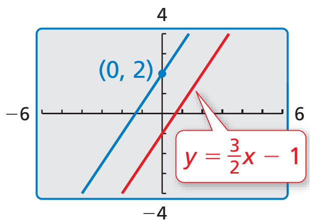
3.5 Equations of Parallel and Perpendicular Lines

Essential Question How can you write an equation of a line that is parallel or perpendicular to a given line and passes through a given point?

EXPLORATION 1 **Writing Equations of Parallel and Perpendicular Lines**

Work with a partner. Write an equation of the line that is parallel or perpendicular to the given line and passes through the given point. Use a graphing calculator to verify your answer. What is the relationship between the slopes?

a.



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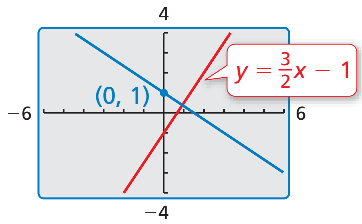

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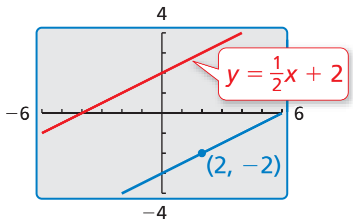

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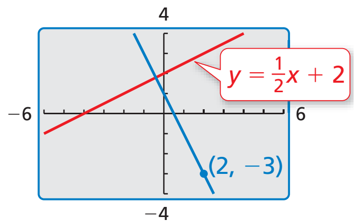

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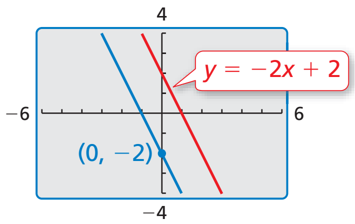

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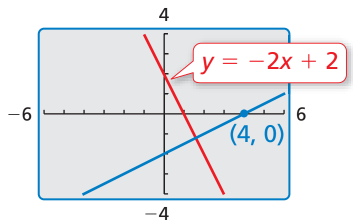
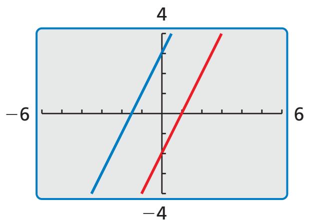


EXPLORATION 2 **Writing Equations of Parallel and Perpendicular Lines**

Work with a partner. Write the equations of the parallel or perpendicular lines. Use a graphing calculator to verify your answers.

a.



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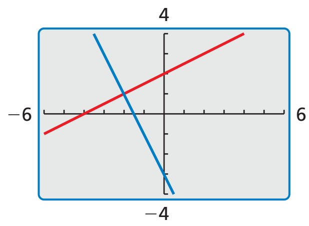


MODELING WITH MATHEMATICS

To be proficient in math, you need to analyze relationships mathematically to draw conclusions.

Communicate Your Answer

3. How can you write an equation of a line that is parallel or perpendicular to a given line and passes through a given point?

4. Write an equation of the line that is (a) parallel and (b) perpendicular to the line $y = 3x + 2$ and passes through the point $(1, -2)$.

3.5 Lesson

Core Vocabulary

directed line segment, *p. 156*

Previous
slope
slope-intercept form
y-intercept

What You Will Learn

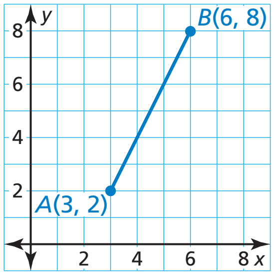
- Use slope to partition directed line segments.
- Identify parallel and perpendicular lines.
- Write equations of parallel and perpendicular lines.
- Use slope to find the distance from a point to a line.

Partitioning a Directed Line Segment

A **directed line segment** AB is a segment that represents moving from point A to point B. The following example shows how to use slope to find a point on a directed line segment that partitions the segment in a given ratio.

EXAMPLE 1 Partitioning a Directed Line Segment

Find the coordinates of point P along the directed line segment AB so that the ratio of AP to PB is 3 to 2.



