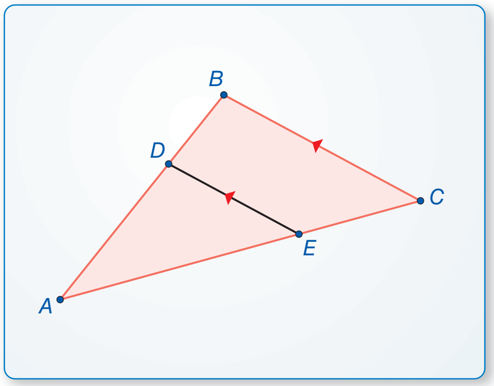
8.4 Proportionality Theorems

Essential Question What proportionality relationships exist in a triangle intersected by an angle bisector or by a line parallel to one of the sides?

EXPLORATION 1 Discovering a Proportionality Relationship

Work with a partner. Use dynamic geometry software to draw any $\triangle ABC$.

a. Construct $\overline{DE}$ parallel to $\overline{BC}$ with endpoints on $\overline{AB}$ and $\overline{AC}$, respectively.



LOOKING FOR STRUCTURE

To be proficient in math, you need to look closely to discern a pattern or structure.

b. Compare the ratios of AD to BD and AE to CE.

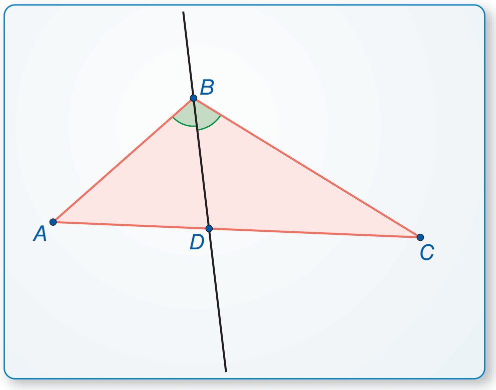
c. Move $\overline{DE}$ to other locations parallel to $\overline{BC}$ with endpoints on $\overline{AB}$ and $\overline{AC}$, and repeat part (b).

d. Change $\triangle ABC$ and repeat parts (a)–(c) several times. Write a conjecture that summarizes your results.

EXPLORATION 2 Discovering a Proportionality Relationship

Work with a partner. Use dynamic geometry software to draw any $\triangle ABC$.

a. Bisect $\angle B$ and plot point D at the intersection of the angle bisector and $\overline{AC}$.



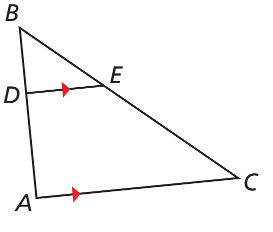
b. Compare the ratios of AD to DC and BA to BC.

c. Change $\triangle ABC$ and repeat parts (a) and (b) several times. Write a conjecture that summarizes your results.

Communicate Your Answer

3. What proportionality relationships exist in a triangle intersected by an angle bisector or by a line parallel to one of the sides?

4. Use the figure at the right to write a proportion.



8.4 Lesson

Core Vocabulary

Previous
corresponding angles
ratio
proportion

What You Will Learn

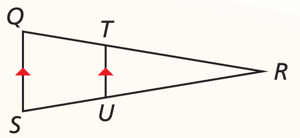
- Use the Triangle Proportionality Theorem and its converse.
- Use other proportionality theorems.

Using the Triangle Proportionality Theorem

Theorems

Theorem 8.6 Triangle Proportionality Theorem

If a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally.

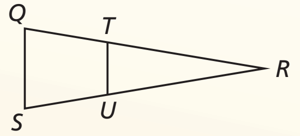


If $\overline{TU} \parallel \overline{QS}$, then $\frac{RT}{TQ} = \frac{RU}{US}$.

Proof Ex. 27, p. 451

Theorem 8.7 Converse of the Triangle Proportionality Theorem

If a line divides two sides of a triangle proportionally, then it is parallel to the third side.

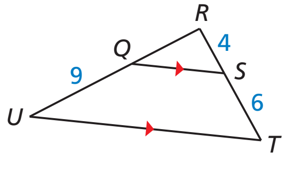


If $\frac{RT}{TQ} = \frac{RU}{US}$, then $\overline{TU} \parallel \overline{QS}$.

