4.1 Translations

Essential Question

How can you translate a figure in a coordinate plane?

EXPLORATION 1 Translating a Triangle in a Coordinate Plane

Work with a partner.

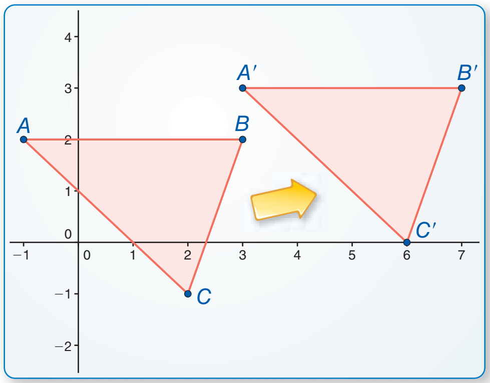
a. Use dynamic geometry software to draw any triangle and label it $\triangle ABC$.

b. Copy the triangle and *translate* (or slide) it to form a new figure, called an *image*, $\triangle A'B'C'$ (read as "triangle A prime, B prime, C prime").

c. What is the relationship between the coordinates of the vertices of $\triangle ABC$ and those of $\triangle A'B'C'$?

d. What do you observe about the side lengths and angle measures of the two triangles?

> **USING TOOLS STRATEGICALLY**
>
> To be proficient in math, you need to use appropriate tools strategically, including dynamic geometry software.



Sample

Points
$A(-1, 2)$
$B(3, 2)$
$C(2, -1)$
Segments
$AB = 4$
$BC = 3.16$
$AC = 4.24$
Angles
$m\angle A = 45°$
$m\angle B = 71.57°$
$m\angle C = 63.43°$

EXPLORATION 2 Translating a Triangle in a Coordinate Plane

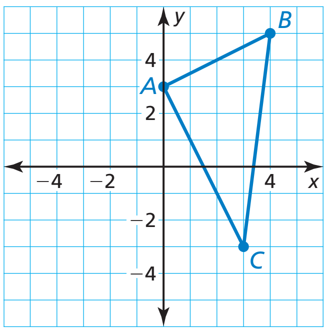
Work with a partner.

a. The point (x, y) is translated a units horizontally and b units vertically. Write a rule to determine the coordinates of the image of (x, y).

$$(x, y) \rightarrow (\square, \square)$$

b. Use the rule you wrote in part (a) to translate $\triangle ABC$ 4 units left and 3 units down. What are the coordinates of the vertices of the image, $\triangle A'B'C'$?

c. Draw $\triangle A'B'C'$. Are its side lengths the same as those of $\triangle ABC$? Justify your answer.



EXPLORATION 3 Comparing Angles of Translations

Work with a partner.

a. In Exploration 2, is $\triangle ABC$ a right triangle? Justify your answer.

b. In Exploration 2, is $\triangle A'B'C'$ a right triangle? Justify your answer.

c. Do you think translations always preserve angle measures? Explain your reasoning.

Communicate Your Answer

4. How can you translate a figure in a coordinate plane?

5. In Exploration 2, translate $\triangle A'B'C'$ 3 units right and 4 units up. What are the coordinates of the vertices of the image, $\triangle A''B''C''$? How are these coordinates related to the coordinates of the vertices of the original triangle, $\triangle ABC$?

4.1 Lesson

What You Will Learn

- Perform translations.
- Perform compositions.
- Solve real-life problems involving compositions.

Core Vocabulary

vector, *p. 174*
initial point, *p. 174*
terminal point, *p. 174*
horizontal component, *p. 174*
vertical component, *p. 174*
component form, *p. 174*
transformation, *p. 174*
image, *p. 174*
preimage, *p. 174*
translation, *p. 174*
rigid motion, *p. 176*
composition of transformations, *p. 176*

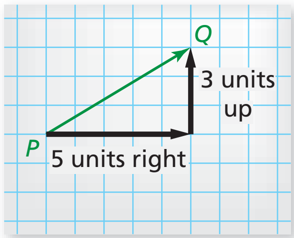
Performing Translations

A **vector** is a quantity that has both direction and *magnitude*, or size, and is represented in the coordinate plane by an arrow drawn from one point to another.