REMEMBER

Recall that the slope of a line or line segment through two points, (x_1, y_1) and (x_2, y_2), is defined as follows.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}}$$

You can choose either of the two points to be (x_1, y_1).

SOLUTION

In order to divide the segment in the ratio 3 to 2, think of dividing, or *partitioning*, the segment into 3 + 2, or 5 congruent pieces.

Point P is the point that is $\frac{3}{5}$ of the way from point A to point B.

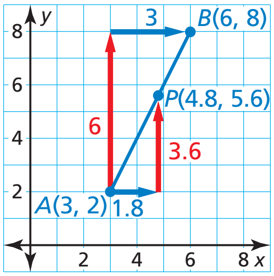
Find the rise and run from point A to point B. Leave the slope in terms of rise and run and do not simplify.

$$\text{slope of } \overline{AB}\text{: } m = \frac{8-2}{6-3} = \frac{6}{3} = \frac{\text{rise}}{\text{run}}$$

To find the coordinates of point P, add $\frac{3}{5}$ of the run to the x-coordinate of A, and add $\frac{3}{5}$ of the rise to the y-coordinate of A.

$$\text{run: } \frac{3}{5} \text{ of } 3 = \frac{3}{5} \cdot 3 = 1.8$$

$$\text{rise: } \frac{3}{5} \text{ of } 6 = \frac{3}{5} \cdot 6 = 3.6$$



So, the coordinates of P are

$$(3 + 1.8, 2 + 3.6) = (4.8, 5.6).$$

The ratio of AP to PB is 3 to 2.

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Find the coordinates of point P along the directed line segment AB so that AP to PB is the given ratio.

1. $A(1, 3)$, $B(8, 4)$; 4 to 1

2. $A(-2, 1)$, $B(4, 5)$; 3 to 7

Identifying Parallel and Perpendicular Lines

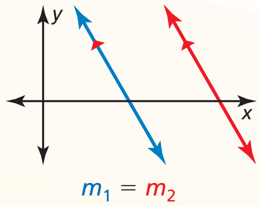
In the coordinate plane, the x-axis and the y-axis are perpendicular. Horizontal lines are parallel to the x-axis, and vertical lines are parallel to the y-axis.

Theorems

Theorem 3.13 Slopes of Parallel Lines

In a coordinate plane, two distinct nonvertical lines are parallel if and only if they have the same slope.

Any two vertical lines are parallel.

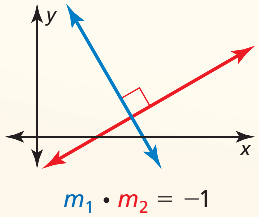


Proof p. 439; Ex. 41, p. 444

Theorem 3.14 Slopes of Perpendicular Lines

In a coordinate plane, two nonvertical lines are perpendicular if and only if the product of their slopes is -1.

Horizontal lines are perpendicular to vertical lines.



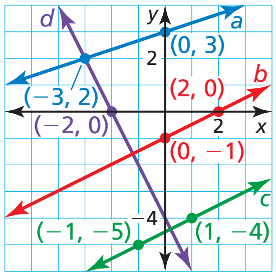
Proof p. 440; Ex. 42, p. 444

READING

If the product of two numbers is -1, then the numbers are called *negative reciprocals*.

EXAMPLE 2 Identifying Parallel and Perpendicular Lines

Determine which of the lines are parallel and which of the lines are perpendicular.



SOLUTION

Find the slope of each line.

$$\text{Line } a\text{: } m = \frac{3-2}{0-(-3)} = \frac{1}{3}$$

$$\text{Line } b\text{: } m = \frac{0-(-1)}{2-0} = \frac{1}{2}$$

$$\text{Line } c\text{: } m = \frac{-4-(-5)}{1-(-1)} = \frac{1}{2}$$

$$\text{Line } d\text{: } m = \frac{2-0}{-3-(-2)} = -2$$

▶ Because lines b and c have the same slope, lines b and c are parallel. Because $\frac{1}{2}(-2) = -1$, lines b and d are perpendicular and lines c and d are perpendicular.

