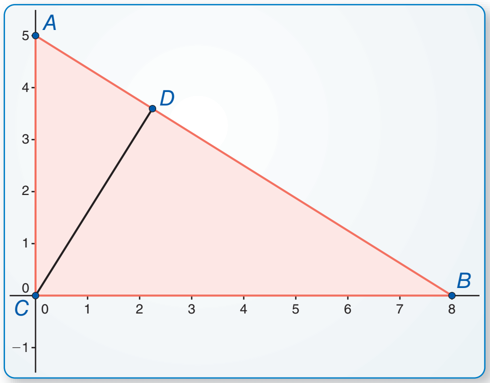
9.3 Similar Right Triangles

Essential Question How are altitudes and geometric means of right triangles related?

EXPLORATION 1 Writing a Conjecture

Work with a partner.

a. Use dynamic geometry software to construct right $\triangle ABC$, as shown. Draw $\overline{CD}$ so that it is an altitude from the right angle to the hypotenuse of $\triangle ABC$.



Points
$A(0, 5)$
$B(8, 0)$
$C(0, 0)$
$D(2.25, 3.6)$
Segments
$AB = 9.43$
$BC = 8$
$AC = 5$

b. The geometric mean of two positive numbers a and b is the positive number x that satisfies

$$\frac{a}{x} = \frac{x}{b}. \qquad x \text{ is the geometric mean of } a \text{ and } b.$$

Write a proportion involving the side lengths of $\triangle CBD$ and $\triangle ACD$ so that CD is the geometric mean of two of the other side lengths. Use similar triangles to justify your steps.

c. Use the proportion you wrote in part (b) to find CD.

d. Generalize the proportion you wrote in part (b). Then write a conjecture about how the geometric mean is related to the altitude from the right angle to the hypotenuse of a right triangle.

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results in constructing arguments.

EXPLORATION 2 Comparing Geometric and Arithmetic Means

Work with a partner. Use a spreadsheet to find the arithmetic mean and the geometric mean of several pairs of positive numbers. Compare the two means. What do you notice?

	A	B	C	D
1	a	b	Arithmetic Mean	Geometric Mean
2	3	4	3.5	3.464
3	4	5		
4	6	7		
5	0.5	0.5		
6	0.4	0.8		
7	2	5		
8	1	4		
9	9	16		
10	10	100		
11				

Communicate Your Answer

3. How are altitudes and geometric means of right triangles related?

9.3 Lesson

What You Will Learn

- Identify similar triangles.
- Solve real-life problems involving similar triangles.
- Use geometric means.

Core Vocabulary

geometric mean, *p. 480*

Previous
altitude of a triangle
similar figures

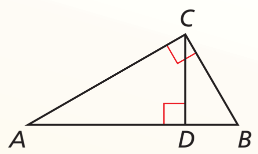
Identifying Similar Triangles

When the altitude is drawn to the hypotenuse of a right triangle, the two smaller triangles are similar to the original triangle and to each other.

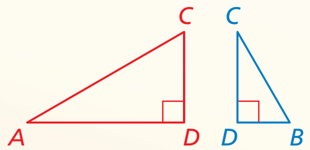
Theorem

Theorem 9.6 Right Triangle Similarity Theorem

If the altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and to each other.



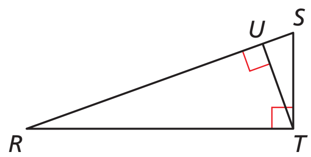
$\triangle CBD \sim \triangle ABC$, $\triangle ACD \sim \triangle ABC$, and $\triangle CBD \sim \triangle ACD$.



Proof Ex. 45, p. 484

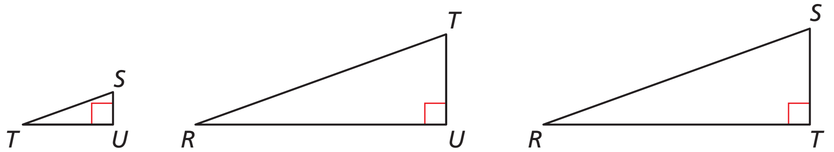
EXAMPLE 1 Identifying Similar Triangles

Identify the similar triangles in the diagram.



SOLUTION

Sketch the three similar right triangles so that the corresponding angles and sides have the same orientation.



