5.8 Coordinate Proofs

Essential Question

How can you use a coordinate plane to write a proof?

EXPLORATION 1 Writing a Coordinate Proof

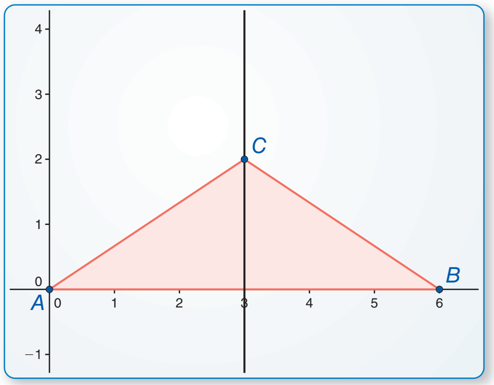
Work with a partner.

a. Use dynamic geometry software to draw $\overline{AB}$ with endpoints $A(0, 0)$ and $B(6, 0)$.

b. Draw the vertical line $x = 3$.

c. Draw $\triangle ABC$ so that C lies on the line $x = 3$.

d. Use your drawing to prove that $\triangle ABC$ is an isosceles triangle.



Sample

Points
$A(0, 0)$
$B(6, 0)$
$C(3, y)$
Segments
$AB = 6$
Line
$x = 3$

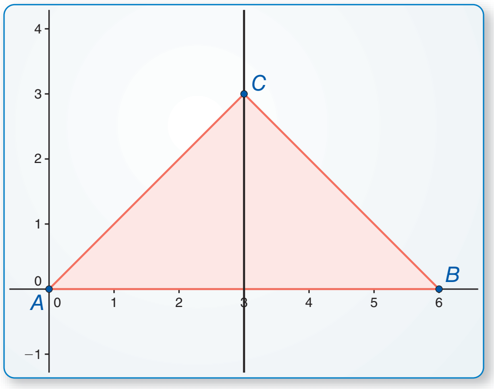
EXPLORATION 2 Writing a Coordinate Proof

Work with a partner.

a. Use dynamic geometry software to draw $\overline{AB}$ with endpoints $A(0, 0)$ and $B(6, 0)$.

b. Draw the vertical line $x = 3$.

c. Plot the point $C(3, 3)$ and draw $\triangle ABC$. Then use your drawing to prove that $\triangle ABC$ is an isosceles right triangle.



Sample

Points
$A(0, 0)$
$B(6, 0)$
$C(3, 3)$
Segments
$AB = 6$
$BC = 4.24$
$AC = 4.24$
Line
$x = 3$

d. Change the coordinates of C so that C lies below the x-axis and $\triangle ABC$ is an isosceles right triangle.

e. Write a coordinate proof to show that if C lies on the line $x = 3$ and $\triangle ABC$ is an isosceles right triangle, then C must be the point $(3, 3)$ or the point found in part (d).

CRITIQUING THE REASONING OF OTHERS

To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results.

Communicate Your Answer

3. How can you use a coordinate plane to write a proof?

4. Write a coordinate proof to prove that $\triangle ABC$ with vertices $A(0, 0)$, $B(6, 0)$, and $C(3, 3\sqrt{3})$ is an equilateral triangle.

5.8 Lesson

Core Vocabulary

coordinate proof, *p. 284*

What You Will Learn

- Place figures in a coordinate plane.
- Write coordinate proofs.

Placing Figures in a Coordinate Plane

A **coordinate proof** involves placing geometric figures in a coordinate plane. When you use variables to represent the coordinates of a figure in a coordinate proof, the results are true for all figures of that type.

EXAMPLE 1 Placing a Figure in a Coordinate Plane

Place each figure in a coordinate plane in a way that is convenient for finding side lengths. Assign coordinates to each vertex.