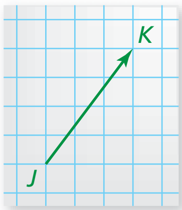
Core Concept

Vectors

The diagram shows a vector. The **initial point**, or starting point, of the vector is P, and the **terminal point**, or ending point, is Q. The vector is named $\overrightarrow{PQ}$, which is read as "vector PQ." The **horizontal component** of $\overrightarrow{PQ}$ is 5, and the **vertical component** is 3. The **component form** of a vector combines the horizontal and vertical components. So, the component form of $\overrightarrow{PQ}$ is $\langle 5, 3\rangle$.



EXAMPLE 1 Identifying Vector Components



In the diagram, name the vector and write its component form.

SOLUTION

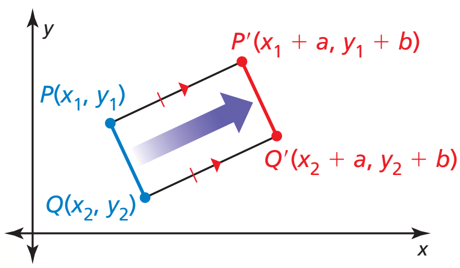
The vector is $\overrightarrow{JK}$. To move from the initial point J to the terminal point K, you move 3 units right and 4 units up. So, the component form is $\langle 3, 4\rangle$.

A **transformation** is a function that moves or changes a figure in some way to produce a new figure called an **image**. Another name for the original figure is the **preimage**. The points on the preimage are the inputs for the transformation, and the points on the image are the outputs.

Core Concept

Translations

A **translation** moves every point of a figure the same distance in the same direction. More specifically, a translation *maps*, or moves, the points P and Q of a plane figure along a vector $\langle a, b\rangle$ to the points P' and Q', so that one of the following statements is true.



- $PP' = QQ'$ and $\overline{PP'} \parallel \overline{QQ'}$, or
- $PP' = QQ'$ and $\overline{PP'}$ and $\overline{QQ'}$ are collinear.

STUDY TIP

You can use *prime notation* to name an image. For example, if the preimage is point P, then its image is point P', read as "point P *prime*."

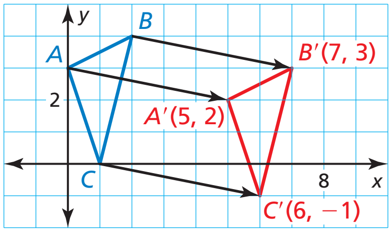
Translations map lines to parallel lines and segments to parallel segments. For instance, in the figure above, $\overline{PQ} \parallel \overline{P'Q'}$.

EXAMPLE 2 Translating a Figure Using a Vector

The vertices of $\triangle ABC$ are $A(0, 3)$, $B(2, 4)$, and $C(1, 0)$. Translate $\triangle ABC$ using the vector $\langle 5, -1 \rangle$.

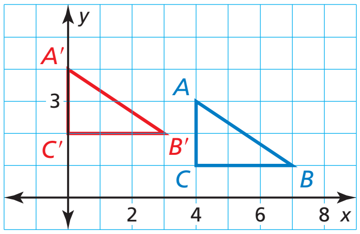
SOLUTION

First, graph $\triangle ABC$. Use $\langle 5, -1 \rangle$ to move each vertex 5 units right and 1 unit down. Label the image vertices. Draw $\triangle A'B'C'$. Notice that the vectors drawn from preimage vertices to image vertices are parallel.



You can also express a translation along the vector $\langle a, b \rangle$ using a rule, which has the notation $(x, y) \rightarrow (x + a, y + b)$.