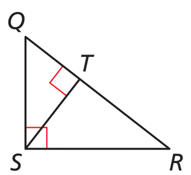
- $\triangle TSU \sim \triangle RTU \sim \triangle RST$

Monitoring Progress

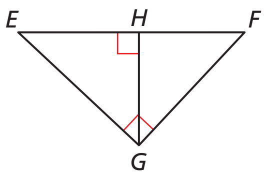
Help in English and Spanish at *BigIdeasMath.com*

Identify the similar triangles.

1.



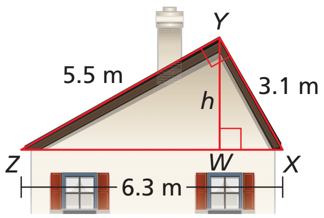
2.



Solving Real-Life Problems

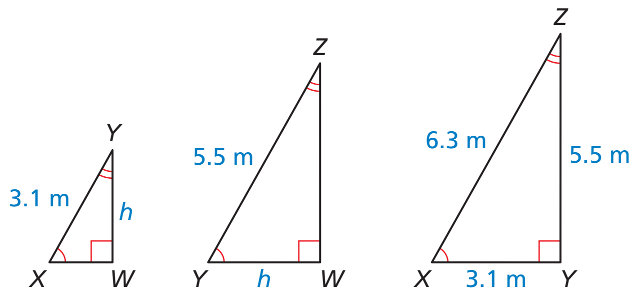
EXAMPLE 2 **Modeling with Mathematics**

A roof has a cross section that is a right triangle. The diagram shows the approximate dimensions of this cross section. Find the height *h* of the roof.



SOLUTION

1. **Understand the Problem** You are given the side lengths of a right triangle. You need to find the height of the roof, which is the altitude drawn to the hypotenuse.
2. **Make a Plan** Identify any similar triangles. Then use the similar triangles to write a proposition involving the height and solve for *h*.
3. **Solve the Problem** Identify the similar triangles and sketch them.



$\triangle XYW \sim \triangle YZW \sim \triangle XZY$

COMMON ERROR

Notice that if you tried to write a proportion using $\triangle XYW$ and $\triangle YZW$, then there would be two unknowns, so you would not be able to solve for *h*.

Because $\triangle XYW \sim \triangle XZY$, you can write a proportion.

$$\frac{YW}{ZY} = \frac{XY}{XZ}$$ Corresponding side lengths of similar triangles are proportional.

$$\frac{h}{5.5} = \frac{3.1}{6.3}$$ Substitute.

$$h \approx 2.7$$ Multiply each side by 5.5.

▶ The height of the roof is about 2.7 meters.

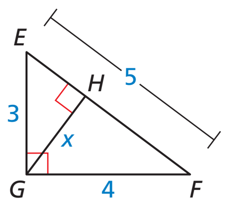
4. **Look Back** Because the height of the roof is a leg of right $\triangle YZW$ and right $\triangle XYW$, it should be shorter than each of their hypotenuses. The lengths of the two hypotenuses are $YZ = 5.5$ and $XY = 3.1$. Because $2.7 < 3.1$, the answer seems reasonable.

Monitoring Progress

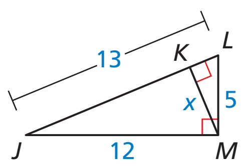
Help in English and Spanish at *BigIdeasMath.com*

Find the value of *x*.

3.



4.



Using a Geometric Mean

Core Concept

Geometric Mean

The **geometric mean** of two positive numbers a and b is the positive number x that satisfies $\frac{a}{x} = \frac{x}{b}$. So, $x^2 = ab$ and $x = \sqrt{ab}$.

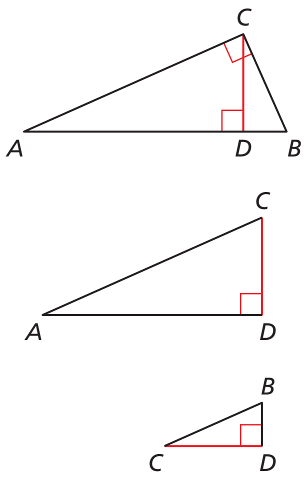
EXAMPLE 3 **Finding a Geometric Mean**

Find the geometric mean of 24 and 48.