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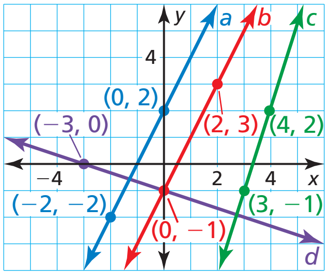
3. Determine which of the lines are parallel and which of the lines are perpendicular.



Writing Equations of Parallel and Perpendicular Lines

You can apply the Slopes of Parallel Lines Theorem and the Slopes of Perpendicular Lines Theorem to write equations of parallel and perpendicular lines.

EXAMPLE 3 Writing an Equation of a Parallel Line

Write an equation of the line passing through the point $(-1, 1)$ that is parallel to the line $y = 2x - 3$.

REMEMBER

The linear equation $y = 2x - 3$ is written in slope-intercept form $y = mx + b$, where m is the slope and b is the y-intercept.

SOLUTION

Step 1 Find the slope m of the parallel line. The line $y = 2x - 3$ has a slope of 2. By the Slopes of Parallel Lines Theorem, a line parallel to this line also has a slope of 2. So, $m = 2$.

Step 2 Find the y-intercept b by using $m = 2$ and $(x, y) = (-1, 1)$.

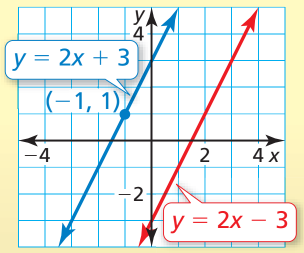
$y = mx + b$ — Use slope-intercept form.

$1 = 2(-1) + b$ — Substitute for m, x, and y.

$3 = b$ — Solve for b.

▶ Because $m = 2$ and $b = 3$, an equation of the line is $y = 2x + 3$. Use a graph to check that the line $y = 2x - 3$ is parallel to the line $y = 2x + 3$.

Check



EXAMPLE 4 Writing an Equation of a Perpendicular Line

Write an equation of the line passing through the point $(2, 3)$ that is perpendicular to the line $2x + y = 2$.

SOLUTION

Step 1 Find the slope m of the perpendicular line. The line $2x + y = 2$, or $y = -2x + 2$, has a slope of -2. Use the Slopes of Perpendicular Lines Theorem.

$-2 \cdot m = -1$ — The product of the slopes of $\perp$ lines is -1.

$m = \frac{1}{2}$ — Divide each side by -2.

Step 2 Find the y-intercept b by using $m = \frac{1}{2}$ and $(x, y) = (2, 3)$.

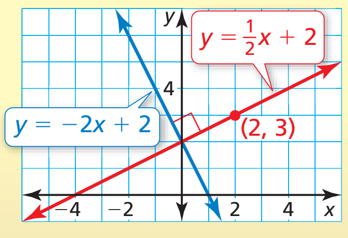
$y = mx + b$ — Use slope-intercept form.

$3 = \frac{1}{2}(2) + b$ — Substitute for m, x, and y.

$2 = b$ — Solve for b.

▶ Because $m = \frac{1}{2}$ and $b = 2$, an equation of the line is $y = \frac{1}{2}x + 2$. Check that the lines are perpendicular by graphing their equations and using a protractor to measure one of the angles formed by their intersection.

Check



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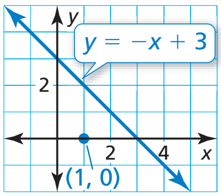
4. Write an equation of the line that passes through the point $(1, 5)$ and is (a) parallel to the line $y = 3x - 5$ and (b) perpendicular to the line $y = 3x - 5$.

5. How do you know that the lines $x = 4$ and $y = 2$ are perpendicular?

Finding the Distance from a Point to a Line

Recall that the distance from a point to a line is the length of the perpendicular segment from the point to the line.