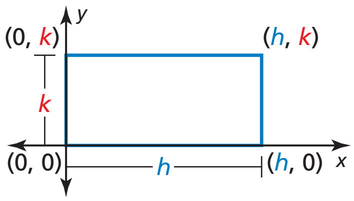
a. a rectangle

b. a scalene triangle

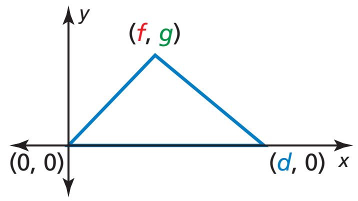
SOLUTION

It is easy to find lengths of horizontal and vertical segments and distances from (0, 0), so place one vertex at the origin and one or more sides on an axis.

a. Let h represent the length and k represent the width.



b. Notice that you need to use three different variables.



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1. Show another way to place the rectangle in Example 1 part (a) that is convenient for finding side lengths. Assign new coordinates.
2. A square has vertices (0, 0), (m, 0), and (0, m). Find the fourth vertex.

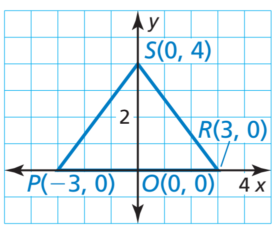
Once a figure is placed in a coordinate plane, you may be able to prove statements about the figure.

EXAMPLE 2 Writing a Plan for a Coordinate Proof

Write a plan to prove that $\overrightarrow{SO}$ bisects $\angle PSR$.

Given Coordinates of vertices of $\triangle POS$ and $\triangle ROS$

Prove $\overrightarrow{SO}$ bisects $\angle PSR$.



SOLUTION

Plan for Proof Use the Distance Formula to find the side lengths of $\triangle POS$ and $\triangle ROS$. Then use the SSS Congruence Theorem (Theorem 5.8) to show that $\triangle POS \cong \triangle ROS$. Finally, use the fact that corresponding parts of congruent triangles are congruent to conclude that $\angle PSO \cong \angle RSO$, which implies that $\overrightarrow{SO}$ bisects $\angle PSR$.

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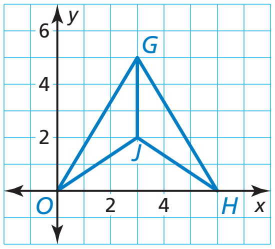
3. Write a plan for the proof.

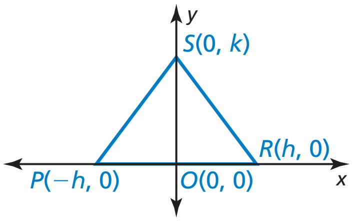
Given $\overrightarrow{GJ}$ bisects $\angle OGH$.

Prove $\triangle GJO \cong \triangle GJH$



The coordinate proof in Example 2 applies to a specific triangle. When you want to prove a statement about a more general set of figures, it is helpful to use variables as coordinates.

For instance, you can use variable coordinates to duplicate the proof in Example 2. Once this is done, you can conclude that $\overrightarrow{SO}$ bisects $\angle PSR$ for any triangle whose coordinates fit the given pattern.

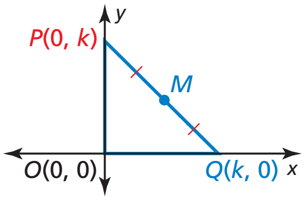


EXAMPLE 3 Applying Variable Coordinates

Place an isosceles right triangle in a coordinate plane. Then find the length of the hypotenuse and the coordinates of its midpoint M.

SOLUTION

Place $\triangle PQO$ with the right angle at the origin. Let the length of the legs be k. Then the vertices are located at $P(0, k)$, $Q(k, 0)$, and $O(0, 0)$.