SOLUTION

$x^2 = ab$	Definition of geometric mean
$x^2 = 24 \cdot 48$	Substitute 24 for a and 48 for b.
$x = \sqrt{24 \cdot 48}$	Take the positive square root of each side.
$x = \sqrt{24 \cdot 24 \cdot 2}$	Factor.
$x = 24\sqrt{2}$	Simplify.

▶ The geometric mean of 24 and 48 is $24\sqrt{2} \approx 33.9$.



In right $\triangle ABC$, altitude $\overline{CD}$ is drawn to the hypotenuse, forming two smaller right triangles that are similar to $\triangle ABC$. From the Right Triangle Similarity Theorem, you know that $\triangle CBD \sim \triangle ACD \sim \triangle ABC$. Because the triangles are similar, you can write and simplify the following proportions involving geometric means.

$$\frac{CD}{AD} = \frac{BD}{CD} \qquad \frac{CB}{DB} = \frac{AB}{CB} \qquad \frac{AC}{AD} = \frac{AB}{AC}$$

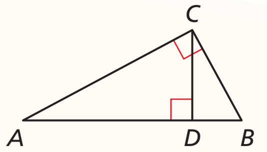
$$CD^2 = AD \cdot BD \qquad CB^2 = DB \cdot AB \qquad AC^2 = AD \cdot AB$$

Theorems

Theorem 9.7 Geometric Mean (Altitude) Theorem

In a right triangle, the altitude from the right angle to the hypotenuse divides the hypotenuse into two segments.

The length of the altitude is the geometric mean of the lengths of the two segments of the hypotenuse.



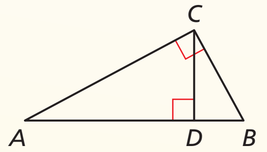
$CD^2 = AD \cdot BD$

Proof Ex. 41, p. 484

Theorem 9.8 Geometric Mean (Leg) Theorem

In a right triangle, the altitude from the right angle to the hypotenuse divides the hypotenuse into two segments.

The length of each leg of the right triangle is the geometric mean of the lengths of the hypotenuse and the segment of the hypotenuse that is adjacent to the leg.



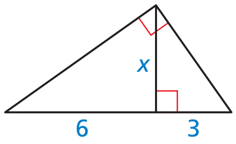
$CB^2 = DB \cdot AB$
$AC^2 = AD \cdot AB$

Proof Ex. 42, p. 484

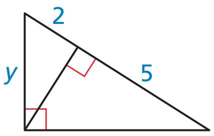
EXAMPLE 4 Using a Geometric Mean

Find the value of each variable.

a.



b.



COMMON ERROR

In Example 4(b), the Geometric Mean (Leg) Theorem gives $y^2 = 2 \cdot (5 + 2)$, not $y^2 = 5 \cdot (5 + 2)$, because the side with length y is adjacent to the segment with length 2.

SOLUTION

a. Apply the Geometric Mean (Altitude) Theorem.

$$x^2 = 6 \cdot 3$$
$$x^2 = 18$$
$$x = \sqrt{18}$$
$$x = \sqrt{9} \cdot \sqrt{2}$$
$$x = 3\sqrt{2}$$

▶ The value of x is $3\sqrt{2}$.

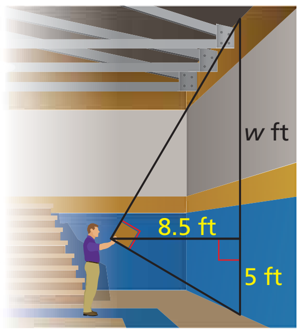
b. Apply the Geometric Mean (Leg) Theorem.

$$y^2 = 2 \cdot (5 + 2)$$
$$y^2 = 2 \cdot 7$$
$$y^2 = 14$$
$$y = \sqrt{14}$$

▶ The value of y is $\sqrt{14}$.

EXAMPLE 5 Using Indirect Measurement

To find the cost of installing a rock wall in your school gymnasium, you need to find the height of the gym wall. You use a cardboard square to line up the top and bottom of the gym wall. Your friend measures the vertical distance from the ground to your eye and the horizontal distance from you to the gym wall. Approximate the height of the gym wall.



