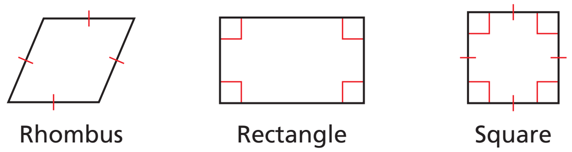
7.4 Properties of Special Parallelograms

Essential Question What are the properties of the diagonals of rectangles, rhombuses, and squares?

Recall the three types of parallelograms shown below.



EXPLORATION 1 Identifying Special Quadrilaterals

Work with a partner. Use dynamic geometry software.

a. Draw a circle with center A.

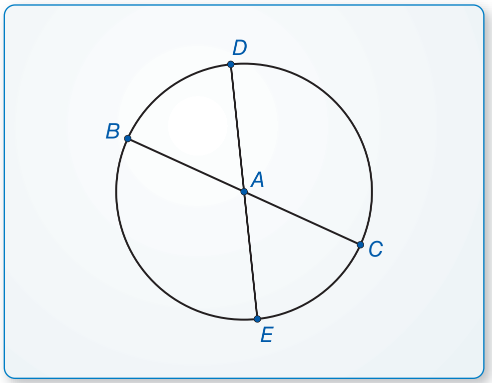
b. Draw two diameters of the circle. Label the endpoints B, C, D, and E.

c. Draw quadrilateral $BDCE$.

d. Is $BDCE$ a parallelogram? rectangle? rhombus? square? Explain your reasoning.

e. Repeat parts (a)–(d) for several other circles. Write a conjecture based on your results.

Sample



EXPLORATION 2 Identifying Special Quadrilaterals

Work with a partner. Use dynamic geometry software.

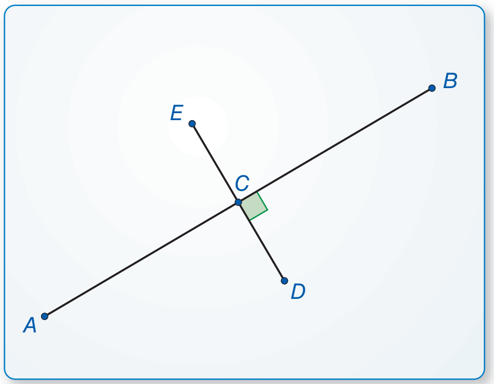
a. Construct two segments that are perpendicular bisectors of each other. Label the endpoints A, B, D, and E. Label the intersection C.

b. Draw quadrilateral $AEBD$.

c. Is $AEBD$ a parallelogram? rectangle? rhombus? square? Explain your reasoning.

d. Repeat parts (a)–(c) for several other segments. Write a conjecture based on your results.

Sample



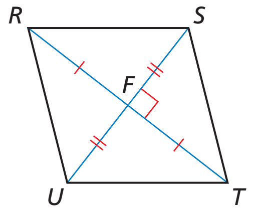
CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to make conjectures and build a logical progression of statements to explore the truth of your conjectures.

Communicate Your Answer

3. What are the properties of the diagonals of rectangles, rhombuses, and squares?

4. Is $RSTU$ a parallelogram? rectangle? rhombus? square? Explain your reasoning.



5. What type of quadrilateral has congruent diagonals that bisect each other?

7.4 Lesson

Core Vocabulary

rhombus, *p. 388*
rectangle, *p. 388*
square, *p. 388*

Previous
quadrilateral
parallelogram
diagonal

What You Will Learn

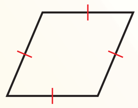
- Use properties of special parallelograms.
- Use properties of diagonals of special parallelograms.
- Use coordinate geometry to identify special types of parallelograms.

Using Properties of Special Parallelograms

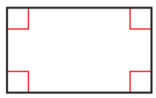
In this lesson, you will learn about three special types of parallelograms: *rhombuses*, *rectangles*, and *squares*.

Core Concept

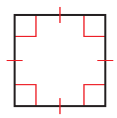
Rhombuses, Rectangles, and Squares



A **rhombus** is a parallelogram with four congruent sides.



A **rectangle** is a parallelogram with four right angles.