EXAMPLE 5 Finding the Distance from a Point to a Line



Find the distance from the point $(1, 0)$ to the line $y = -x + 3$.

SOLUTION

Step 1 Find an equation of the line perpendicular to the line $y = -x + 3$ that passes through the point $(1, 0)$.

First, find the slope m of the perpendicular line. The line $y = -x + 3$ has a slope of -1. Use the Slopes of Perpendicular Lines Theorem.

$-1 \cdot m = -1$ — The product of the slopes of $\perp$ lines is -1.

$m = 1$ — Divide each side by -1.

Then find the y-intercept b by using $m = 1$ and $(x, y) = (1, 0)$.

$y = mx + b$ — Use slope-intercept form.

$0 = 1(1) + b$ — Substitute for x, y, and m.

$-1 = b$ — Solve for b.

Because $m = 1$ and $b = -1$, an equation of the line is $y = x - 1$.

REMEMBER

Recall that the solution of a system of two linear equations in two variables gives the coordinates of the point of intersection of the graphs of the equations.

There are two special cases when the lines have the same slope.

- When the system has no solution, the lines are parallel.
- When the system has infinitely many solutions, the lines coincide.

Step 2 Use the two equations to write and solve a system of equations to find the point where the two lines intersect.

$y = -x + 3$ — Equation 1

$y = x - 1$ — Equation 2

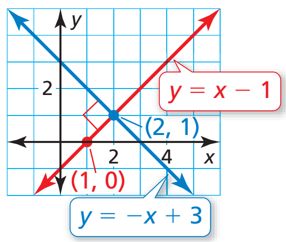
Substitute $-x + 3$ for y in Equation 2.

$y = x - 1$ — Equation 2

$-x + 3 = x - 1$ — Substitute $-x + 3$ for y.

$x = 2$ — Solve for x.

Substitute 2 for x in Equation 1 and solve for y.



$y = -x + 3$ — Equation 1

$y = -2 + 3$ — Substitute 2 for x.

$y = 1$ — Simplify.

So, the perpendicular lines intersect at $(2, 1)$.

Step 3 Use the Distance Formula to find the distance from $(1, 0)$ to $(2, 1)$.

$$\text{distance} = \sqrt{(1-2)^2 + (0-1)^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2} \approx 1.4$$

So, the distance from the point $(1, 0)$ to the line $y = -x + 3$ is about 1.4 units.

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6. Find the distance from the point $(6, 4)$ to the line $y = x + 4$.

7. Find the distance from the point $(-1, 6)$ to the line $y = -2x$.

3.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A ________ line segment AB is a segment that represents moving from point A to point B.

2. **WRITING** How are the slopes of perpendicular lines related?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, find the coordinates of point P along the directed line segment AB so that AP to PB is the given ratio. *(See Example 1.)*

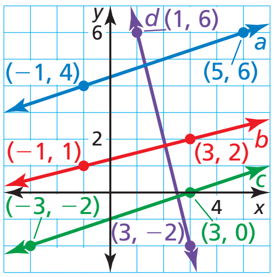
3. $A(8, 0)$, $B(3, -2)$; 1 to 4

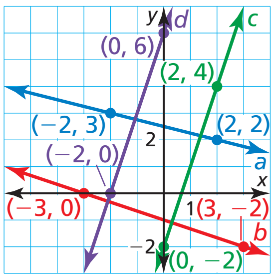
4. $A(-2, -4)$, $B(6, 1)$; 3 to 2

5. $A(1, 6)$, $B(-2, -3)$; 5 to 1

6. $A(-3, 2)$, $B(5, -4)$; 2 to 6

In Exercises 7 and 8, determine which of the lines are parallel and which of the lines are perpendicular. *(See Example 2.)*

7.


8.


In Exercises 9–12, tell whether the lines through the given points are *parallel*, *perpendicular*, or *neither*. Justify your answer.