EXAMPLE 3 Writing a Translation Rule



Write a rule for the translation of $\triangle ABC$ to $\triangle A'B'C'$.

SOLUTION

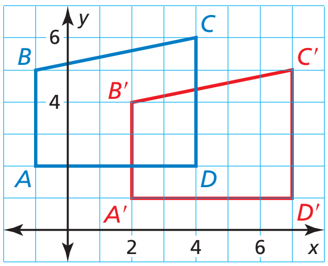
To go from A to A', you move 4 units left and 1 unit up, so you move along the vector $\langle -4, 1 \rangle$.

▶ So, a rule for the translation is $(x, y) \rightarrow (x - 4, y + 1)$.

EXAMPLE 4 Translating a Figure in the Coordinate Plane

Graph quadrilateral $ABCD$ with vertices $A(-1, 2)$, $B(-1, 5)$, $C(4, 6)$, and $D(4, 2)$ and its image after the translation $(x, y) \rightarrow (x + 3, y - 1)$.

SOLUTION



Graph quadrilateral $ABCD$. To find the coordinates of the vertices of the image, add 3 to the x-coordinates and subtract 1 from the y-coordinates of the vertices of the preimage. Then graph the image, as shown at the left.

$$(x, y) \rightarrow (x + 3, y - 1)$$

$$A(-1, 2) \rightarrow A'(2, 1)$$

$$B(-1, 5) \rightarrow B'(2, 4)$$

$$C(4, 6) \rightarrow C'(7, 5)$$

$$D(4, 2) \rightarrow D'(7, 1)$$

Monitoring Progress

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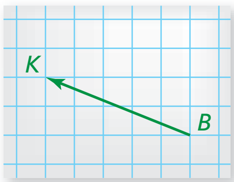


1. Name the vector and write its component form.
2. The vertices of $\triangle LMN$ are $L(2, 2)$, $M(5, 3)$, and $N(9, 1)$. Translate $\triangle LMN$ using the vector $\langle -2, 6 \rangle$.
3. In Example 3, write a rule to translate $\triangle A'B'C'$ back to $\triangle ABC$.
4. Graph $\triangle RST$ with vertices $R(2, 2)$, $S(5, 2)$, and $T(3, 5)$ and its image after the translation $(x, y) \rightarrow (x + 1, y + 2)$.

Performing Compositions

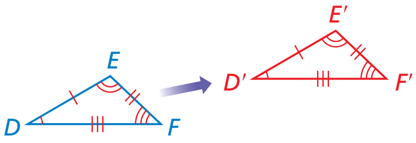
A **rigid motion** is a transformation that preserves length and angle measure. Another name for a rigid motion is an *isometry*. A rigid motion maps lines to lines, rays to rays, and segments to segments.

Postulate

Postulate 4.1 Translation Postulate

A translation is a rigid motion.

Because a translation is a rigid motion, and a rigid motion preserves length and angle measure, the following statements are true for the translation shown.



- $DE = D'E'$, $EF = E'F'$, $FD = F'D'$
- $m\angle D = m\angle D'$, $m\angle E = m\angle E'$, $m\angle F = m\angle F'$

When two or more transformations are combined to form a single transformation, the result is a **composition of transformations**.

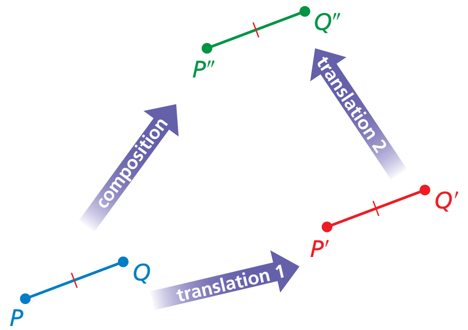
Theorem

Theorem 4.1 Composition Theorem

The composition of two (or more) rigid motions is a rigid motion.

Proof Ex. 35, p. 180

The theorem above is important because it states that no matter how many rigid motions you perform, lengths and angle measures will be preserved in the final image. For instance, the composition of two or more translations is a translation, as shown.



