11.1 Using Normal Distributions

Essential Question In a normal distribution, about what percent of the data lies within one, two, and three standard deviations of the mean?

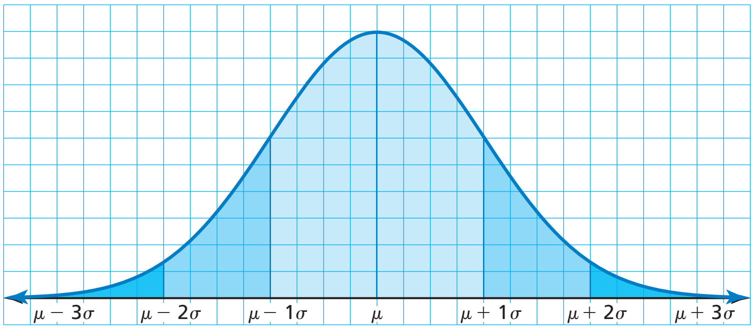
Recall that the standard deviation σ of a numerical data set is given by

$$\sigma = \sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_n - \mu)^2}{n}}$$

where n is the number of values in the data set and μ is the mean of the data set.

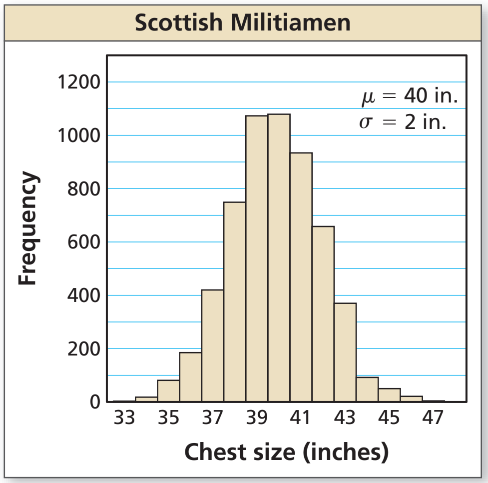
EXPLORATION 1 Analyzing a Normal Distribution

Work with a partner. In many naturally occurring data sets, the histogram of the data is bell-shaped. In statistics, such data sets are said to have a *normal distribution*. For the normal distribution shown below, estimate the percent of the data that lies within one, two, and three standard deviations of the mean. Each square on the grid represents 1%.



EXPLORATION 2 Analyzing a Data Set

Work with a partner. A famous data set was collected in Scotland in the mid-1800s. It contains the chest sizes (in inches) of 5738 men in the Scottish Militia. Do the data fit a normal distribution? Explain.



MODELING WITH MATHEMATICS

To be proficient in math, you need to analyze relationships mathematically to draw conclusions.

Chest size	Number of men
33	3
34	18
35	81
36	185
37	420
38	749
39	1073
40	1079
41	934
42	658
43	370
44	92
45	50
46	21
47	4
48	1

Communicate Your Answer

3. In a normal distribution, about what percent of the data lies within one, two, and three standard deviations of the mean?

4. Use the Internet or some other reference to find another data set that is normally distributed. Display your data in a histogram.

11.1 Lesson

Core Vocabulary

normal distribution, *p. 596*
normal curve, *p. 596*
standard normal distribution, *p. 597*
z-score, *p. 597*

Previous
probability distribution
symmetric
mean
standard deviation
skewed
median

What You Will Learn

- Calculate probabilities using normal distributions.
- Use *z*-scores and the standard normal table to find probabilities.
- Recognize data sets that are normal.

Normal Distributions

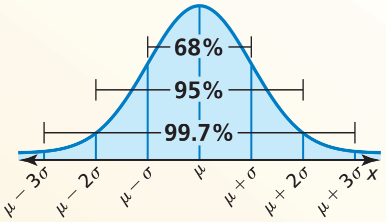
You have studied probability distributions. One type of probability distribution is a *normal distribution*. The graph of a **normal distribution** is a bell-shaped curve called a **normal curve** that is symmetric about the mean.

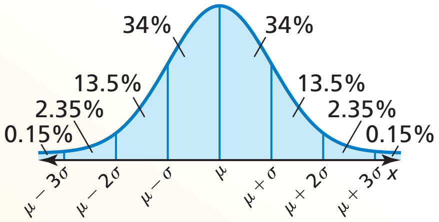
Core Concept

Areas Under a Normal Curve

A normal distribution with mean μ (the Greek letter *mu*) and standard deviation σ (the Greek letter *sigma*) has these properties.

- The total area under the related normal curve is 1.
- About 68% of the area lies within 1 standard deviation of the mean.
- About 95% of the area lies within 2 standard deviations of the mean.
- About 99.7% of the area lies within 3 standard deviations of the mean.



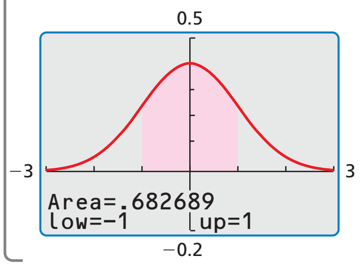


From the second bulleted statement above and the symmetry of a normal curve, you can deduce that 34% of the area lies within 1 standard deviation to the left of the mean, and 34% of the area lies within 1 standard deviation to the right of the mean. The second diagram above shows other partial areas based on the properties of a normal curve.

