6.1 Perpendicular and Angle Bisectors

Essential Question

What conjectures can you make about a point on the perpendicular bisector of a segment and a point on the bisector of an angle?

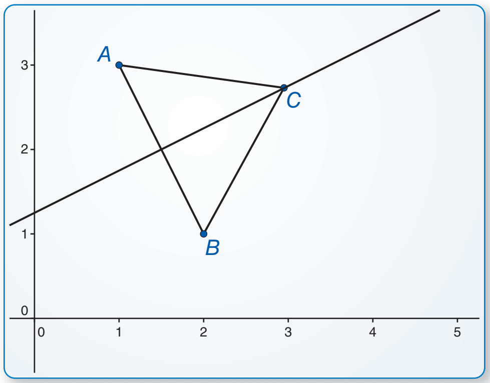
EXPLORATION 1 Points on a Perpendicular Bisector

Work with a partner. Use dynamic geometry software.

a. Draw any segment and label it $\overline{AB}$. Construct the perpendicular bisector of $\overline{AB}$.

b. Label a point C that is on the perpendicular bisector of $\overline{AB}$ but is not on $\overline{AB}$.

c. Draw $\overline{CA}$ and $\overline{CB}$ and find their lengths. Then move point C to other locations on the perpendicular bisector and note the lengths of $\overline{CA}$ and $\overline{CB}$.



Sample
Points
$A(1, 3)$
$B(2, 1)$
$C(2.95, 2.73)$
Segments
$AB = 2.24$
$CA = ?$
$CB = ?$
Line
$-x + 2y = 2.5$

d. Repeat parts (a)–(c) with other segments. Describe any relationship(s) you notice.

USING TOOLS STRATEGICALLY

To be proficient in math, you need to visualize the results of varying assumptions, explore consequences, and compare predictions with data.

EXPLORATION 2 Points on an Angle Bisector

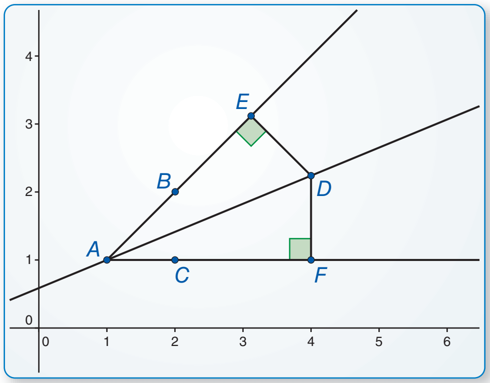
Work with a partner. Use dynamic geometry software.

a. Draw two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ to form $\angle BAC$. Construct the bisector of $\angle BAC$.

b. Label a point D on the bisector of $\angle BAC$.

c. Construct and find the lengths of the perpendicular segments from D to the sides of $\angle BAC$. Move point D along the angle bisector and note how the lengths change.

d. Repeat parts (a)–(c) with other angles. Describe any relationship(s) you notice.



Sample
Points
$A(1, 1)$
$B(2, 2)$
$C(2, 1)$
$D(4, 2.24)$
Rays
$AB = -x + y = 0$
$AC = y = 1$
Line
$-0.38x + 0.92y = 0.54$

Communicate Your Answer

3. What conjectures can you make about a point on the perpendicular bisector of a segment and a point on the bisector of an angle?

4. In Exploration 2, what is the distance from point D to $\overrightarrow{AB}$ when the distance from D to $\overrightarrow{AC}$ is 5 units? Justify your answer.

6.1 Lesson

What You Will Learn

- Use perpendicular bisectors to find measures.
- Use angle bisectors to find measures and distance relationships.
- Write equations for perpendicular bisectors.

Core Vocabulary

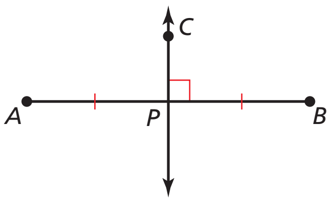
equidistant, *p. 302*

Previous
perpendicular bisector
angle bisector

Using Perpendicular Bisectors

In Section 3.4, you learned that a *perpendicular bisector* of a line segment is the line that is perpendicular to the segment at its midpoint.