EXAMPLE 5 Performing a Composition

Graph $\overline{RS}$ with endpoints $R(-8, 5)$ and $S(-6, 8)$ and its image after the composition.

Translation: $(x, y) \rightarrow (x + 5, y - 2)$

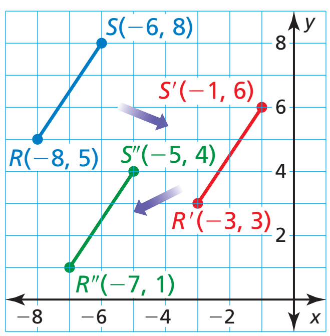
Translation: $(x, y) \rightarrow (x - 4, y - 2)$

SOLUTION

Step 1 Graph $\overline{RS}$.

Step 2 Translate $\overline{RS}$ 5 units right and 2 units down. $\overline{R'S'}$ has endpoints $R'(-3, 3)$ and $S'(-1, 6)$.

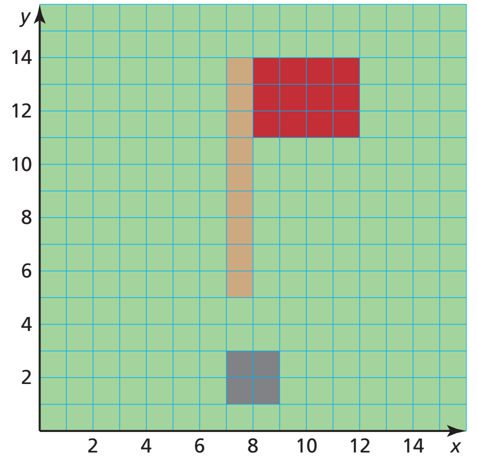
Step 3 Translate $\overline{R'S'}$ 4 units left and 2 units down. $\overline{R''S''}$ has endpoints $R''(-7, 1)$ and $S''(-5, 4)$.



Solving Real-Life Problems

EXAMPLE 6 **Modeling with Mathematics**

You are designing a favicon for a golf website. In an image-editing program, you move the red rectangle 2 units left and 3 units down. Then you move the red rectangle 1 unit right and 1 unit up. Rewrite the composition as a single translation.



SOLUTION

1. **Understand the Problem** You are given two translations. You need to rewrite the result of the composition of the two translations as a single translation.

2. **Make a Plan** You can choose an arbitrary point (x, y) in the red rectangle and determine the horizontal and vertical shift in the coordinates of the point after both translations. This tells you how much you need to shift each coordinate to map the original figure to the final image.

3. **Solve the Problem** Let $A(x, y)$ be an arbitrary point in the red rectangle. After the first translation, the coordinates of its image are

   $A'(x - 2, y - 3)$.

   The second translation maps $A'(x - 2, y - 3)$ to

   $A''(x - 2 + 1, y - 3 + 1) = A''(x - 1, y - 2)$.

   The composition of translations uses the original point (x, y) as the input and returns the point $(x - 1, y - 2)$ as the output.

   ▶ So, the single translation rule for the composition is $(x, y) \to (x - 1, y - 2)$.

4. **Look Back** Check that the rule is correct by testing a point. For instance, (10, 12) is a point in the red rectangle. Apply the two translations to (10, 12).

   $(10, 12) \to (8, 9) \to (9, 10)$

   Does the final result match the rule you found in Step 3?

   $(10, 12) \to (10 - 1, 12 - 2) = (9, 10)$ ✓

Monitoring Progress

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5. Graph $\overline{TU}$ with endpoints $T(1, 2)$ and $U(4, 6)$ and its image after the composition.