9. **Line 1:** $(1, 0)$, $(7, 4)$
   Line 2: $(7, 0)$, $(3, 6)$

10. **Line 1:** $(-3, 1)$, $(-7, -2)$
    Line 2: $(2, -1)$, $(8, 4)$

11. **Line 1:** $(-9, 3)$, $(-5, 7)$
    Line 2: $(-11, 6)$, $(-7, 2)$

12. **Line 1:** $(10, 5)$, $(-8, 9)$
    Line 2: $(2, -4)$, $(11, -6)$

In Exercises 13–16, write an equation of the line passing through point P that is parallel to the given line. Graph the equations of the lines to check that they are parallel. *(See Example 3.)*

13. $P(0, -1)$, $y = -2x + 3$

14. $P(3, 8)$, $y = \frac{1}{5}(x + 4)$

15. $P(-2, 6)$, $x = -5$

16. $P(4, 0)$, $-x + 2y = 12$

In Exercises 17–20, write an equation of the line passing through point P that is perpendicular to the given line. Graph the equations of the lines to check that they are perpendicular. *(See Example 4.)*

17. $P(0, 0)$, $y = -9x - 1$

18. $P(4, -6)$, $y = -3$

19. $P(2, 3)$, $y - 4 = -2(x + 3)$

20. $P(-8, 0)$, $3x - 5y = 6$

In Exercises 21–24, find the distance from point A to the given line. *(See Example 5.)*

21. $A(-1, 7)$, $y = 3x$

22. $A(-9, -3)$, $y = x - 6$

23. $A(15, -21)$, $5x + 2y = 4$

24. $A\left(-\frac{1}{4}, 5\right)$, $-x + 2y = 14$

25. **ERROR ANALYSIS** Describe and correct the error in determining whether the lines are parallel, perpendicular, or neither.

Line 1: $(3, -5), (2, -1)$
Line 2: $(0, 3), (1, 7)$

$m_1 = \frac{-1-(-5)}{2-3} = -4 \qquad m_2 = \frac{7-3}{1-0} = 4$

Lines 1 and 2 are perpendicular.

26. **ERROR ANALYSIS** Describe and correct the error in writing an equation of the line that passes through the point (3, 4) and is parallel to the line $y = 2x + 1$.

$y = 2x + 1, (3, 4)$

$4 = m(3) + 1$

$1 = m$

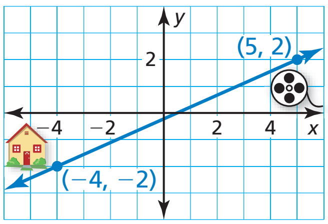
The line $y = x + 1$ is parallel to the line $y = 2x + 1$.

In Exercises 27–30, find the midpoint of $\overline{PQ}$. Then write an equation of the line that passes through the midpoint and is perpendicular to $\overline{PQ}$. This line is called the *perpendicular bisector*.

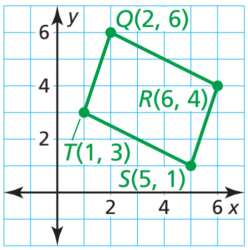
27. $P(-4, 3)$, $Q(4, -1)$ **28.** $P(-5, -5)$, $Q(3, 3)$

29. $P(0, 2)$, $Q(6, -2)$ **30.** $P(-7, 0)$, $Q(1, 8)$

31. **MODELING WITH MATHEMATICS** Your school lies directly between your house and the movie theater. The distance from your house to the school is one-fourth of the distance from the school to the movie theater. What point on the graph represents your school?

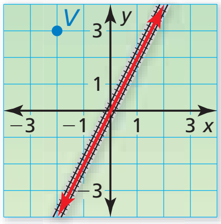


32. **REASONING** Is quadrilateral $QRST$ a parallelogram? Explain your reasoning.



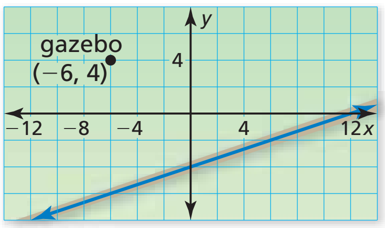
33. **REASONING** A triangle has vertices $L(0, 6)$, $M(5, 8)$, and $N(4, -1)$. Is the triangle a right triangle? Explain your reasoning.