Use the Distance Formula to find PQ, the length of the hypotenuse.

$$PQ = \sqrt{(k-0)^2 + (0-k)^2} = \sqrt{k^2 + (-k)^2} = \sqrt{k^2 + k^2} = \sqrt{2k^2} = k\sqrt{2}$$

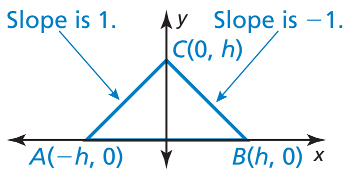
Use the Midpoint Formula to find the midpoint M of the hypotenuse.

$$M\left(\frac{0+k}{2}, \frac{k+0}{2}\right) = M\left(\frac{k}{2}, \frac{k}{2}\right)$$

▶ So, the length of the hypotenuse is $k\sqrt{2}$ and the midpoint of the hypotenuse is $\left(\frac{k}{2}, \frac{k}{2}\right)$.

FINDING AN ENTRY POINT

Another way to solve Example 3 is to place a triangle with point C at $(0, h)$ on the y-axis and hypotenuse $\overline{AB}$ on the x-axis. To make $\angle ACB$ a right angle, position A and B so that legs $\overline{CA}$ and $\overline{CB}$ have slopes of 1 and -1, respectively.



Length of hypotenuse $= 2h$

$$M\left(\frac{-h+h}{2}, \frac{0+0}{2}\right) = M(0, 0)$$

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4. Graph the points $O(0, 0)$, $H(m, n)$, and $J(m, 0)$. Is $\triangle OHJ$ a right triangle? Find the side lengths and the coordinates of the midpoint of each side.

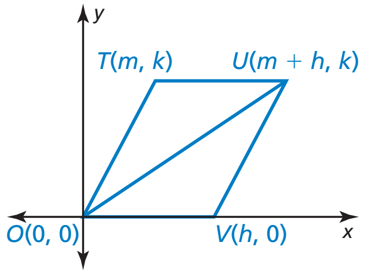
Writing Coordinate Proofs

EXAMPLE 4 Writing a Coordinate Proof

Write a coordinate proof.

Given Coordinates of vertices of quadrilateral $OTUV$

Prove $\triangle OTU \cong \triangle UVO$



SOLUTION

Segments $\overline{OV}$ and $\overline{UT}$ have the same length.

$$OV = |h - 0| = h$$

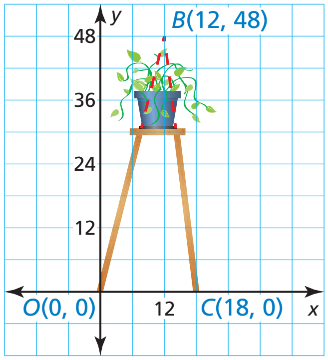
$$UT = |(m + h) - m| = h$$

Horizontal segments $\overline{UT}$ and $\overline{OV}$ each have a slope of 0, which implies that they are parallel. Segment $\overline{OU}$ intersects $\overline{UT}$ and $\overline{OV}$ to form congruent alternate interior angles, $\angle TUO$ and $\angle VOU$. By the Reflexive Property of Congruence (Theorem 2.1), $\overline{OU} \cong \overline{OU}$.

▶ So, you can apply the SAS Congruence Theorem (Theorem 5.5) to conclude that $\triangle OTU \cong \triangle UVO$.

EXAMPLE 5 Writing a Coordinate Proof

You buy a tall, three-legged plant stand. When you place a plant on the stand, the stand appears to be unstable under the weight of the plant. The diagram at the right shows a coordinate plane superimposed on one pair of the plant stand's legs. The legs are extended to form $\triangle OBC$. Prove that $\triangle OBC$ is a scalene triangle. Explain why the plant stand may be unstable.



SOLUTION

First, find the side lengths of $\triangle OBC$.

$$OB = \sqrt{(48 - 0)^2 + (12 - 0)^2} = \sqrt{2448} \approx 49.5$$

$$BC = \sqrt{(18 - 12)^2 + (0 - 48)^2} = \sqrt{2340} \approx 48.4$$

$$OC = |18 - 0| = 18$$

▶ Because $\triangle OBC$ has no congruent sides, $\triangle OBC$ is a scalene triangle by definition. The plant stand may be unstable because $\overline{OB}$ is longer than $\overline{BC}$, so the plant stand is leaning to the right.