A point is **equidistant** from two figures when the point is the *same distance* from each figure.



$\overleftrightarrow{CP}$ is a $\perp$ bisector of $\overline{AB}$.

STUDY TIP

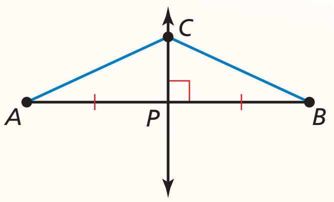
A perpendicular bisector can be a segment, a ray, a line, or a plane.

Theorems

Theorem 6.1 Perpendicular Bisector Theorem

In a plane, if a point lies on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

If $\overleftrightarrow{CP}$ is the $\perp$ bisector of $\overline{AB}$, then $CA = CB$.

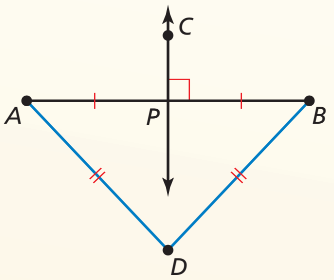


Proof p. 302

Theorem 6.2 Converse of the Perpendicular Bisector Theorem

In a plane, if a point is equidistant from the endpoints of a segment, then it lies on the perpendicular bisector of the segment.

If $DA = DB$, then point D lies on the $\perp$ bisector of $\overline{AB}$.

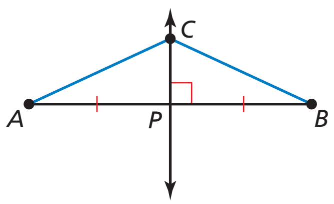


Proof Ex. 32, p. 308

PROOF Perpendicular Bisector Theorem

Given $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$.

Prove $CA = CB$

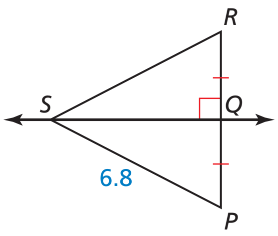


Paragraph Proof Because $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$, $\overleftrightarrow{CP}$ is perpendicular to $\overline{AB}$ and point P is the midpoint of $\overline{AB}$. By the definition of midpoint, $AP = BP$, and by the definition of perpendicular lines, $m\angle CPA = m\angle CPB = 90°$. Then by the definition of segment congruence, $\overline{AP} \cong \overline{BP}$, and by the definition of angle congruence, $\angle CPA \cong \angle CPB$. By the Reflexive Property of Congruence (Theorem 2.1), $\overline{CP} \cong \overline{CP}$. So, $\triangle CPA \cong \triangle CPB$ by the SAS Congruence Theorem (Theorem 5.5), and $\overline{CA} \cong \overline{CB}$ because corresponding parts of congruent triangles are congruent. So, $CA = CB$ by the definition of segment congruence.

EXAMPLE 1 Using the Perpendicular Bisector Theorems

Find each measure.

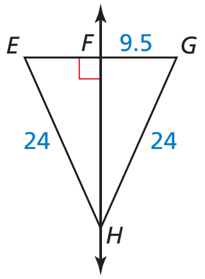
a. *RS*



From the figure, $\overleftrightarrow{SQ}$ is the perpendicular bisector of $\overline{PR}$. By the Perpendicular Bisector Theorem, $PS = RS$.

▶ So, $RS = PS = 6.8$.

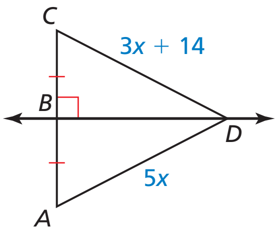
b. *EG*



Because $EH = GH$ and $\overleftrightarrow{HF} \perp \overline{EG}$, $\overleftrightarrow{HF}$ is the perpendicular bisector of $\overline{EG}$ by the Converse of the Perpendicular Bisector Theorem. By the definition of segment bisector, $EG = 2GF$.