A **square** is a parallelogram with four congruent sides and four right angles.

You can use the corollaries below to prove that a quadrilateral is a rhombus, rectangle, or square, without first proving that the quadrilateral is a parallelogram.

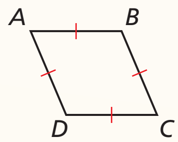
Corollaries

Corollary 7.2 Rhombus Corollary

A quadrilateral is a rhombus if and only if it has four congruent sides.

ABCD is a rhombus if and only if $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{AD}$.

Proof Ex. 81, p. 396

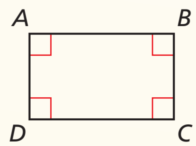


Corollary 7.3 Rectangle Corollary

A quadrilateral is a rectangle if and only if it has four right angles.

ABCD is a rectangle if and only if $\angle A$, $\angle B$, $\angle C$, and $\angle D$ are right angles.

Proof Ex. 82, p. 396

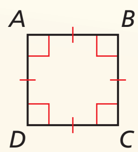


Corollary 7.4 Square Corollary

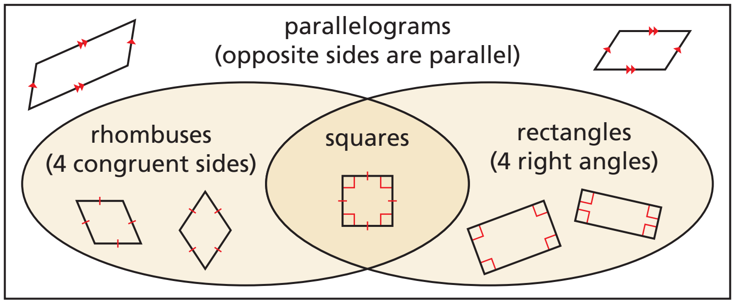
A quadrilateral is a square if and only if it is a rhombus and a rectangle.

ABCD is a square if and only if $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{AD}$ and $\angle A$, $\angle B$, $\angle C$, and $\angle D$ are right angles.

Proof Ex. 83, p. 396



The Venn diagram below illustrates some important relationships among parallelograms, rhombuses, rectangles, and squares. For example, you can see that a square is a rhombus because it is a parallelogram with four congruent sides. Because it has four right angles, a square is also a rectangle.



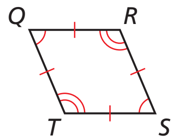
EXAMPLE 1 Using Properties of Special Quadrilaterals

For any rhombus $QRST$, decide whether the statement is *always* or *sometimes* true. Draw a diagram and explain your reasoning.

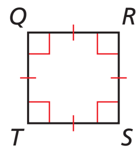
a. $\angle Q \cong \angle S$ **b.** $\angle Q \cong \angle R$

SOLUTION

a. By definition, a rhombus is a parallelogram with four congruent sides. By the Parallelogram Opposite Angles Theorem (Theorem 7.4), opposite angles of a parallelogram are congruent. So, $\angle Q \cong \angle S$. The statement is *always* true.

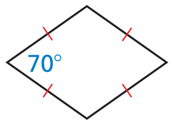


b. If rhombus $QRST$ is a square, then all four angles are congruent right angles. So, $\angle Q \cong \angle R$ when $QRST$ is a square. Because not all rhombuses are also squares, the statement is *sometimes* true.



EXAMPLE 2 Classifying Special Quadrilaterals

Classify the special quadrilateral. Explain your reasoning.



SOLUTION

The quadrilateral has four congruent sides. By the Rhombus Corollary, the quadrilateral is a rhombus. Because one of the angles is not a right angle, the rhombus cannot be a square.

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1. For any square $JKLM$, is it *always* or *sometimes* true that $\overline{JK} \perp \overline{KL}$? Explain your reasoning.
2. For any rectangle $EFGH$, is it *always* or *sometimes* true that $\overline{FG} \cong \overline{GH}$? Explain your reasoning.
3. A quadrilateral has four congruent sides and four congruent angles. Sketch the quadrilateral and classify it.

Using Properties of Diagonals

READING

Recall that biconditionals, such as the Rhombus Diagonals Theorem, can be rewritten as two parts. To prove a biconditional, you must prove both parts.