SOLUTION

By the Geometric Mean (Altitude) Theorem, you know that 8.5 is the geometric mean of w and 5.

$8.5^2 = w \cdot 5$	Geometric Mean (Altitude) Theorem
$72.25 = 5w$	Square 8.5.
$14.45 = w$	Divide each side by 5.

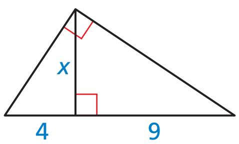
▶ The height of the wall is $5 + w = 5 + 14.45 = 19.45$ feet.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Find the geometric mean of the two numbers.

5. 12 and 27 **6.** 18 and 54 **7.** 16 and 18



8. Find the value of x in the triangle at the left.

9. **WHAT IF?** In Example 5, the vertical distance from the ground to your eye is 5.5 feet and the distance from you to the gym wall is 9 feet. Approximate the height of the gym wall.

9.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

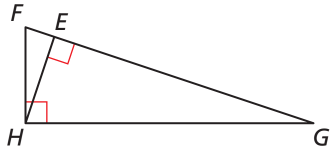
Vocabulary and Core Concept Check

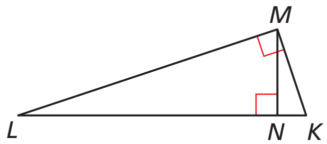
1. **COMPLETE THE SENTENCE** If the altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and ________________.

2. **WRITING** In your own words, explain *geometric mean*.

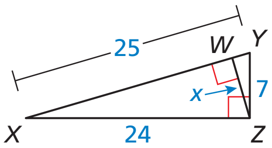
Monitoring Progress and Modeling with Mathematics

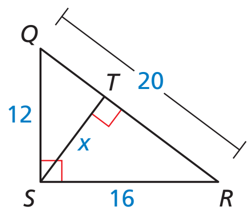
In Exercises 3 and 4, identify the similar triangles. *(See Example 1.)*

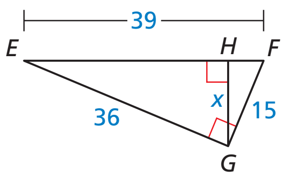
3. 

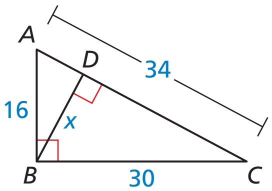
4. 

In Exercises 5–10, find the value of x. *(See Example 2.)*

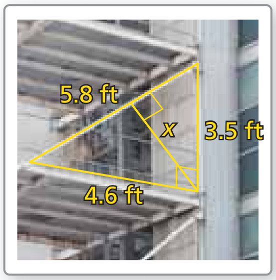
5. 

6. 

7. 

8. 

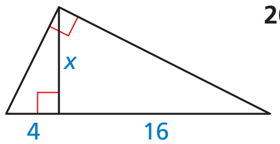
9. 

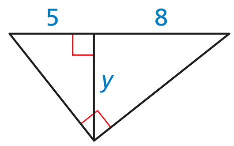
10. 

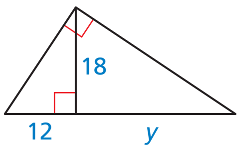
In Exercises 11–18, find the geometric mean of the two numbers. *(See Example 3.)*

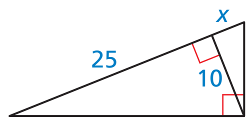
11. 8 and 32
12. 9 and 16
13. 14 and 20
14. 25 and 35
15. 16 and 25
16. 8 and 28
17. 17 and 36
18. 24 and 45

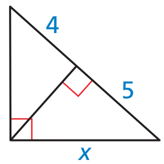
In Exercises 19–26, find the value of the variable. *(See Example 4.)*

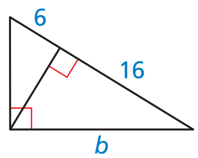
19. 

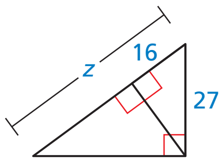
20. 

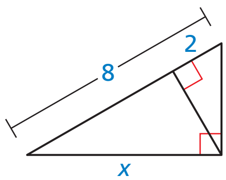
21. 

22. 

23. 