Proof Ex. 28, p. 451

EXAMPLE 1 Finding the Length of a Segment

In the diagram, $\overline{QS} \parallel \overline{UT}$, $RS = 4$, $ST = 6$, and $QU = 9$. What is the length of $\overline{RQ}$?



SOLUTION

$\frac{RQ}{QU} = \frac{RS}{ST}$ Triangle Proportionality Theorem

$\frac{RQ}{9} = \frac{4}{6}$ Substitute.

$RQ = 6$ Multiply each side by 9 and simplify.

▶ The length of $\overline{RQ}$ is 6 units.

Monitoring Progress

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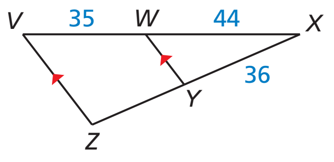
1. Find the length of $\overline{YZ}$.



The theorems on the previous page also imply the following:

Contrapositive of the Triangle Proportionality Theorem

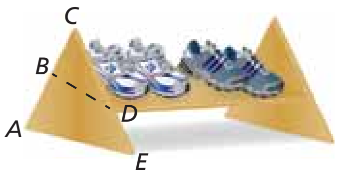
If $\frac{RT}{TQ} \neq \frac{RU}{US}$, then $\overline{TU} \nparallel \overline{QS}$.

Inverse of the Triangle Proportionality Theorem

If $\overline{TU} \nparallel \overline{QS}$, then $\frac{RT}{TQ} \neq \frac{RU}{US}$.

EXAMPLE 2 Solving a Real-Life Problem

On the shoe rack shown, $BA = 33$ centimeters, $CB = 27$ centimeters, $CD = 44$ centimeters, and $DE = 25$ centimeters. Explain why the shelf is not parallel to the floor.



SOLUTION

Find and simplify the ratios of the lengths.

$$\frac{CD}{DE} = \frac{44}{25} \qquad \frac{CB}{BA} = \frac{27}{33} = \frac{9}{11}$$

▶ Because $\frac{44}{25} \neq \frac{9}{11}$, $\overline{BD}$ is not parallel to $\overline{AE}$. So, the shelf is not parallel to the floor.

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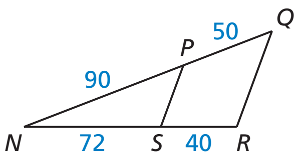


2. Determine whether $\overline{PS} \parallel \overline{QR}$.

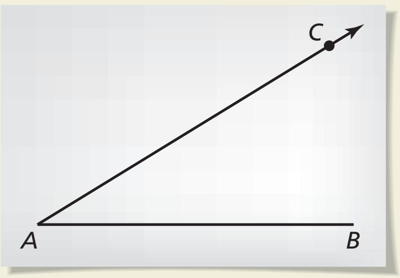
Recall that you partitioned a directed line segment in the coordinate plane in Section 3.5. You can apply the Triangle Proportionality Theorem to construct a point along a directed line segment that partitions the segment in a given ratio.

CONSTRUCTION Constructing a Point along a Directed Line Segment

Construct the point L on $\overline{AB}$ so that the ratio of AL to LB is 3 to 1.

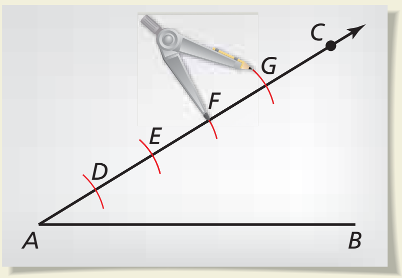
SOLUTION

Step 1



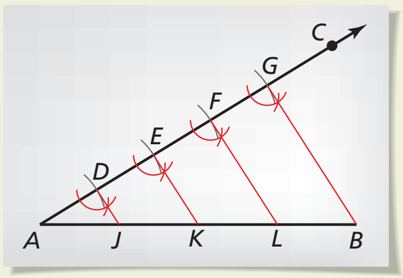
Draw a segment and a ray Draw $\overline{AB}$ of any length. Choose any point C not on $\overleftrightarrow{AB}$. Draw $\overrightarrow{AC}$.

Step 2



Draw arcs Place the point of a compass at A and make an arc of any radius intersecting $\overrightarrow{AC}$. Label the point of intersection D. Using the same compass setting, make three more arcs on $\overrightarrow{AC}$, as shown. Label the points of intersection E, F, and G and note that $AD = DE = EF = FG$.