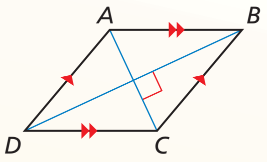
Theorems

Theorem 7.11 Rhombus Diagonals Theorem

A parallelogram is a rhombus if and only if its diagonals are perpendicular.

$\square ABCD$ is a rhombus if and only if $\overline{AC} \perp \overline{BD}$.

Proof p. 390; Ex. 72, p. 395

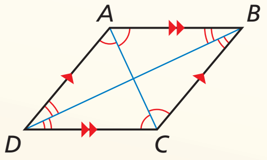


Theorem 7.12 Rhombus Opposite Angles Theorem

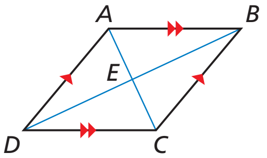
A parallelogram is a rhombus if and only if each diagonal bisects a pair of opposite angles.

$\square ABCD$ is a rhombus if and only if $\overline{AC}$ bisects $\angle BCD$ and $\angle BAD$, and $\overline{BD}$ bisects $\angle ABC$ and $\angle ADC$.

Proof Exs. 73 and 74, p. 395



PROOF **Part of Rhombus Diagonals Theorem**



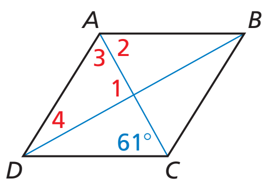
Given $ABCD$ is a rhombus.

Prove $\overline{AC} \perp \overline{BD}$

$ABCD$ is a rhombus. By the definition of a rhombus, $\overline{AB} \cong \overline{BC}$. Because a rhombus is a parallelogram and the diagonals of a parallelogram bisect each other, $\overline{BD}$ bisects $\overline{AC}$ at E. So, $\overline{AE} \cong \overline{EC}$. $\overline{BE} \cong \overline{BE}$ by the Reflexive Property of Congruence (Theorem 2.1). So, $\triangle AEB \cong \triangle CEB$ by the SSS Congruence Theorem (Theorem 5.8). $\angle AEB \cong \angle CEB$ because corresponding parts of congruent triangles are congruent. Then by the Linear Pair Postulate (Postulate 2.8), $\angle AEB$ and $\angle CEB$ are supplementary. Two congruent angles that form a linear pair are right angles, so $m\angle AEB = m\angle CEB = 90°$ by the definition of a right angle. So, $\overline{AC} \perp \overline{BD}$ by the definition of perpendicular lines.

EXAMPLE 3 **Finding Angle Measures in a Rhombus**

Find the measures of the numbered angles in rhombus $ABCD$.



SOLUTION

Use the Rhombus Diagonals Theorem and the Rhombus Opposite Angles Theorem to find the angle measures.

$m\angle 1 = 90°$	The diagonals of a rhombus are perpendicular.
$m\angle 2 = 61°$	Alternate Interior Angles Theorem (Theorem 3.2)
$m\angle 3 = 61°$	Each diagonal of a rhombus bisects a pair of opposite angles, and $m\angle 2 = 61°$.
$m\angle 1 + m\angle 3 + m\angle 4 = 180°$	Triangle Sum Theorem (Theorem 5.1)
$90° + 61° + m\angle 4 = 180°$	Substitute 90° for $m\angle 1$ and 61° for $m\angle 3$.
$m\angle 4 = 29°$	Solve for $m\angle 4$.

So, $m\angle 1 = 90°$, $m\angle 2 = 61°$, $m\angle 3 = 61°$, and $m\angle 4 = 29°$.

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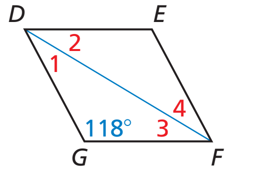
4. In Example 3, what is $m\angle ADC$ and $m\angle BCD$?