▶ So, $EG = 2(9.5) = 19$.

c. *AD*

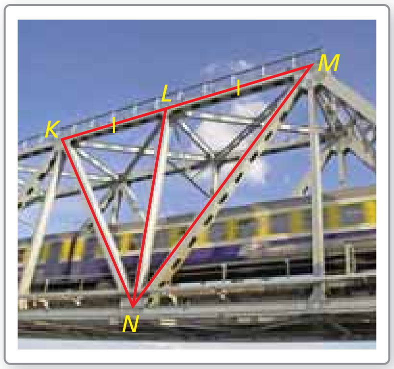


From the figure, $\overleftrightarrow{BD}$ is the perpendicular bisector of $\overline{AC}$.

$AD = CD$	Perpendicular Bisector Theorem
$5x = 3x + 14$	Substitute.
$x = 7$	Solve for *x*.

▶ So, $AD = 5x = 5(7) = 35$.

EXAMPLE 2 Solving a Real-Life Problem



Is there enough information in the diagram to conclude that point *N* lies on the perpendicular bisector of $\overline{KM}$?

SOLUTION

It is given that $\overline{KL} \cong \overline{ML}$. So, $\overline{LN}$ is a segment bisector of $\overline{KM}$. You do not know whether $\overline{LN}$ is perpendicular to $\overline{KM}$ because it is not indicated in the diagram.

▶ So, you cannot conclude that point *N* lies on the perpendicular bisector of $\overline{KM}$.

Monitoring Progress

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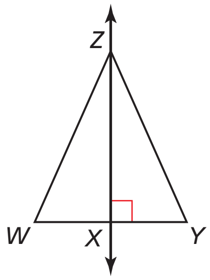
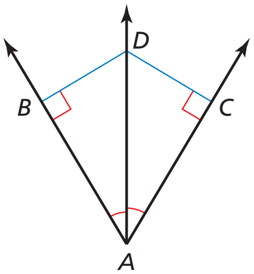
Use the diagram and the given information to find the indicated measure.



1. $\overleftrightarrow{ZX}$ is the perpendicular bisector of $\overline{WY}$, and $YZ = 13.75$. Find *WZ*.
2. $\overleftrightarrow{ZX}$ is the perpendicular bisector of $\overline{WY}$, $WZ = 4n - 13$, and $YZ = n + 17$. Find *YZ*.
3. Find *WX* when $WZ = 20.5$, $WY = 14.8$, and $YZ = 20.5$.

Using Angle Bisectors



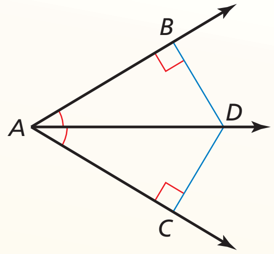
In Section 1.5, you learned that an *angle bisector* is a ray that divides an angle into two congruent adjacent angles. You also know that the *distance from a point to a line* is the length of the perpendicular segment from the point to the line. So, in the figure, $\overrightarrow{AD}$ is the bisector of $\angle BAC$, and the distance from point D to $\overrightarrow{AB}$ is DB, where $\overline{DB} \perp \overrightarrow{AB}$.

Theorems

Theorem 6.3 Angle Bisector Theorem

If a point lies on the bisector of an angle, then it is equidistant from the two sides of the angle.

If $\overrightarrow{AD}$ bisects $\angle BAC$ and $\overline{DB} \perp \overrightarrow{AB}$ and $\overline{DC} \perp \overrightarrow{AC}$, then $DB = DC$.

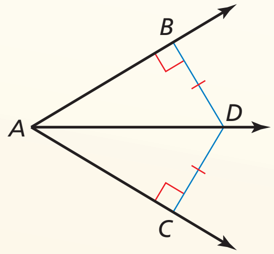


Proof Ex. 33(a), p. 308

Theorem 6.4 Converse of the Angle Bisector Theorem

If a point is in the interior of an angle and is equidistant from the two sides of the angle, then it lies on the bisector of the angle.

If $\overline{DB} \perp \overrightarrow{AB}$ and $\overline{DC} \perp \overrightarrow{AC}$ and $DB = DC$, then $\overrightarrow{AD}$ bisects $\angle BAC$.