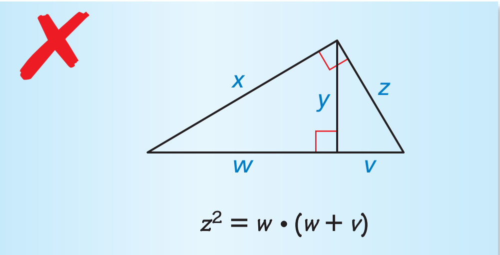
24. 

25. 

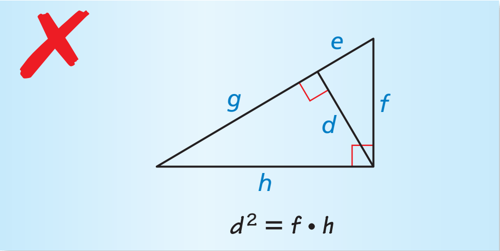
26. 

ERROR ANALYSIS In Exercises 27 and 28, describe and correct the error in writing an equation for the given diagram.

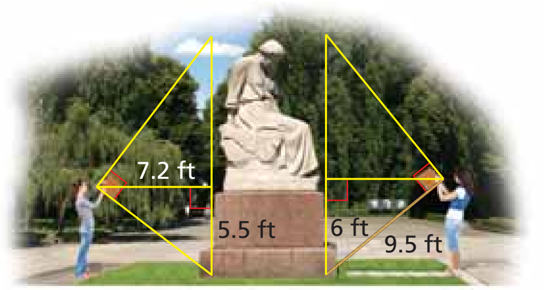
27.



28.



MODELING WITH MATHEMATICS In Exercises 29 and 30, use the diagram. *(See Example 5.)*



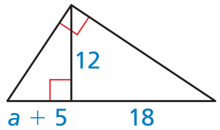
Ex. 29 Ex. 30

29. You want to determine the height of a monument at a local park. You use a cardboard square to line up the top and bottom of the monument, as shown at the above left. Your friend measures the vertical distance from the ground to your eye and the horizontal distance from you to the monument. Approximate the height of the monument.

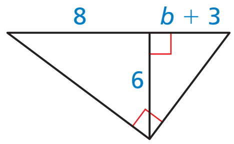
30. Your classmate is standing on the other side of the monument. She has a piece of rope staked at the base of the monument. She extends the rope to the cardboard square she is holding lined up to the top and bottom of the monument. Use the information in the diagram above to approximate the height of the monument. Do you get the same answer as in Exercise 29? Explain your reasoning.

MATHEMATICAL CONNECTIONS In Exercises 31–34, find the value(s) of the variable(s).

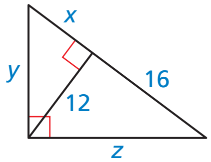
31.



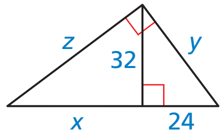
32.



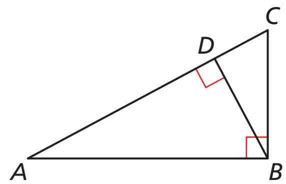
33.



34.



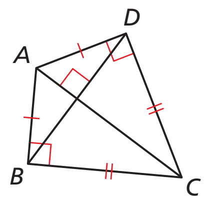
35. REASONING Use the diagram. Decide which proportions are true. Select all that apply.



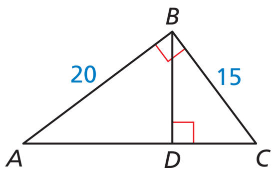
Ⓐ $\dfrac{DB}{DC} = \dfrac{DA}{DB}$ Ⓑ $\dfrac{BA}{CB} = \dfrac{CB}{BD}$

Ⓒ $\dfrac{CA}{BA} = \dfrac{BA}{CA}$ Ⓓ $\dfrac{DB}{BC} = \dfrac{DA}{BA}$

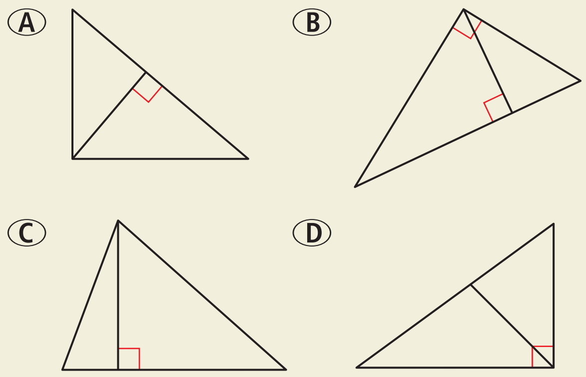
36. ANALYZING RELATIONSHIPS You are designing a diamond-shaped kite. You know that $AD = 44.8$ centimeters, $DC = 72$ centimeters, and $AC = 84.8$ centimeters. You want to use a straight crossbar $\overline{BD}$. About how long should it be? Explain your reasoning.