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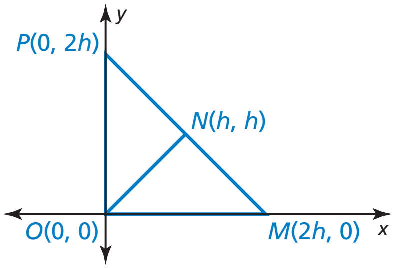
5. Write a coordinate proof.

Given Coordinates of vertices of $\triangle NPO$ and $\triangle NMO$

Prove $\triangle NPO \cong \triangle NMO$



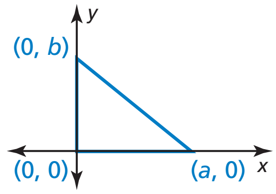
5.8 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** How is a *coordinate proof* different from other types of proofs you have studied? How is it the same?

2. **WRITING** Explain why it is convenient to place a right triangle on the grid as shown when writing a coordinate proof.



Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, place the figure in a coordinate plane in a convenient way. Assign coordinates to each vertex. Explain the advantages of your placement. *(See Example 1.)*

3. a right triangle with leg lengths of 3 units and 2 units

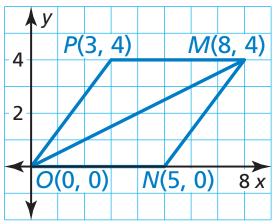
4. a square with a side length of 3 units

5. an isosceles right triangle with leg length p

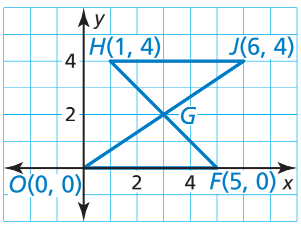
6. a scalene triangle with one side length of $2m$

In Exercises 7 and 8, write a plan for the proof. *(See Example 2.)*

7. **Given** Coordinates of vertices of $\triangle OPM$ and $\triangle ONM$
   Prove $\triangle OPM$ and $\triangle ONM$ are isosceles triangles.



8. **Given** G is the midpoint of $\overline{HF}$.
   Prove $\triangle GHJ \cong \triangle GFO$



In Exercises 9–12, place the figure in a coordinate plane and find the indicated length.

9. a right triangle with leg lengths of 7 and 9 units; Find the length of the hypotenuse.

10. an isosceles triangle with a base length of 60 units and a height of 50 units; Find the length of one of the legs.

11. a rectangle with a length of 5 units and a width of 4 units; Find the length of the diagonal.

12. a square with side length n; Find the length of the diagonal.

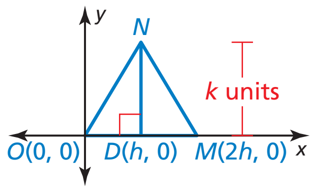
In Exercises 13 and 14, graph the triangle with the given vertices. Find the length and the slope of each side of the triangle. Then find the coordinates of the midpoint of each side. Is the triangle a right triangle? isosceles? Explain. (Assume all variables are positive and $m \neq n$.) *(See Example 3.)*

13. $A(0, 0)$, $B(h, h)$, $C(2h, 0)$

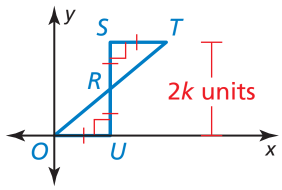
14. $D(0, n)$, $E(m, n)$, $F(m, 0)$

In Exercises 15 and 16, find the coordinates of any unlabeled vertices. Then find the indicated length(s).

15. Find ON and MN.



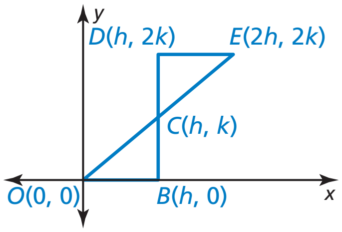
16. Find OT.



