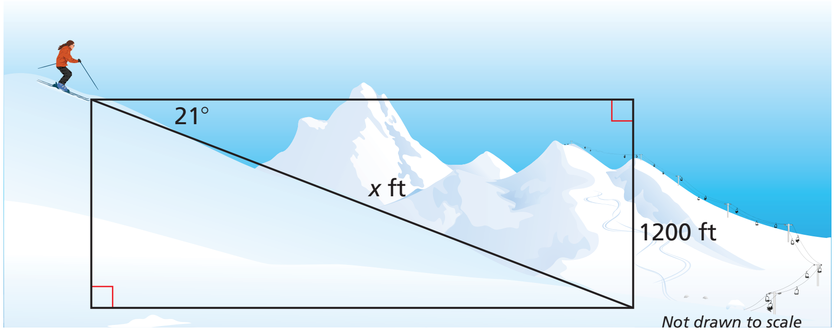
4. Find the sine and cosine of a 60° angle.

Solving Real-Life Problems

Recall from the previous lesson that the angle an upward line of sight makes with a horizontal line is called the *angle of elevation*. The angle that a downward line of sight makes with a horizontal line is called the **angle of depression**.

EXAMPLE 6 Modeling with Mathematics

You are skiing on a mountain with an altitude of 1200 feet. The angle of depression is 21°. Find the distance x you ski down the mountain to the nearest foot.



SOLUTION

1. **Understand the Problem** You are given the angle of depression and the altitude of the mountain. You need to find the distance that you ski down the mountain.
2. **Make a Plan** Write a trigonometric ratio for the sine of the angle of depression involving the distance x. Then solve for x.
3. **Solve the Problem**

$$\sin 21^\circ = \frac{\text{opp.}}{\text{hyp.}} \quad \text{Write ratio for sine of } 21^\circ.$$

$$\sin 21^\circ = \frac{1200}{x} \quad \text{Substitute.}$$

$$x \cdot \sin 21^\circ = 1200 \quad \text{Multiply each side by } x.$$

$$x = \frac{1200}{\sin 21^\circ} \quad \text{Divide each side by } \sin 21^\circ.$$

$$x \approx 3348.5 \quad \text{Use a calculator.}$$

▶ You ski about 3349 feet down the mountain.

4. **Look Back** Check your answer. The value of sin 21° is about 0.3584. Substitute for x in the sine ratio and compare the values.

$$\frac{1200}{x} \approx \frac{1200}{3348.5}$$

$$\approx 0.3584$$

This value is approximately the same as the value of sin 21°. ✓

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5. **WHAT IF?** In Example 6, the angle of depression is 28°. Find the distance x you ski down the mountain to the nearest foot.

9.5 Exercises

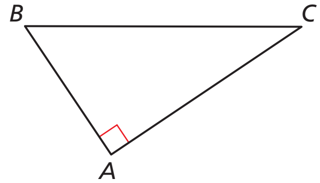
Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. VOCABULARY The sine ratio compares the length of ____________ to the length of _____________.

2. WHICH ONE DOESN'T BELONG? Which ratio does *not* belong with the other three? Explain your reasoning.

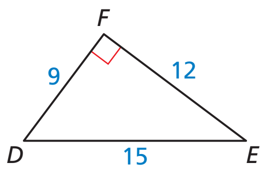
$\sin B$	$\cos C$
$\tan B$	$\frac{AC}{BC}$



Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, find sin *D*, sin *E*, cos *D*, and cos *E*. Write each answer as a fraction and as a decimal rounded to four places. *(See Example 1.)*

3.



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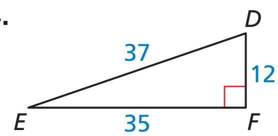

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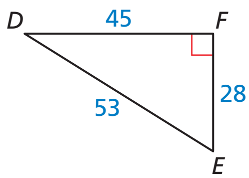

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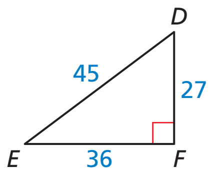

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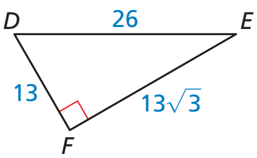

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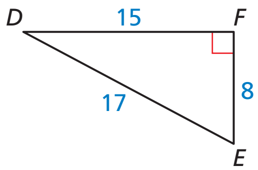


In Exercises 9–12, write the expression in terms of cosine. *(See Example 2.)*

9. $\sin 37^\circ$ **10.** $\sin 81^\circ$

11. $\sin 29^\circ$ **12.** $\sin 64^\circ$

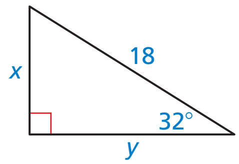
In Exercises 13–16, write the expression in terms of sine.