5. Find the measures of the numbered angles in rhombus $DEFG$.



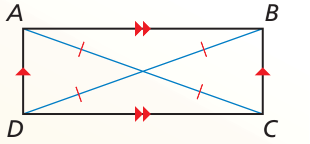
Theorem

Theorem 7.13 Rectangle Diagonals Theorem

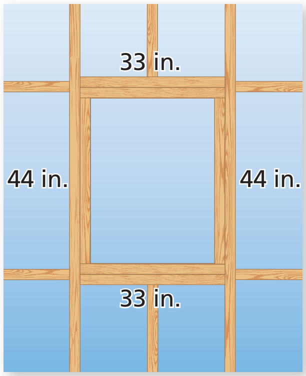
A parallelogram is a rectangle if and only if its diagonals are congruent.

$\square ABCD$ is a rectangle if and only if $\overline{AC} \cong \overline{BD}$.

Proof Exs. 87 and 88, p. 396



EXAMPLE 4 Identifying a Rectangle



You are building a frame for a window. The window will be installed in the opening shown in the diagram.

a. The opening must be a rectangle. Given the measurements in the diagram, can you assume that it is? Explain.

b. You measure the diagonals of the opening. The diagonals are 54.8 inches and 55.3 inches. What can you conclude about the shape of the opening?

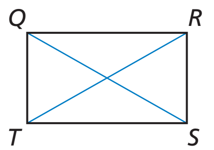
SOLUTION

a. No, you cannot. The boards on opposite sides are the same length, so they form a parallelogram. But you do not know whether the angles are right angles.

b. By the Rectangle Diagonals Theorem, the diagonals of a rectangle are congruent. The diagonals of the quadrilateral formed by the boards are not congruent, so the boards do not form a rectangle.

EXAMPLE 5 Finding Diagonal Lengths in a Rectangle

In rectangle $QRST$, $QS = 5x - 31$ and $RT = 2x + 11$. Find the lengths of the diagonals of $QRST$.



SOLUTION

By the Rectangle Diagonals Theorem, the diagonals of a rectangle are congruent. Find x so that $\overline{QS} \cong \overline{RT}$.

$QS = RT$	Set the diagonal lengths equal.
$5x - 31 = 2x + 11$	Substitute $5x - 31$ for QS and $2x + 11$ for RT.
$3x - 31 = 11$	Subtract $2x$ from each side.
$3x = 42$	Add 31 to each side.
$x = 14$	Divide each side by 3.

When $x = 14$, $QS = 5(14) - 31 = 39$ and $RT = 2(14) + 11 = 39$.

▶ Each diagonal has a length of 39 units.

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6. Suppose you measure only the diagonals of the window opening in Example 4 and they have the same measure. Can you conclude that the opening is a rectangle? Explain.

7. **WHAT IF?** In Example 5, $QS = 4x - 15$ and $RT = 3x + 8$. Find the lengths of the diagonals of $QRST$.

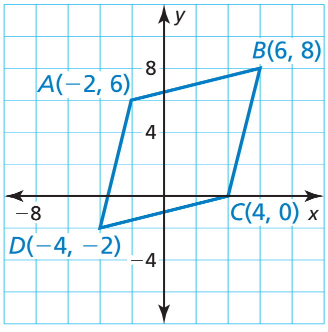
Using Coordinate Geometry

EXAMPLE 6 Identifying a Parallelogram in the Coordinate Plane

Decide whether $\square ABCD$ with vertices $A(-2, 6)$, $B(6, 8)$, $C(4, 0)$, and $D(-4, -2)$ is a *rectangle*, a *rhombus*, or a *square*. Give all names that apply.

SOLUTION

1. **Understand the Problem** You know the vertices of $\square ABCD$. You need to identify the type of parallelogram.



2. **Make a Plan** Begin by graphing the vertices. From the graph, it appears that all four sides are congruent and there are no right angles.

   Check the lengths and slopes of the diagonals of $\square ABCD$. If the diagonals are congruent, then $\square ABCD$ is a rectangle. If the diagonals are perpendicular, then $\square ABCD$ is a rhombus. If they are both congruent and perpendicular, then $\square ABCD$ is a rectangle, a rhombus, and a square.