   Translation: $(x, y) \to (x - 2, y - 3)$
   Translation: $(x, y) \to (x - 4, y + 5)$

6. Graph $\overline{VW}$ with endpoints $V(-6, -4)$ and $W(-3, 1)$ and its image after the composition.

   Translation: $(x, y) \to (x + 3, y + 1)$
   Translation: $(x, y) \to (x - 6, y - 4)$

7. In Example 6, you move the gray square 2 units right and 3 units up. Then you move the gray square 1 unit left and 1 unit down. Rewrite the composition as a single transformation.

4.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

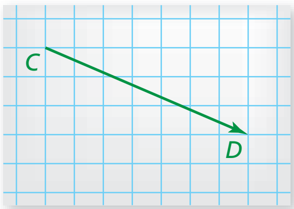
Vocabulary and Core Concept Check

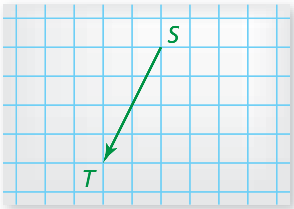
1. **VOCABULARY** Name the preimage and image of the transformation $\triangle ABC \rightarrow \triangle A'B'C'$.

2. **COMPLETE THE SENTENCE** A _______ moves every point of a figure the same distance in the same direction.

Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, name the vector and write its component form. *(See Example 1.)*

3. 

4. 

In Exercises 5–8, the vertices of $\triangle DEF$ are $D(2, 5)$, $E(6, 3)$, and $F(4, 0)$. Translate $\triangle DEF$ using the given vector. Graph $\triangle DEF$ and its image. *(See Example 2.)*

5. $\langle 6, 0\rangle$

6. $\langle 5, -1\rangle$

7. $\langle -3, -7\rangle$

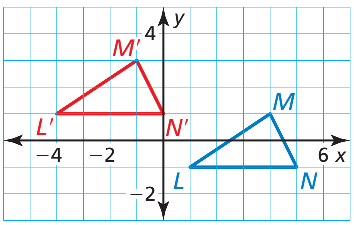
8. $\langle -2, -4\rangle$

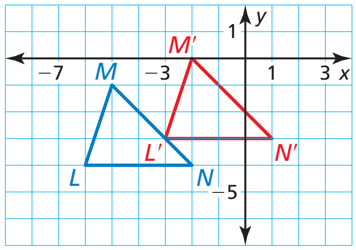
In Exercises 9 and 10, find the component form of the vector that translates $P(-3, 6)$ to P'.

9. $P'(0, 1)$

10. $P'(-4, 8)$

In Exercises 11 and 12, write a rule for the translation of $\triangle LMN$ to $\triangle L'M'N'$. *(See Example 3.)*

11. 

12. 

In Exercises 13–16, use the translation.

$(x, y) \rightarrow (x - 8, y + 4)$

13. What is the image of $A(2, 6)$?

14. What is the image of $B(-1, 5)$?

15. What is the preimage of $C'(-3, -10)$?

16. What is the preimage of $D'(4, -3)$?

In Exercises 17–20, graph $\triangle PQR$ with vertices $P(-2, 3)$, $Q(1, 2)$, and $R(3, -1)$ and its image after the translation. *(See Example 4.)*

17. $(x, y) \rightarrow (x + 4, y + 6)$

18. $(x, y) \rightarrow (x + 9, y - 2)$

19. $(x, y) \rightarrow (x - 2, y - 5)$

20. $(x, y) \rightarrow (x - 1, y + 3)$

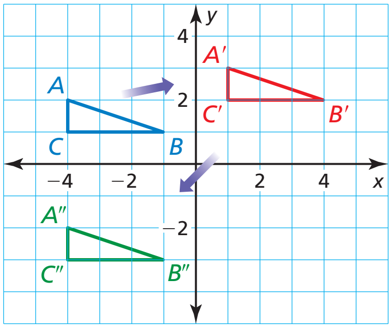
In Exercises 21 and 22, graph $\triangle XYZ$ with vertices $X(2, 4)$, $Y(6, 0)$, and $Z(7, 2)$ and its image after the composition. *(See Example 5.)*