Step 3



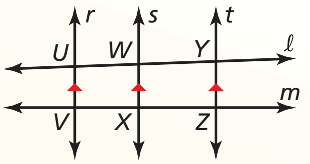
Draw a segment Draw $\overline{GB}$. Copy $\angle AGB$ and construct congruent angles at D, E, and F with sides that intersect $\overline{AB}$ at J, K, and L. Sides $\overline{DJ}$, $\overline{EK}$, and $\overline{FL}$ are all parallel, and they divide $\overline{AB}$ equally. So, $AJ = JK = KL = LB$. Point L divides directed line segment AB in the ratio 3 to 1.

Using Other Proportionality Theorems

Theorem

Theorem 8.8 Three Parallel Lines Theorem

If three parallel lines intersect two transversals, then they divide the transversals proportionally.

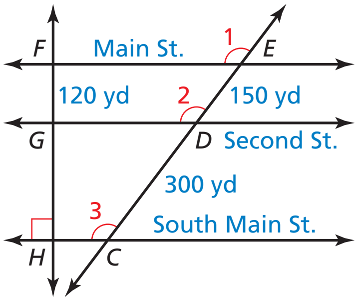


$$\frac{UW}{WY} = \frac{VX}{XZ}$$

Proof Ex. 32, p. 451

EXAMPLE 3 **Using the Three Parallel Lines Theorem**

In the diagram, $\angle 1$, $\angle 2$, and $\angle 3$ are all congruent, $GF = 120$ yards, $DE = 150$ yards, and $CD = 300$ yards. Find the distance HF between Main Street and South Main Street.



SOLUTION

Corresponding angles are congruent, so $\overleftrightarrow{FE}$, $\overleftrightarrow{GD}$, and $\overleftrightarrow{HC}$ are parallel. There are different ways you can write a proportion to find HG.

Method 1 Use the Three Parallel Lines Theorem to set up a proportion.

$$\frac{HG}{GF} = \frac{CD}{DE}$$ Three Parallel Lines Theorem

$$\frac{HG}{120} = \frac{300}{150}$$ Substitute.

$$HG = 240$$ Multiply each side by 120 and simplify.

By the Segment Addition Postulate (Postulate 1.2), $HF = HG + GF = 240 + 120 = 360$.

▶ The distance between Main Street and South Main Street is 360 yards.

Method 2 Set up a proportion involving total and partial distances.

Step 1 Make a table to compare the distances.

	$\overleftrightarrow{CE}$	$\overleftrightarrow{HF}$
Total distance	$CE = 300 + 150 = 450$	HF
Partial distance	$DE = 150$	$GF = 120$

Step 2 Write and solve a proportion.

$$\frac{450}{150} = \frac{HF}{120}$$ Write proportion.

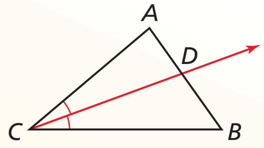
$$360 = HF$$ Multiply each side by 120 and simplify.

▶ The distance between Main Street and South Main Street is 360 yards.

Theorem

Theorem 8.9 Triangle Angle Bisector Theorem

If a ray bisects an angle of a triangle, then it divides the opposite side into segments whose lengths are proportional to the lengths of the other two sides.

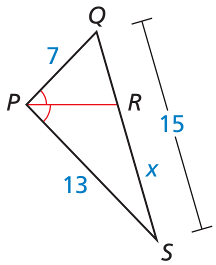


$\frac{AD}{DB} = \frac{CA}{CB}$

Proof Ex. 35, p. 452

EXAMPLE 4 Using the Triangle Angle Bisector Theorem

In the diagram, $\angle QPR \cong \angle RPS$. Use the given side lengths to find the length of $\overline{RS}$.



SOLUTION

Because $\overrightarrow{PR}$ is an angle bisector of $\angle QPS$, you can apply the Triangle Angle Bisector Theorem. Let $RS = x$. Then $RQ = 15 - x$.

$\frac{RQ}{RS} = \frac{PQ}{PS}$	Triangle Angle Bisector Theorem
$\frac{15 - x}{x} = \frac{7}{13}$	Substitute.
$195 - 13x = 7x$	Cross Products Property
$9.75 = x$	Solve for *x*.

▶ The length of $\overline{RS}$ is 9.75 units.