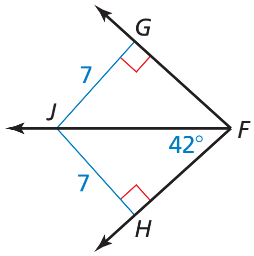


Proof Ex. 33(b), p. 308

EXAMPLE 3 Using the Angle Bisector Theorems

Find each measure.

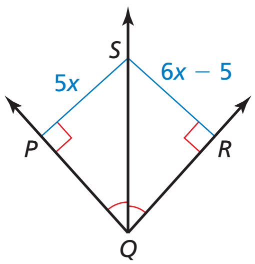
a. $m\angle GFJ$



Because $\overline{JG} \perp \overrightarrow{FG}$ and $\overline{JH} \perp \overrightarrow{FH}$ and $JG = JH = 7$, $\overrightarrow{FJ}$ bisects $\angle GFH$ by the Converse of the Angle Bisector Theorem.

▶ So, $m\angle GFJ = m\angle HFJ = 42°$.

b. RS



$PS = RS$	Angle Bisector Theorem
$5x = 6x - 5$	Substitute.
$5 = x$	Solve for x.

▶ So, $RS = 6x - 5 = 6(5) - 5 = 25$.

Monitoring Progress

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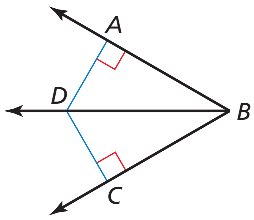
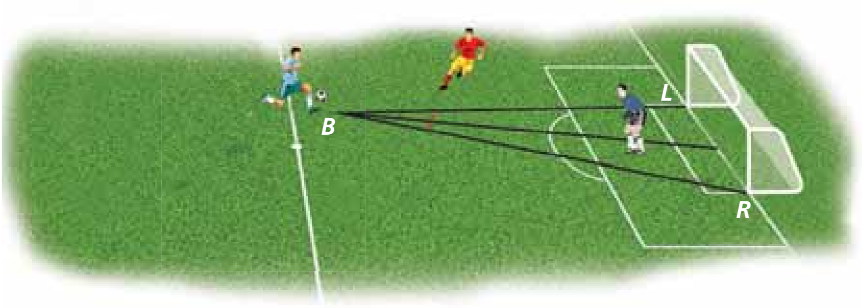
Use the diagram and the given information to find the indicated measure.



4. $\overrightarrow{BD}$ bisects $\angle ABC$, and $DC = 6.9$. Find DA.
5. $\overrightarrow{BD}$ bisects $\angle ABC$, $AD = 3z + 7$, and $CD = 2z + 11$. Find CD.
6. Find $m\angle ABC$ when $AD = 3.2$, $CD = 3.2$, and $m\angle DBC = 39°$.

EXAMPLE 4 Solving a Real-Life Problem

A soccer goalie's position relative to the ball and goalposts forms congruent angles, as shown. Will the goalie have to move farther to block a shot toward the right goalpost R or the left goalpost L?



SOLUTION

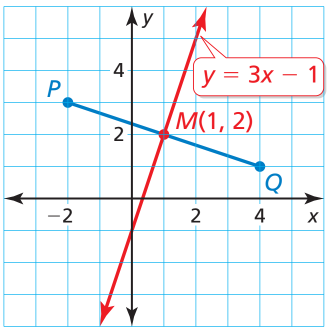
The congruent angles tell you that the goalie is on the bisector of $\angle LBR$. By the Angle Bisector Theorem, the goalie is equidistant from $\overrightarrow{BR}$ and $\overrightarrow{BL}$.

▶ So, the goalie must move the same distance to block either shot.

Writing Equations for Perpendicular Bisectors

EXAMPLE 5 Writing an Equation for a Bisector

Write an equation of the perpendicular bisector of the segment with endpoints $P(-2, 3)$ and $Q(4, 1)$.



SOLUTION

Step 1 Graph $\overline{PQ}$. By definition, the perpendicular bisector of $\overline{PQ}$ is perpendicular to $\overline{PQ}$ at its midpoint.