3. **Solve the Problem** Use the Distance Formula to find AC and BD.

$$AC = \sqrt{(-2-4)^2 + (6-0)^2} = \sqrt{72} = 6\sqrt{2}$$

$$BD = \sqrt{[6-(-4)]^2 + [8-(-2)]^2} = \sqrt{200} = 10\sqrt{2}$$

   Because $6\sqrt{2} \neq 10\sqrt{2}$, the diagonals are not congruent. So, $\square ABCD$ is not a rectangle. Because it is not a rectangle, it also cannot be a square.

   Use the slope formula to find the slopes of the diagonals $\overline{AC}$ and $\overline{BD}$.

$$\text{slope of } \overline{AC} = \frac{6-0}{-2-4} = \frac{6}{-6} = -1 \qquad \text{slope of } \overline{BD} = \frac{8-(-2)}{6-(-4)} = \frac{10}{10} = 1$$

   Because the product of the slopes of the diagonals is -1, the diagonals are perpendicular.

   ▶ So, $\square ABCD$ is a rhombus.

4. **Look Back** Check the side lengths of $\square ABCD$. Each side has a length of $2\sqrt{17}$ units, so $\square ABCD$ is a rhombus. Check the slopes of two consecutive sides.

$$\text{slope of } \overline{AB} = \frac{8-6}{6-(-2)} = \frac{2}{8} = \frac{1}{4} \qquad \text{slope of } \overline{BC} = \frac{8-0}{6-4} = \frac{8}{2} = 4$$

   Because the product of these slopes is not -1, $\overline{AB}$ is not perpendicular to $\overline{BC}$.

   So, $\angle ABC$ is not a right angle, and $\square ABCD$ cannot be a rectangle or a square. ✓

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8. Decide whether $\square PQRS$ with vertices $P(-5, 2)$, $Q(0, 4)$, $R(2, -1)$, and $S(-3, -3)$ is a *rectangle*, a *rhombus*, or a *square*. Give all names that apply.

7.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** What is another name for an equilateral rectangle?

2. **WRITING** What should you look for in a parallelogram to know if the parallelogram is also a rhombus?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, for any rhombus *JKLM*, decide whether the statement is *always* or *sometimes* true. Draw a diagram and explain your reasoning. *(See Example 1.)*

3. $\angle L \cong \angle M$

4. $\angle K \cong \angle M$

5. $\overline{JM} \cong \overline{KL}$

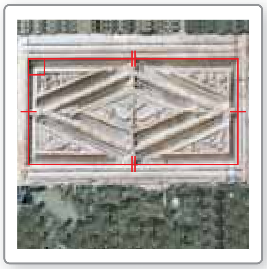
6. $\overline{JK} \cong \overline{KL}$

7. $\overline{JL} \cong \overline{KM}$

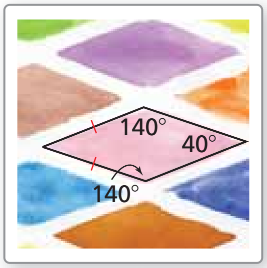
8. $\angle JKM \cong \angle LKM$

In Exercises 9–12, classify the quadrilateral. Explain your reasoning. *(See Example 2.)*

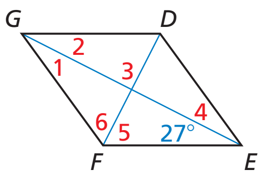
9. 

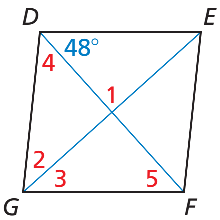
10. 

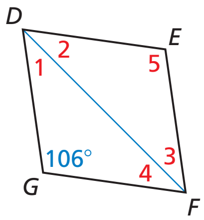
11. 

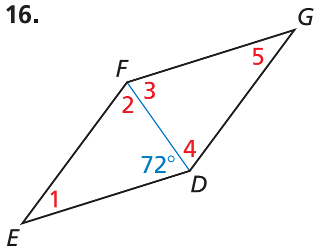
12. 

In Exercises 13–16, find the measures of the numbered angles in rhombus *DEFG*. *(See Example 3.)*

13. 

14. 

15. 

16. 

In Exercises 17–22, for any rectangle *WXYZ*, decide whether the statement is *always* or *sometimes* true. Draw a diagram and explain your reasoning.