PROOF **In Exercises 17 and 18, write a coordinate proof.** *(See Example 4.)*

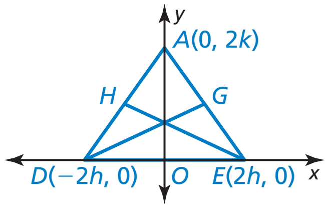
17. **Given** Coordinates of vertices of $\triangle DEC$ and $\triangle BOC$

Prove $\triangle DEC \cong \triangle BOC$



18. **Given** Coordinates of $\triangle DEA$, H is the midpoint of $\overline{DA}$, G is the midpoint of $\overline{EA}$.

Prove $\overline{DG} \cong \overline{EH}$



19. **MODELING WITH MATHEMATICS** You and your cousin are camping in the woods. You hike to a point that is 500 meters east and 1200 meters north of the campsite. Your cousin hikes to a point that is 1000 meters east of the campsite. Use a coordinate proof to prove that the triangle formed by your position, your cousin's position, and the campsite is isosceles. *(See Example 5.)*

20. **MAKING AN ARGUMENT** Two friends see a drawing of quadrilateral $PQRS$ with vertices $P(0, 2)$, $Q(3, -4)$, $R(1, -5)$, and $S(-2, 1)$. One friend says the quadrilateral is a parallelogram but not a rectangle. The other friend says the quadrilateral is a rectangle. Which friend is correct? Use a coordinate proof to support your answer.

21. **MATHEMATICAL CONNECTIONS** Write an algebraic expression for the coordinates of each endpoint of a line segment whose midpoint is the origin.

22. **REASONING** The vertices of a parallelogram are $(w, 0)$, $(0, v)$, $(-w, 0)$, and $(0, -v)$. What is the midpoint of the side in Quadrant III?

Ⓐ $\left(\frac{w}{2}, \frac{v}{2}\right)$ Ⓑ $\left(-\frac{w}{2}, -\frac{v}{2}\right)$

Ⓒ $\left(-\frac{w}{2}, \frac{v}{2}\right)$ Ⓓ $\left(\frac{w}{2}, -\frac{v}{2}\right)$

23. **REASONING** A rectangle with a length of $3h$ and a width of k has a vertex at $(-h, k)$. Which point cannot be a vertex of the rectangle?

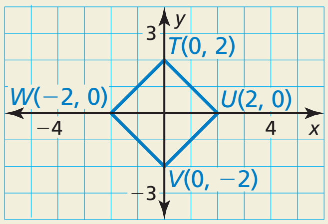
Ⓐ (h, k) Ⓑ $(-h, 0)$

Ⓒ $(2h, 0)$ Ⓓ $(2h, k)$

24. **THOUGHT PROVOKING** Choose one of the theorems you have encountered up to this point that you think would be easier to prove with a coordinate proof than with another type of proof. Explain your reasoning. Then write a coordinate proof.

25. **CRITICAL THINKING** The coordinates of a triangle are $(5d, -5d)$, $(0, -5d)$, and $(5d, 0)$. How should the coordinates be changed to make a coordinate proof easier to complete?

26. **HOW DO YOU SEE IT?** Without performing any calculations, how do you know that the diagonals of square $TUVW$ are perpendicular to each other? How can you use a similar diagram to show that the diagonals of any square are perpendicular to each other?



27. **PROOF** Write a coordinate proof for each statement.

a. The midpoint of the hypotenuse of a right triangle is the same distance from each vertex of the triangle.

b. Any two congruent right isosceles triangles can be combined to form a single isosceles triangle.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

$\overrightarrow{YW}$ **bisects** $\angle XYZ$ **such that** $m\angle XYW = (3x - 7)°$ **and** $m\angle WYZ = (2x + 1)°$. *(Section 1.5)*

28. Find the value of x.

29. Find $m\angle XYZ$.