13. $\cos 59^\circ$ **14.** $\cos 42^\circ$

15. $\cos 73^\circ$ **16.** $\cos 18^\circ$

In Exercises 17–22, find the value of each variable using sine and cosine. Round your answers to the nearest tenth. *(See Example 3.)*

17.



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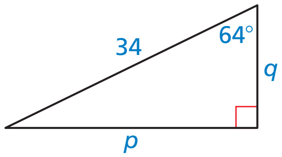

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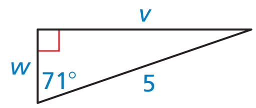

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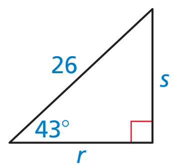

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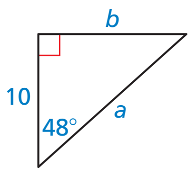

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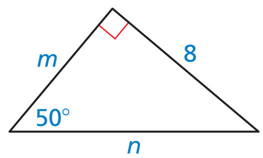
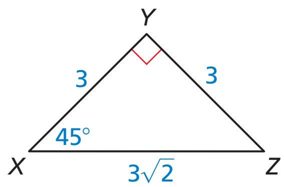
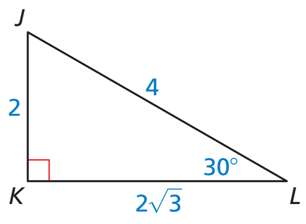


23. REASONING Which ratios are equal? Select all that apply. *(See Example 4.)*



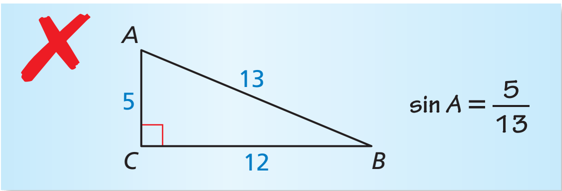
$\sin X$	$\cos X$	$\sin Z$	$\cos Z$

24. REASONING Which ratios are equal to $\frac{1}{2}$? Select all that apply. *(See Example 5.)*



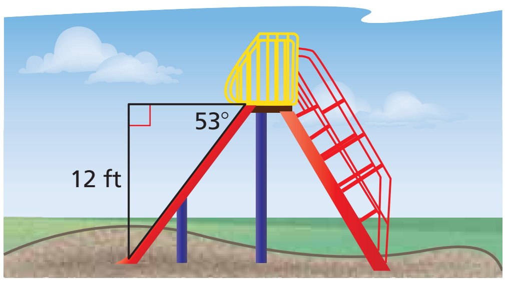
$\sin L$	$\cos L$	$\sin J$	$\cos J$

25. ERROR ANALYSIS Describe and correct the error in finding $\sin A$.

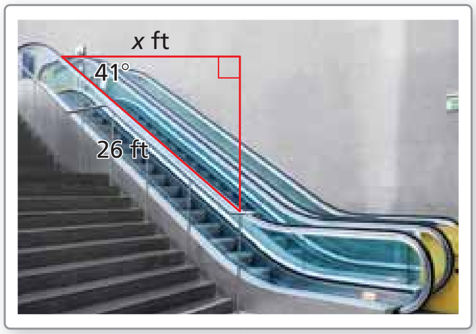


26. WRITING Explain how to tell which side of a right triangle is adjacent to an angle and which side is the hypotenuse.

27. MODELING WITH MATHEMATICS The top of the slide is 12 feet from the ground and has an angle of depression of 53°. What is the length of the slide? *(See Example 6.)*



28. MODELING WITH MATHEMATICS Find the horizontal distance x the escalator covers.

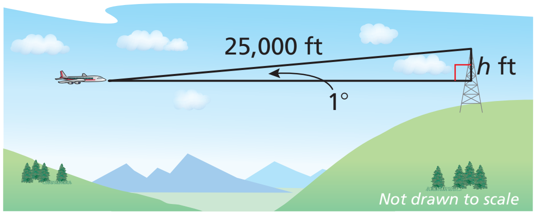


29. PROBLEM SOLVING You are flying a kite with 20 feet of string extended. The angle of elevation from the spool of string to the kite is 67°.

a. Draw and label a diagram that represents the situation.

b. How far off the ground is the kite if you hold the spool 5 feet off the ground? Describe how the height where you hold the spool affects the height of the kite.