Step 2 Find the midpoint M of $\overline{PQ}$.

$$M\left(\frac{-2+4}{2}, \frac{3+1}{2}\right) = M\left(\frac{2}{2}, \frac{4}{2}\right) = M(1, 2)$$

Step 3 Find the slope of the perpendicular bisector.

$$\text{slope of } \overline{PQ} = \frac{1-3}{4-(-2)} = \frac{-2}{6} = -\frac{1}{3}$$

Because the slopes of perpendicular lines are negative reciprocals, the slope of the perpendicular bisector is 3.

Step 4 Write an equation. The bisector of $\overline{PQ}$ has slope 3 and passes through (1, 2).

$y = mx + b$	Use slope-intercept form.
$2 = 3(1) + b$	Substitute for m, x, and y.
$-1 = b$	Solve for b.

▶ So, an equation of the perpendicular bisector of $\overline{PQ}$ is $y = 3x - 1$.

Monitoring Progress

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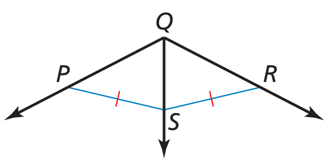


7. Do you have enough information to conclude that $\overrightarrow{QS}$ bisects $\angle PQR$? Explain.

8. Write an equation of the perpendicular bisector of the segment with endpoints $(-1, -5)$ and $(3, -1)$.

6.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** Point C is in the interior of $\angle DEF$. If $\angle DEC$ and $\angle CEF$ are congruent, then $\overrightarrow{EC}$ is the ________ of $\angle DEF$.

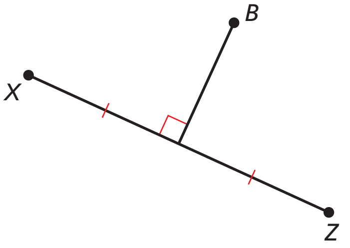
2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

Is point B the same distance from both X and Z?

Is point B equidistant from X and Z?

Is point B collinear with X and Z?

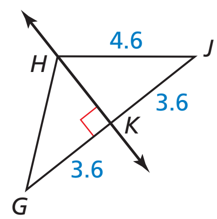
Is point B on the perpendicular bisector of $\overline{XZ}$?



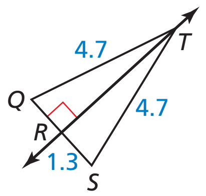
Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, find the indicated measure. Explain your reasoning. *(See Example 1.)*

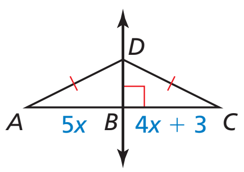
3. GH



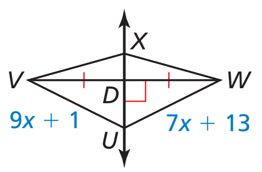
4. QR



5. AB

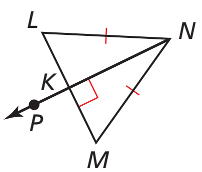


6. UW



In Exercises 7–10, tell whether the information in the diagram allows you to conclude that point P lies on the perpendicular bisector of $\overline{LM}$. Explain your reasoning. *(See Example 2.)*

7.



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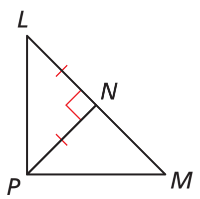
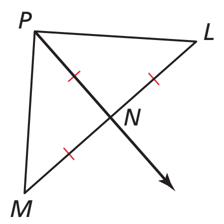
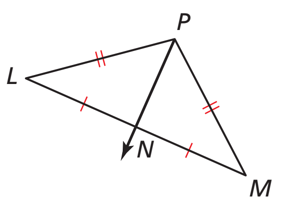


9.

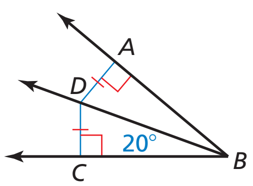


10.