Monitoring Progress

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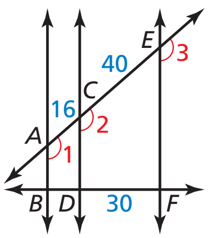
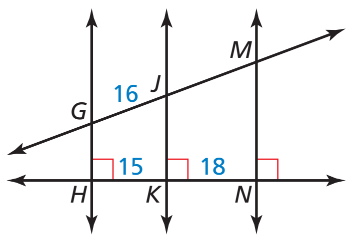
Find the length of the given line segment.

3. $\overline{BD}$

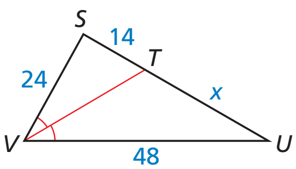


4. $\overline{JM}$

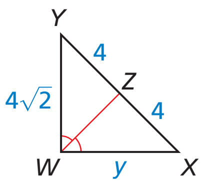


Find the value of the variable.

5.



6.



8.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

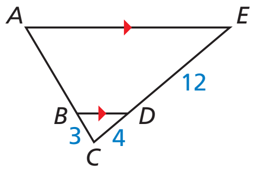
1. **COMPLETE THE STATEMENT** If a line divides two sides of a triangle proportionally, then it is __________ to the third side. This theorem is known as the ____________.

2. **VOCABULARY** In $\triangle ABC$, point R lies on $\overline{BC}$ and $\overrightarrow{AR}$ bisects $\angle CAB$. Write the proportionality statement for the triangle that is based on the Triangle Angle Bisector Theorem (Theorem 8.9).

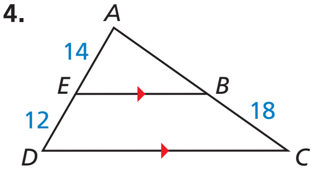
Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, find the length of $\overline{AB}$. *(See Example 1.)*

3.

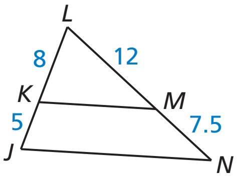


4.

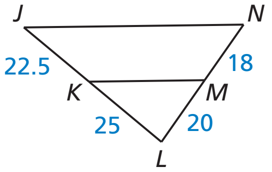


In Exercises 5–8, determine whether $\overline{KM} \parallel \overline{JN}$. *(See Example 2.)*

5.



6.



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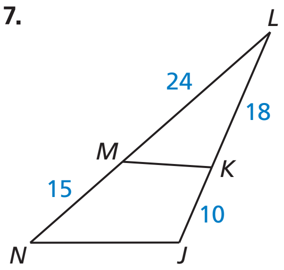
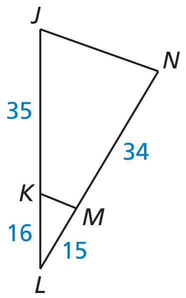


8.



CONSTRUCTION **In Exercises 9–12, draw a segment with the given length. Construct the point that divides the segment in the given ratio.**

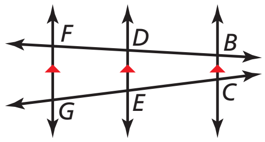
9. 3 in.; 1 to 4

10. 2 in.; 2 to 3

11. 12 cm; 1 to 3

12. 9 cm; 2 to 5

In Exercises 13–16, use the diagram to complete the proportion.



13. $\dfrac{BD}{BF} = \dfrac{\square}{CG}$

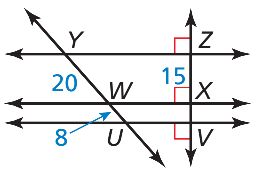
14. $\dfrac{CG}{\square} = \dfrac{BF}{DF}$

15. $\dfrac{EG}{CE} = \dfrac{DF}{\square}$

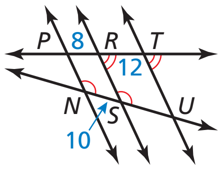
16. $\dfrac{\square}{BD} = \dfrac{CG}{CE}$

In Exercises 17 and 18, find the length of the indicated line segment. *(See Example 3.)*

17. $\overline{VX}$

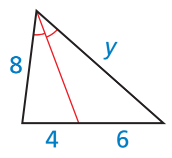


18. $\overline{SU}$

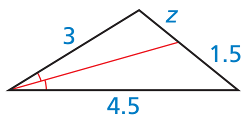


In Exercises 19–22, find the value of the variable. *(See Example 4.)*

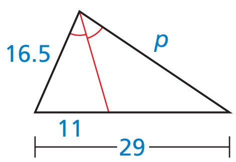
19.