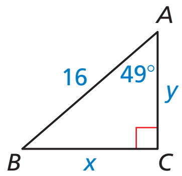
30. MODELING WITH MATHEMATICS Planes that fly at high speeds and low elevations have radar systems that can determine the range of an obstacle and the angle of elevation to the top of the obstacle. The radar of a plane flying at an altitude of 20,000 feet detects a tower that is 25,000 feet away, with an angle of elevation of 1°.



a. How many feet must the plane rise to pass over the tower?

b. Planes cannot come closer than 1000 feet vertically to any object. At what altitude must the plane fly in order to pass over the tower?

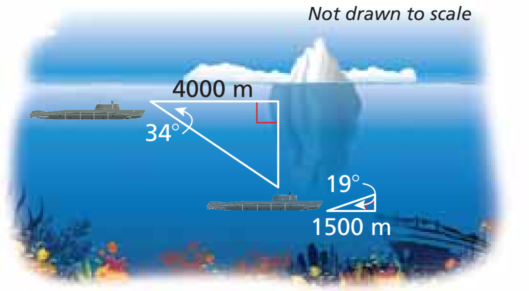
31. MAKING AN ARGUMENT Your friend uses the equation $\sin 49° = \frac{x}{16}$ to find BC. Your cousin uses the equation $\cos 41° = \frac{x}{16}$ to find BC. Who is correct? Explain your reasoning.



32. WRITING Describe what you must know about a triangle in order to use the sine ratio and what you must know about a triangle in order to use the cosine ratio.

33. MATHEMATICAL CONNECTIONS If $\triangle EQU$ is equilateral and $\triangle RGT$ is a right triangle with $RG = 2$, $RT = 1$, and $m\angle T = 90°$, show that $\sin E = \cos G$.

34. MODELING WITH MATHEMATICS Submarines use sonar systems, which are similar to radar systems, to detect obstacles. Sonar systems use sound to detect objects under water.

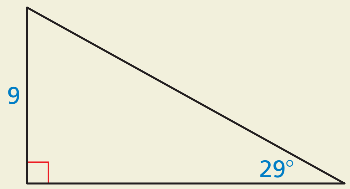


a. You are traveling underwater in a submarine. The sonar system detects an iceberg 4000 meters ahead, with an angle of depression of 34° to the bottom of the iceberg. How many meters must the submarine lower to pass under the iceberg?

b. The sonar system then detects a sunken ship 1500 meters ahead, with an angle of elevation of 19° to the highest part of the sunken ship. How many meters must the submarine rise to pass over the sunken ship?

35. ABSTRACT REASONING Make a conjecture about how you could use trigonometric ratios to find angle measures in a triangle.

36. HOW DO YOU SEE IT? Using only the given information, would you use a sine ratio or a cosine ratio to find the length of the hypotenuse? Explain your reasoning.



37. MULTIPLE REPRESENTATIONS You are standing on a cliff above an ocean. You see a sailboat from your vantage point 30 feet above the ocean.

a. Draw and label a diagram of the situation.

b. Make a table showing the angle of depression and the length of your line of sight. Use the angles 40°, 50°, 60°, 70°, and 80°.

c. Graph the values you found in part (b), with the angle measures on the x-axis.

d. Predict the length of the line of sight when the angle of depression is 30°.

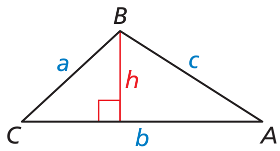
38. THOUGHT PROVOKING One of the following infinite series represents $\sin x$ and the other one represents $\cos x$ (where x is measured in radians). Which is which? Justify your answer. Then use each series to approximate the sine and cosine of $\frac{\pi}{6}$. (*Hints*: $\pi = 180°$; $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$; Find the values that the sine and cosine ratios approach as the angle measure approaches zero.)

a. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

b. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

39. CRITICAL THINKING Let A be any acute angle of a right triangle. Show that (a) $\tan A = \frac{\sin A}{\cos A}$ and (b) $(\sin A)^2 + (\cos A)^2 = 1$.

40. CRITICAL THINKING Explain why the area of $\triangle ABC$ in the diagram can be found using the formula Area $= \frac{1}{2}ab \sin C$. Then calculate the area when $a = 4$, $b = 7$, and $m\angle C = 40°$.

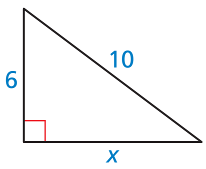


Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the value of x. Tell whether the side lengths form a Pythagorean triple. *(Section 9.1)*

41.



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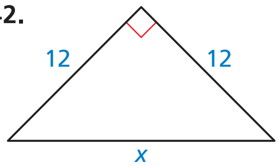
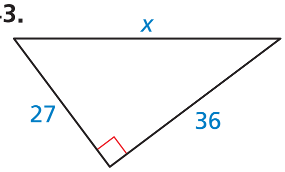


43.



44.