17. $\angle W \cong \angle X$

18. $\overline{WX} \cong \overline{YZ}$

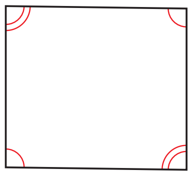
19. $\overline{WX} \cong \overline{XY}$

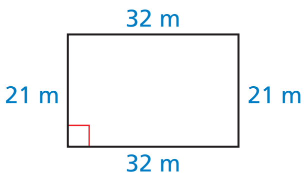
20. $\overline{WY} \cong \overline{XZ}$

21. $\overline{WY} \perp \overline{XZ}$

22. $\angle WXZ \cong \angle YXZ$

In Exercises 23 and 24, determine whether the quadrilateral is a rectangle. *(See Example 4.)*

23. 

24. 

In Exercises 25–28, find the lengths of the diagonals of rectangle *WXYZ*. *(See Example 5.)*

25. $WY = 6x - 7$
$XZ = 3x + 2$

26. $WY = 14x + 10$
$XZ = 11x + 22$

27. $WY = 24x - 8$
$XZ = -18x + 13$

28. $WY = 16x + 2$
$XZ = 36x - 6$

In Exercises 29–34, name each quadrilateral—*parallelogram*, *rectangle*, *rhombus*, or *square*—for which the statement is always true.

29. It is equiangular.

30. It is equiangular and equilateral.

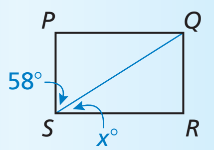
31. The diagonals are perpendicular.

32. Opposite sides are congruent.

33. The diagonals bisect each other.

34. The diagonals bisect opposite angles.

35. **ERROR ANALYSIS** Quadrilateral $PQRS$ is a rectangle. Describe and correct the error in finding the value of x.

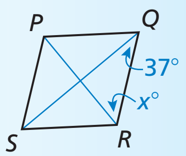


$m\angle QSR = m\angle QSP$

$x° = 58°$

$x = 58$

36. **ERROR ANALYSIS** Quadrilateral $PQRS$ is a rhombus. Describe and correct the error in finding the value of x.

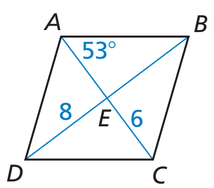


$m\angle QRP = m\angle SQR$

$x° = 37°$

$x = 37$

In Exercises 37–42, the diagonals of rhombus $ABCD$ intersect at E. Given that $m\angle BAC = 53°$, $DE = 8$, and $EC = 6$, find the indicated measure.

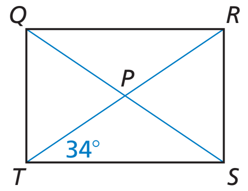


37. $m\angle DAC$ **38.** $m\angle AED$

39. $m\angle ADC$ **40.** DB

41. AE **42.** AC

In Exercises 43–48, the diagonals of rectangle $QRST$ intersect at P. Given that $m\angle PTS = 34°$ and $QS = 10$, find the indicated measure.

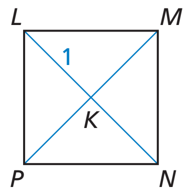


43. $m\angle QTR$ **44.** $m\angle QRT$

45. $m\angle SRT$ **46.** QP

47. RT **48.** RP

In Exercises 49–54, the diagonals of square $LMNP$ intersect at K. Given that $LK = 1$, find the indicated measure.



49. $m\angle MKN$ **50.** $m\angle LMK$

51. $m\angle LPK$ **52.** KN

53. LN **54.** MP

In Exercises 55–60, decide whether $\square JKLM$ is a rectangle, a rhombus, or a square. Give all names that apply. Explain your reasoning. *(See Example 6.)*

55. $J(-4, 2)$, $K(0, 3)$, $L(1, -1)$, $M(-3, -2)$

56. $J(-2, 7)$, $K(7, 2)$, $L(-2, -3)$, $M(-11, 2)$

57. $J(3, 1)$, $K(3, -3)$, $L(-2, -3)$, $M(-2, 1)$

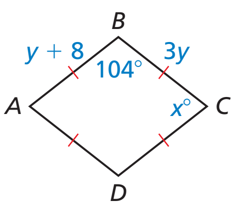
58. $J(-1, 4)$, $K(-3, 2)$, $L(2, -3)$, $M(4, -1)$