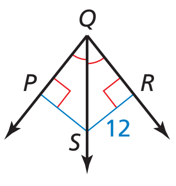


In Exercises 11–14, find the indicated measure. Explain your reasoning. *(See Example 3.)*

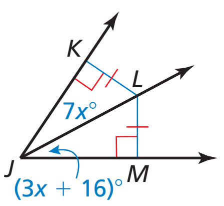
11. $m\angle ABD$



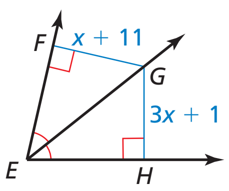
12. PS



13. $m\angle KJL$

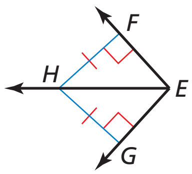


14. FG

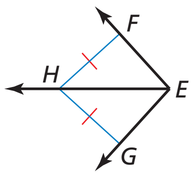


In Exercises 15 and 16, tell whether the information in the diagram allows you to conclude that $\overrightarrow{EH}$ bisects $\angle FEG$. Explain your reasoning. *(See Example 4.)*

15.

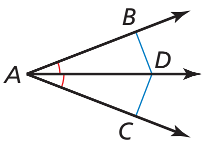


16.

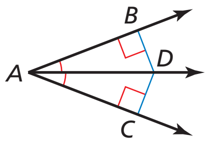


In Exercises 17 and 18, tell whether the information in the diagram allows you to conclude that $DB = DC$. Explain your reasoning.

17.



18.



In Exercises 19–22, write an equation of the perpendicular bisector of the segment with the given endpoints. *(See Example 5.)*

19. $M(1, 5)$, $N(7, -1)$

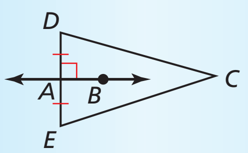
20. $Q(-2, 0)$, $R(6, 12)$

21. $U(-3, 4)$, $V(9, 8)$

22. $Y(10, -7)$, $Z(-4, 1)$

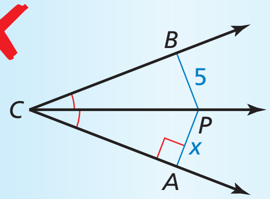
ERROR ANALYSIS **In Exercises 23 and 24, describe and correct the error in the student's reasoning.**

23.



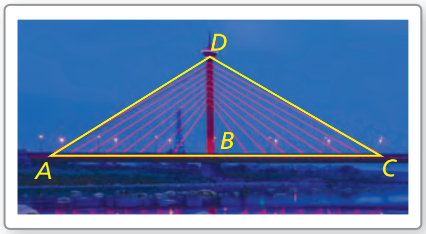
Because $AD = AE$, $\overrightarrow{AB}$ will pass through point C.

24.

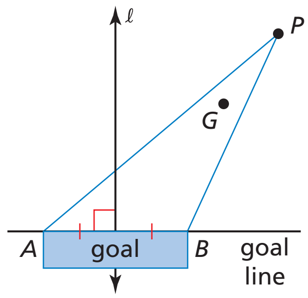


By the Angle Bisector Theorem (Theorem 6.3), $x = 5$.

25. **MODELING MATHEMATICS** In the photo, the road is perpendicular to the support beam and $\overline{AB} \cong \overline{CB}$. Which theorem allows you to conclude that $\overline{AD} \cong \overline{CD}$?



26. **MODELING WITH MATHEMATICS** The diagram shows the position of the goalie and the puck during a hockey game. The goalie is at point G, and the puck is at point P.



a. What should be the relationship between $\overrightarrow{PG}$ and $\angle APB$ to give the goalie equal distances to travel on each side of $\overrightarrow{PG}$?

b. How does $m\angle APB$ change as the puck gets closer to the goal? Does this change make it easier or more difficult for the goalie to defend the goal? Explain your reasoning.

27. **CONSTRUCTION** Use a compass and straightedge to construct a copy of $\overline{XY}$. Construct a perpendicular bisector and plot a point Z on the bisector so that the distance between point Z and $\overline{XY}$ is 3 centimeters. Measure $\overline{XZ}$ and $\overline{YZ}$. Which theorem does this construction demonstrate?

28. **WRITING** Explain how the Converse of the Perpendicular Bisector Theorem (Theorem 6.2) is related to the construction of a perpendicular bisector.

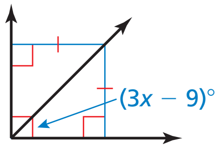
29. **REASONING** What is the value of x in the diagram?