37. ANALYZING RELATIONSHIPS Use the Geometric Mean Theorems (Theorems 9.7 and 9.8) to find AC and BD.



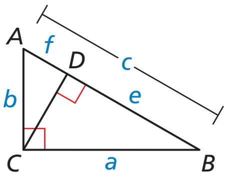
38. HOW DO YOU SEE IT? In which of the following triangles does the Geometric Mean (Altitude) Theorem (Theorem 9.7) apply?



39. PROVING A THEOREM Use the diagram of $\triangle ABC$. Copy and complete the proof of the Pythagorean Theorem (Theorem 9.1).

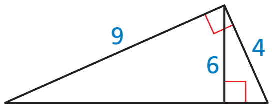
Given In $\triangle ABC$, $\angle BCA$ is a right angle.

Prove $c^2 = a^2 + b^2$



STATEMENTS	REASONS
1. In $\triangle ABC$, $\angle BCA$ is a right angle.	1. ________
2. Draw a perpendicular segment (altitude) from C to $\overline{AB}$.	2. Perpendicular Postulate (Postulate 3.2)
3. $ce = a^2$ and $cf = b^2$	3. ________
4. $ce + b^2 =$ ____ $+ b^2$	4. Addition Property of Equality
5. $ce + cf = a^2 + b^2$	5. ________
6. $c(e + f) = a^2 + b^2$	6. ________
7. $e + f =$ ____	7. Segment Addition Postulate (Postulate 1.2)
8. $c \cdot c = a^2 + b^2$	8. ________
9. $c^2 = a^2 + b^2$	9. Simplify.

40. MAKING AN ARGUMENT Your friend claims the geometric mean of 4 and 9 is 6, and then labels the triangle, as shown. Is your friend correct? Explain your reasoning.



In Exercises 41 and 42, use the given statements to prove the theorem.

Given $\triangle ABC$ is a right triangle.
Altitude $\overline{CD}$ is drawn to hypotenuse $\overline{AB}$.

41. PROVING A THEOREM Prove the Geometric Mean (Altitude) Theorem (Theorem 9.7) by showing that $CD^2 = AD \cdot BD$.

42. PROVING A THEOREM Prove the Geometric Mean (Leg) Theorem (Theorem 9.8) by showing that $CB^2 = DB \cdot AB$ and $AC^2 = AD \cdot AB$.

43. CRITICAL THINKING Draw a right isosceles triangle and label the two leg lengths x. Then draw the altitude to the hypotenuse and label its length y. Now, use the Right Triangle Similarity Theorem (Theorem 9.6) to draw the three similar triangles from the image and label any side length that is equal to either x or y. What can you conclude about the relationship between the two smaller triangles? Explain your reasoning.

44. THOUGHT PROVOKING The arithmetic mean and geometric mean of two nonnegative numbers x and y are shown.

$$\text{arithmetic mean} = \frac{x + y}{2}$$

$$\text{geometric mean} = \sqrt{xy}$$

Write an inequality that relates these two means. Justify your answer.

45. PROVING A THEOREM Prove the Right Triangle Similarity Theorem (Theorem 9.6) by proving three similarity statements.

Given $\triangle ABC$ is a right triangle.
Altitude $\overline{CD}$ is drawn to hypotenuse $\overline{AB}$.

Prove $\triangle CBD \sim \triangle ABC$, $\triangle ACD \sim \triangle ABC$, $\triangle CBD \sim \triangle ACD$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation for x. *(Skills Review Handbook)*

46. $13 = \dfrac{x}{5}$ **47.** $29 = \dfrac{x}{4}$ **48.** $9 = \dfrac{78}{x}$ **49.** $30 = \dfrac{115}{x}$