59. $J(5, 2)$, $K(1, 9)$, $L(-3, 2)$, $M(1, -5)$

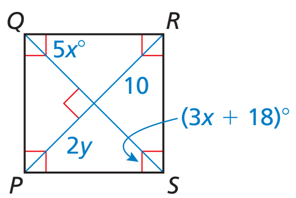
60. $J(5, 2)$, $K(2, 5)$, $L(-1, 2)$, $M(2, -1)$

MATHEMATICAL CONNECTIONS **In Exercises 61 and 62, classify the quadrilateral. Explain your reasoning. Then find the values of x and y.**

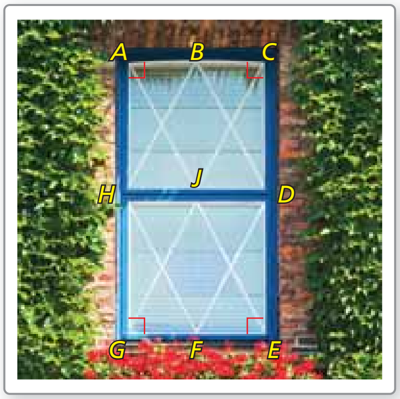
61.



62.



63. **DRAWING CONCLUSIONS** In the window, $\overline{BD} \cong \overline{DF} \cong \overline{BH} \cong \overline{HF}$. Also, $\angle HAB$, $\angle BCD$, $\angle DEF$, and $\angle FGH$ are right angles.



a. Classify $HBDF$ and $ACEG$. Explain your reasoning.

b. What can you conclude about the lengths of the diagonals $\overline{AE}$ and $\overline{GC}$? Given that these diagonals intersect at J, what can you conclude about the lengths of $\overline{AJ}$, $\overline{JE}$, $\overline{CJ}$, and $\overline{JG}$? Explain.

64. **ABSTRACT REASONING** Order the terms in a diagram so that each term builds off the previous term(s). Explain why each figure is in the location you chose.

quadrilateral	square
rectangle	rhombus
parallelogram	

CRITICAL THINKING In Exercises 65–70, complete each statement with *always*, *sometimes*, or *never*. Explain your reasoning.

65. A square is _________ a rhombus.

66. A rectangle is _________ a square.

67. A rectangle _________ has congruent diagonals.

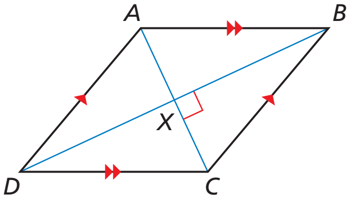
68. The diagonals of a square _________ bisect its angles.

69. A rhombus _________ has four congruent angles.

70. A rectangle _________ has perpendicular diagonals.

71. **USING TOOLS** You want to mark off a square region for a garden at school. You use a tape measure to mark off a quadrilateral on the ground. Each side of the quadrilateral is 2.5 meters long. Explain how you can use the tape measure to make sure that the quadrilateral is a square.

72. **PROVING A THEOREM** Use the plan for proof below to write a paragraph proof for one part of the Rhombus Diagonals Theorem (Theorem 7.11).



Given $ABCD$ is a parallelogram.
$\overline{AC} \perp \overline{BD}$

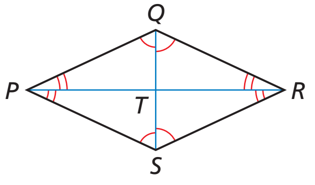
Prove $ABCD$ is a rhombus.

Plan for Proof Because $ABCD$ is a parallelogram, its diagonals bisect each other at X. Use $\overline{AC} \perp \overline{BD}$ to show that $\triangle BXC \cong \triangle DXC$. Then show that $\overline{BC} \cong \overline{DC}$. Use the properties of a parallelogram to show that $ABCD$ is a rhombus.