21. **Translation:** $(x, y) \rightarrow (x + 12, y + 4)$
Translation: $(x, y) \rightarrow (x - 5, y - 9)$

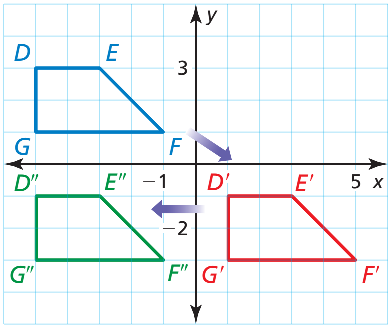
22. **Translation:** $(x, y) \rightarrow (x - 6, y)$
Translation: $(x, y) \rightarrow (x + 2, y + 7)$

In Exercises 23 and 24, describe the composition of translations.

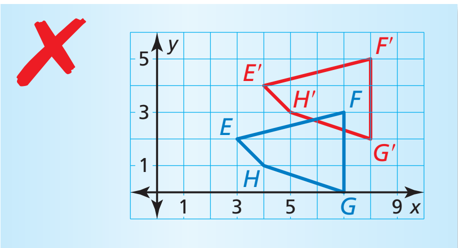
23.



24.

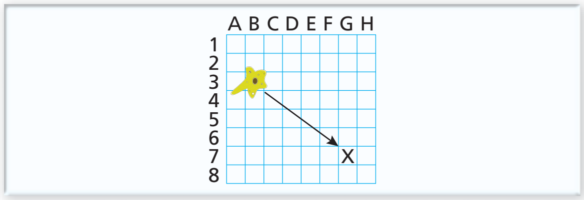


25. ERROR ANALYSIS Describe and correct the error in graphing the image of quadrilateral *EFGH* after the translation $(x, y) \rightarrow (x - 1, y - 2)$.



26. MODELING WITH MATHEMATICS In chess, the knight (the piece shaped like a horse) moves in an L pattern. The board shows two consecutive moves of a black knight during a game. Write a composition of translations for the moves. Then rewrite the composition as a single translation that moves the knight from its original position to its ending position. *(See Example 6.)*

27. PROBLEM SOLVING You are studying an amoeba through a microscope. Suppose the amoeba moves on a grid-indexed microscope slide in a straight line from square B3 to square G7.



a. Describe the translation.

b. The side length of each grid square is 2 millimeters. How far does the amoeba travel?

c. The amoeba moves from square B3 to square G7 in 24.5 seconds. What is its speed in millimeters per second?

28. MATHEMATICAL CONNECTIONS Translation A maps (x, y) to $(x + n, y + t)$. Translation B maps (x, y) to $(x + s, y + m)$.

a. Translate a point using Translation A, followed by Translation B. Write an algebraic rule for the final image of the point after this composition.

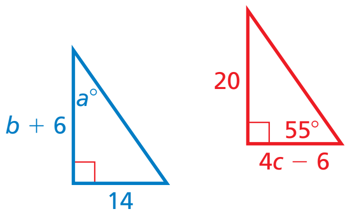
b. Translate a point using Translation B, followed by Translation A. Write an algebraic rule for the final image of the point after this composition.

c. Compare the rules you wrote for parts (a) and (b). Does it matter which translation you do first? Explain your reasoning.

MATHEMATICAL CONNECTIONS In Exercises 29 and 30, a translation maps the blue figure to the red figure. Find the value of each variable.

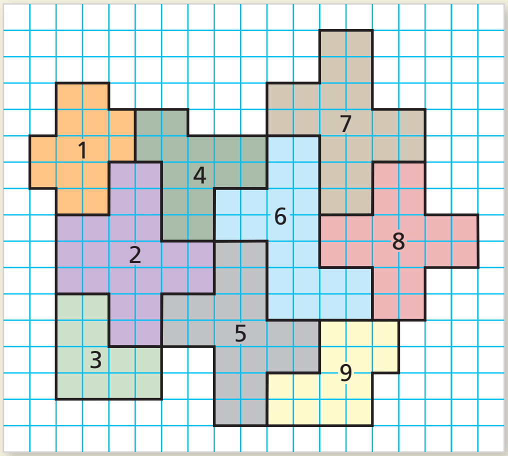
29.