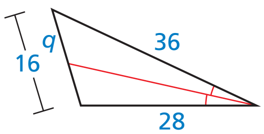
20.



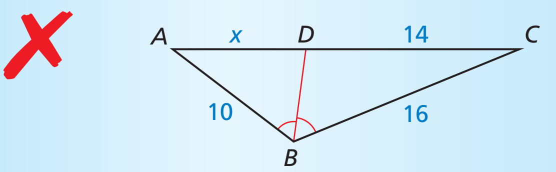
21.



22.



23. ERROR ANALYSIS Describe and correct the error in solving for x.

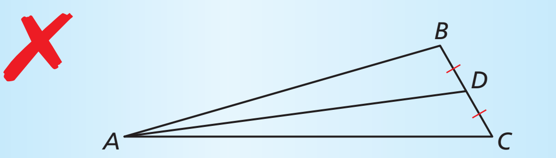


$$\frac{AB}{BC} = \frac{CD}{AD} \Rightarrow \frac{10}{16} = \frac{14}{x}$$

$$10x = 224$$

$$x = 22.4$$

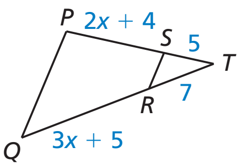
24. ERROR ANALYSIS Describe and correct the error in the student's reasoning.



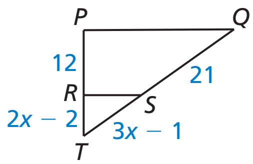
Because $\frac{BD}{CD} = \frac{AB}{AC}$ and $BD = CD$,
it follows that $AB = AC$.

MATHEMATICAL CONNECTIONS In Exercises 25 and 26, find the value of x for which $\overline{PQ} \parallel \overline{RS}$.

25.



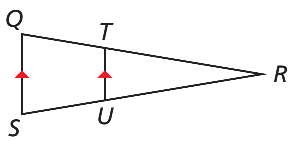
26.



27. PROVING A THEOREM Prove the Triangle Proportionality Theorem (Theorem 8.6).

Given $\overline{QS} \parallel \overline{TU}$

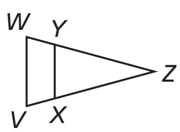
Prove $\frac{QT}{TR} = \frac{SU}{UR}$



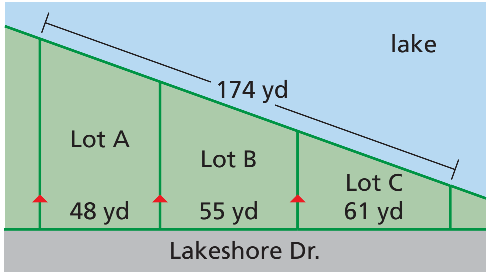
28. PROVING A THEOREM Prove the Converse of the Triangle Proportionality Theorem (Theorem 8.7).

Given $\frac{ZY}{YW} = \frac{ZX}{XV}$

Prove $\overline{YX} \parallel \overline{WV}$



29. MODELING WITH MATHEMATICS The real estate term *lake frontage* refers to the distance along the edge of a piece of property that touches a lake.

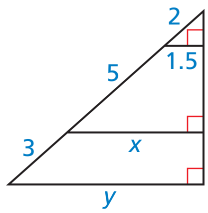


a. Find the lake frontage (to the nearest tenth) of each lot shown.

b. In general, the more lake frontage a lot has, the higher its selling price. Which lot(s) should be listed for the highest price?

c. Suppose that lot prices are in the same ratio as lake frontages. If the least expensive lot is $250,000, what are the prices of the other lots? Explain your reasoning.

30. USING STRUCTURE Use the diagram to find the values of x and y.

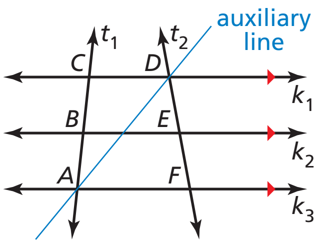


31. REASONING In the construction on page 447, explain why you can apply the Triangle Proportionality Theorem (Theorem 8.6) in Step 3.