PROVING A THEOREM In Exercises 73 and 74, write a proof for part of the Rhombus Opposite Angles Theorem (Theorem 7.12).

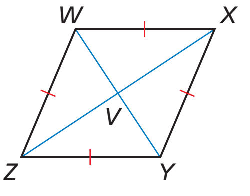
73. **Given** $PQRS$ is a parallelogram.
$\overline{PR}$ bisects $\angle SPQ$ and $\angle QRS$.
$\overline{SQ}$ bisects $\angle PSR$ and $\angle RQP$.

Prove $PQRS$ is a rhombus.



74. **Given** $WXYZ$ is a rhombus.

Prove $\overline{WY}$ bisects $\angle ZWX$ and $\angle XYZ$.
$\overline{ZX}$ bisects $\angle WZY$ and $\angle YXW$.

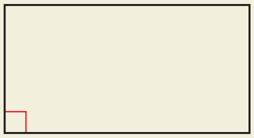


75. **ABSTRACT REASONING** Will a diagonal of a square ever divide the square into two equilateral triangles? Explain your reasoning.

76. **ABSTRACT REASONING** Will a diagonal of a rhombus ever divide the rhombus into two equilateral triangles? Explain your reasoning.

77. **CRITICAL THINKING** Which quadrilateral could be called a regular quadrilateral? Explain your reasoning.

78. **HOW DO YOU SEE IT?** What other information do you need to determine whether the figure is a rectangle?



79. **REASONING** Are all rhombuses similar? Are all squares similar? Explain your reasoning.

80. **THOUGHT PROVOKING** Use the Rhombus Diagonals Theorem (Theorem 7.11) to explain why every rhombus has at least two lines of symmetry.

PROVING A COROLLARY In Exercises 81–83, write the corollary as a conditional statement and its converse. Then explain why each statement is true.

81. Rhombus Corollary (Corollary 7.2)

82. Rectangle Corollary (Corollary 7.3)

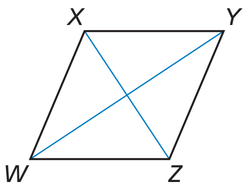
83. Square Corollary (Corollary 7.4)

84. **MAKING AN ARGUMENT** Your friend claims a rhombus will never have congruent diagonals because it would have to be a rectangle. Is your friend correct? Explain your reasoning.

85. **PROOF** Write a proof in the style of your choice.

Given $\triangle XYZ \cong \triangle XWZ$, $\angle XYW \cong \angle ZWY$

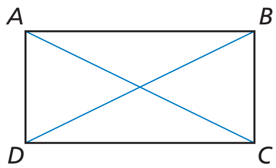
Prove *WXYZ* is a rhombus.



86. **PROOF** Write a proof in the style of your choice.

Given $\overline{BC} \cong \overline{AD}, \overline{BC} \perp \overline{DC}, \overline{AD} \perp \overline{DC}$

Prove *ABCD* is a rectangle.



PROVING A THEOREM In Exercises 87 and 88, write a proof for part of the Rectangle Diagonals Theorem (Theorem 7.13).

87. **Given** *PQRS* is a rectangle.

Prove $\overline{PR} \cong \overline{SQ}$

88. **Given** *PQRS* is a parallelogram.
$\overline{PR} \cong \overline{SQ}$

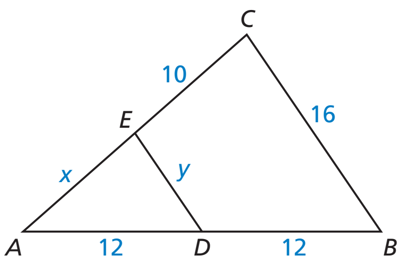
Prove *PQRS* is a rectangle.

Maintaining Mathematical Proficiency

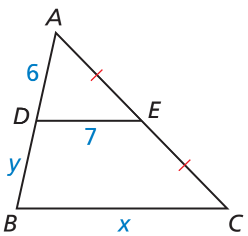
Reviewing what you learned in previous grades and lessons

$\overline{DE}$ is a midsegment of $\triangle ABC$. Find the values of x and y. *(Section 6.4)*

89.



90.



91.