30.



31. **USING STRUCTURE** Quadrilateral $DEFG$ has vertices $D(-1, 2)$, $E(-2, 0)$, $F(-1, -1)$, and $G(1, 3)$. A translation maps quadrilateral $DEFG$ to quadrilateral $D'E'F'G'$. The image of D is $D'(-2, -2)$. What are the coordinates of E', F', and G'?

32. **HOW DO YOU SEE IT?** Which two figures represent a translation? Describe the translation.



33. **REASONING** The translation $(x, y) \rightarrow (x + m, y + n)$ maps $\overline{PQ}$ to $\overline{P'Q'}$. Write a rule for the translation of $\overline{P'Q'}$ to $\overline{PQ}$. Explain your reasoning.

34. **DRAWING CONCLUSIONS** The vertices of a rectangle are $Q(2, -3)$, $R(2, 4)$, $S(5, 4)$, and $T(5, -3)$.

a. Translate rectangle $QRST$ 3 units left and 3 units down to produce rectangle $Q'R'S'T'$. Find the area of rectangle $QRST$ and the area of rectangle $Q'R'S'T'$.

b. Compare the areas. Make a conjecture about the areas of a preimage and its image after a translation.

35. **PROVING A THEOREM** Prove the Composition Theorem (Theorem 4.1).

36. **PROVING A THEOREM** Use properties of translations to prove each theorem.

a. Corresponding Angles Theorem (Theorem 3.1)

b. Corresponding Angles Converse (Theorem 3.5)

37. **WRITING** Explain how to use translations to draw a rectangular prism.

38. **MATHEMATICAL CONNECTIONS** The vector $\overrightarrow{PQ} = \langle 4, 1 \rangle$ describes the translation of $A(-1, w)$ onto $A'(2x + 1, 4)$ and $B(8y - 1, 1)$ onto $B'(3, 3z)$. Find the values of w, x, y, and z.

39. **MAKING AN ARGUMENT** A translation maps $\overline{GH}$ to $\overline{G'H'}$. Your friend claims that if you draw segments connecting G to G' and H to H', then the resulting quadrilateral is a parallelogram. Is your friend correct? Explain your reasoning.

40. **THOUGHT PROVOKING** You are a graphic designer for a company that manufactures floor tiles. Design a floor tile in a coordinate plane. Then use translations to show how the tiles cover an entire floor. Describe the translations that map the original tile to four other tiles.

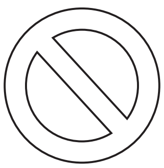
41. **REASONING** The vertices of $\triangle ABC$ are $A(2, 2)$, $B(4, 2)$, and $C(3, 4)$. Graph the image of $\triangle ABC$ after the transformation $(x, y) \rightarrow (x + y, y)$. Is this transformation a translation? Explain your reasoning.

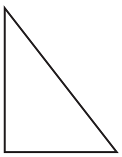
42. **PROOF** $\overline{MN}$ is perpendicular to line ℓ. $\overline{M'N'}$ is the translation of $\overline{MN}$ 2 units to the left. Prove that $\overline{M'N'}$ is perpendicular to ℓ.

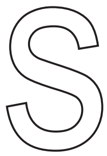
Maintaining Mathematical Proficiency

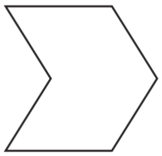
Reviewing what you learned in previous grades and lessons

Tell whether the figure can be folded in half so that one side matches the other. *(Skills Review Handbook)*

43. 

44. 

45. 

46. 

Simplify the expression. *(Skills Review Handbook)*

47. $-(-x)$

48. $-(x + 3)$

49. $x - (12 - 5x)$

50. $x - (-2x + 4)$