32. PROVING A THEOREM Use the diagram with the auxiliary line drawn to write a paragraph proof of the Three Parallel Lines Theorem (Theorem 8.8).

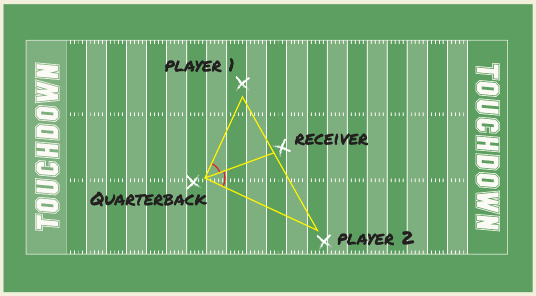
Given $k_1 \parallel k_2 \parallel k_3$

Prove $\frac{CB}{BA} = \frac{DE}{EF}$



33. CRITICAL THINKING In $\triangle LMN$, the angle bisector of $\angle M$ also bisects $\overline{LN}$. Classify $\triangle LMN$ as specifically as possible. Justify your answer.

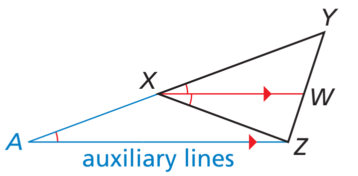
34. HOW DO YOU SEE IT? During a football game, the quarterback throws the ball to the receiver. The receiver is between two defensive players, as shown. If Player 1 is closer to the quarterback when the ball is thrown and both defensive players move at the same speed, which player will reach the receiver first? Explain your reasoning.



35. PROVING A THEOREM Use the diagram with the auxiliary lines drawn to write a paragraph proof of the Triangle Angle Bisector Theorem (Theorem 8.9).

Given $\angle YXW \cong \angle WXZ$

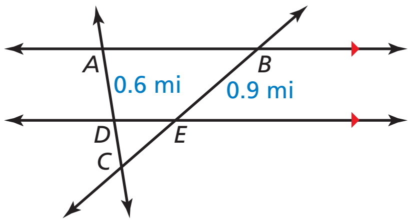
Prove $\dfrac{YW}{WZ} = \dfrac{XY}{XZ}$



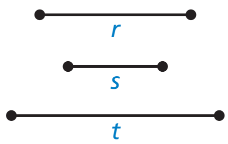
36. THOUGHT PROVOKING Write the converse of the Triangle Angle Bisector Theorem (Theorem 8.9). Is the converse true? Justify your answer.

37. REASONING How is the Triangle Midsegment Theorem (Theorem 6.8) related to the Triangle Proportionality Theorem (Theorem 8.6)? Explain your reasoning.

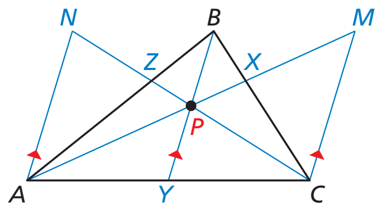
38. MAKING AN ARGUMENT Two people leave points A and B at the same time. They intend to meet at point C at the same time. The person who leaves point A walks at a speed of 3 miles per hour. You and a friend are trying to determine how fast the person who leaves point B must walk. Your friend claims you need to know the length of $\overline{AC}$. Is your friend correct? Explain your reasoning.



39. CONSTRUCTION Given segments with lengths r, s, and t, construct a segment of length x, such that $\frac{r}{s} = \frac{t}{x}$.



40. PROOF Prove *Ceva's Theorem*: If P is any point inside $\triangle ABC$, then $\dfrac{AY}{YC} \cdot \dfrac{CX}{XB} \cdot \dfrac{BZ}{ZA} = 1$.



(*Hint*: Draw segments parallel to $\overline{BY}$ through A and C, as shown. Apply the Triangle Proportionality Theorem (Theorem 8.6) to $\triangle ACM$. Show that $\triangle APN \sim \triangle MPC$, $\triangle CXM \sim \triangle BXP$, and $\triangle BZP \sim \triangle AZN$.)

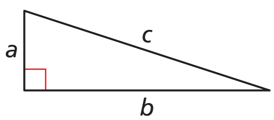
Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Use the triangle. *(Section 5.5)*

41. Which sides are the legs?

42. Which side is the hypotenuse?



Solve the equation. *(Skills Review Handbook)*

43. $x^2 = 121$

44. $x^2 + 16 = 25$

45. $36 + x^2 = 85$