34. **MODELING WITH MATHEMATICS** A new road is being constructed parallel to the train tracks through point V. An equation of the line representing the train tracks is $y = 2x$. Find an equation of the line representing the new road.



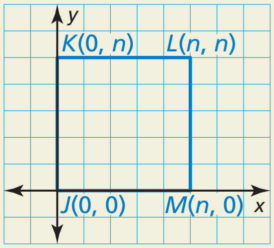
35. **MODELING WITH MATHEMATICS** A bike path is being constructed perpendicular to Washington Boulevard through point $P(2, 2)$. An equation of the line representing Washington Boulevard is $y = -\frac{2}{3}x$. Find an equation of the line representing the bike path.

36. **PROBLEM SOLVING** A gazebo is being built near a nature trail. An equation of the line representing the nature trail is $y = \frac{1}{3}x - 4$. Each unit in the coordinate plane corresponds to 10 feet. Approximately how far is the gazebo from the nature trail?



37. **CRITICAL THINKING** The slope of line ℓ is greater than 0 and less than 1. Write an inequality for the slope of a line perpendicular to ℓ. Explain your reasoning.

38. **HOW DO YOU SEE IT?** Determine whether quadrilateral $JKLM$ is a square. Explain your reasoning.



39. **CRITICAL THINKING** Suppose point P divides the directed line segment XY so that the ratio of XP to PY is 3 to 5. Describe the point that divides the directed line segment YX so that the ratio of YP to PX is 5 to 3.

40. **MAKING AN ARGUMENT** Your classmate claims that no two nonvertical parallel lines can have the same y-intercept. Is your classmate correct? Explain.

41. **MATHEMATICAL CONNECTIONS** Solve each system of equations algebraically. Make a conjecture about what the solution(s) can tell you about whether the lines intersect, are parallel, or are the same line.

a. $y = 4x + 9$
$4x - y = 1$

b. $3y + 4x = 16$
$2x - y = 18$

c. $y = -5x + 6$
$10x + 2y = 12$

42. **THOUGHT PROVOKING** Find a formula for the distance from the point (x_0, y_0) to the line $ax + by = 0$. Verify your formula using a point and a line.

MATHEMATICAL CONNECTIONS In Exercises 43 and 44, find a value for k based on the given description.

43. The line through $(-1, k)$ and $(-7, -2)$ is parallel to the line $y = x + 1$.

44. The line through $(k, 2)$ and $(7, 0)$ is perpendicular to the line $y = x - \frac{28}{5}$.

45. **ABSTRACT REASONING** Make a conjecture about how to find the coordinates of a point that lies beyond point B along $\overrightarrow{AB}$. Use an example to support your conjecture.

46. **PROBLEM SOLVING** What is the distance between the lines $y = 2x$ and $y = 2x + 5$? Verify your answer.

PROVING A THEOREM In Exercises 47 and 48, use the slopes of lines to write a paragraph proof of the theorem.

47. Lines Perpendicular to a Transversal Theorem (Theorem 3.12): In a plane, if two lines are perpendicular to the same line, then they are parallel to each other.

48. Transitive Property of Parallel Lines Theorem (Theorem 3.9): If two lines are parallel to the same line, then they are parallel to each other.

49. **PROOF** Prove the statement: If two lines are vertical, then they are parallel.

50. **PROOF** Prove the statement: If two lines are horizontal, then they are parallel.

51. **PROOF** Prove that horizontal lines are perpendicular to vertical lines.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Plot the point in a coordinate plane. *(Skills Review Handbook)*

52. $A(3, 6)$

53. $B(0, -4)$

54. $C(5, 0)$

55. $D(-1, -2)$

Copy and complete the table. *(Skills Review Handbook)*

56.

x	-2	-1	0	1	2
$y = x + 9$					

57.

x	-2	-1	0	1	2
$y = x - \frac{3}{4}$					