Ⓐ 13

Ⓑ 18

Ⓒ 33

Ⓓ not enough information



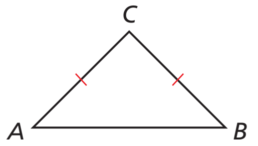
30. **REASONING** Which point lies on the perpendicular bisector of the segment with endpoints $M(7, 5)$ and $N(-1, 5)$?

Ⓐ (2, 0) Ⓑ (3, 9)

Ⓒ (4, 1) Ⓓ (1, 3)

31. **MAKING AN ARGUMENT** Your friend says it is impossible for an angle bisector of a triangle to be the same line as the perpendicular bisector of the opposite side. Is your friend correct? Explain your reasoning.

32. **PROVING A THEOREM** Prove the Converse of the Perpendicular Bisector Theorem (Thm. 6.2). (*Hint:* Construct a line through point C perpendicular to $\overline{AB}$ at point P.)



Given $CA = CB$

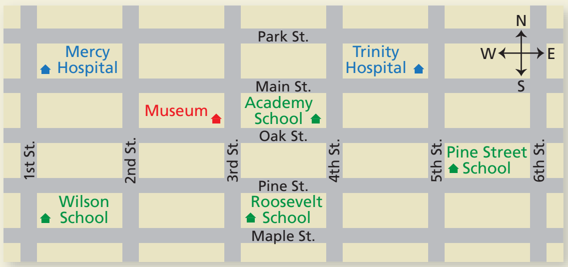
Prove Point C lies on the perpendicular bisector of $\overline{AB}$.

33. **PROVING A THEOREM** Use a congruence theorem to prove each theorem.

a. Angle Bisector Theorem (Thm. 6.3)

b. Converse of the Angle Bisector Theorem (Thm. 6.4)

34. **HOW DO YOU SEE IT?** The figure shows a map of a city. The city is arranged so each block north to south is the same length and each block east to west is the same length.



a. Which school is approximately equidistant from both hospitals? Explain your reasoning.

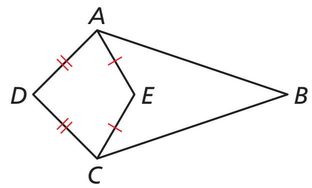
b. Is the museum approximately equidistant from Wilson School and Roosevelt School? Explain your reasoning.

35. **MATHEMATICAL CONNECTIONS** Write an equation whose graph consists of all the points in the given quadrants that are equidistant from the x- and y-axes.

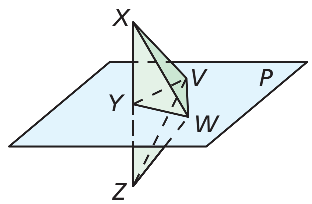
a. I and III **b.** II and IV **c.** I and II

36. **THOUGHT PROVOKING** The postulates and theorems in this book represent Euclidean geometry. In spherical geometry, all points are on the surface of a sphere. A line is a circle on the sphere whose diameter is equal to the diameter of the sphere. In spherical geometry, is it possible for two lines to be perpendicular but not bisect each other? Explain your reasoning.

37. **PROOF** Use the information in the diagram to prove that $\overline{AB} \cong \overline{CB}$ if and only if points D, E, and B are collinear.



38. **PROOF** Prove the statements in parts (a)–(c).



Given Plane P is a perpendicular bisector of $\overline{XZ}$ at point Y.

Prove **a.** $\overline{XW} \cong \overline{ZW}$

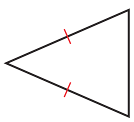
b. $\overline{XV} \cong \overline{ZV}$

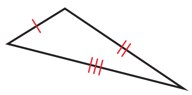
c. $\angle VXW \cong \angle VZW$

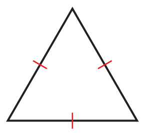
Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

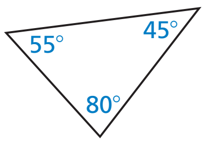
Classify the triangle by its sides. *(Section 5.1)*

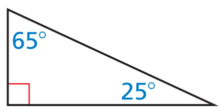
39. 

40. 

41. 

Classify the triangle by its angles. *(Section 5.1)*

42. 

43. 

44.