The areas under a normal curve can be interpreted as probabilities in a normal distribution. So, in a normal distribution, the probability that a randomly chosen *x*-value is between *a* and *b* is given by the area under the normal curve between *a* and *b*.

USING A GRAPHING CALCULATOR

A graphing calculator can be used to find areas under normal curves. For example, the normal distribution shown below has mean 0 and standard deviation 1. The graphing calculator screen shows that the area within 1 standard deviation of the mean is about 0.68, or 68%.



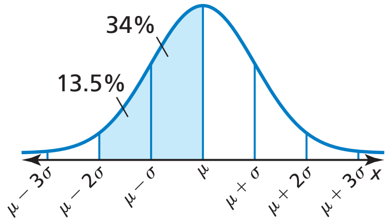
EXAMPLE 1 Finding a Normal Probability

A normal distribution has mean μ and standard deviation σ. An *x*-value is randomly selected from the distribution. Find $P(\mu - 2\sigma \le x \le \mu)$.

SOLUTION

The probability that a randomly selected *x*-value lies between $\mu - 2\sigma$ and μ is the shaded area under the normal curve shown.

$P(\mu - 2\sigma \le x \le \mu) = 0.135 + 0.34 = 0.475$



EXAMPLE 2 Interpreting Normally Distributed Data

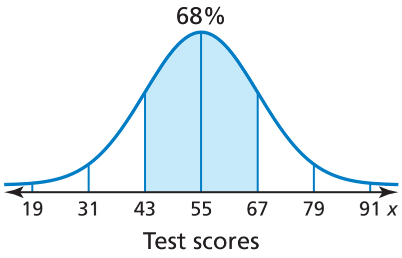
The scores for a state's peace officer standards and training test are normally distributed with a mean of 55 and a standard deviation of 12. The test scores range from 0 to 100.

a. About what percent of the people taking the test have scores between 43 and 67?

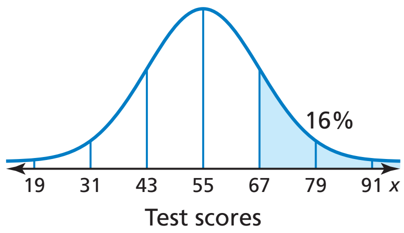
b. An agency in the state will only hire applicants with test scores of 67 or greater. About what percent of the people have test scores that make them eligible to be hired by the agency?

SOLUTION

a. The scores of 43 and 67 represent one standard deviation on either side of the mean, as shown. So, about 68% of the people taking the test have scores between 43 and 67.

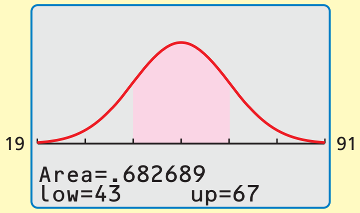


b. A score of 67 is one standard deviation to the right of the mean, as shown. So, the percent of the people who have test scores that make them eligible to be hired by the agency is about 13.5% + 2.35% + 0.15%, or 16%.

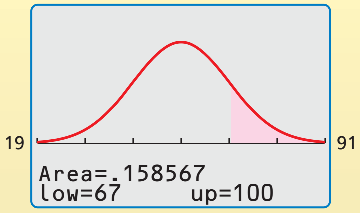


Check

a.



b.



Monitoring Progress

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A normal distribution has mean μ and standard deviation σ. Find the indicated probability for a randomly selected x-value from the distribution.

1. $P(x \le \mu)$

2. $P(x \ge \mu)$

3. $P(\mu \le x \le \mu + 2\sigma)$

4. $P(\mu - \sigma \le x \le \mu)$

5. $P(x \le \mu - 3\sigma)$

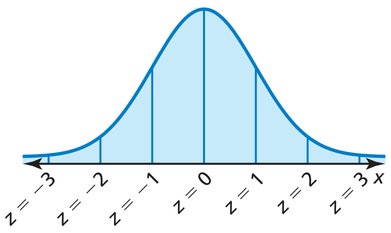
6. $P(x \ge \mu + \sigma)$

7. **WHAT IF?** In Example 2, about what percent of the people taking the test have scores between 43 and 79?

The Standard Normal Distribution

The **standard normal distribution** is the normal distribution with mean 0 and standard deviation 1. The formula below can be used to transform x-values from a normal distribution with mean μ and standard deviation σ into z-values having a standard normal distribution.

Formula $z = \dfrac{x - \mu}{\sigma}$ ← Subtract the mean from the given x-value, then divide by the standard deviation.



The z-value for a particular x-value is called the **z-score** for the x-value and is the number of standard deviations the x-value lies above or below the mean μ.

READING

In the table, the value .0000+ means "slightly more than 0" and the value 1.0000− means "slightly less than 1."

For a randomly selected z-value from a standard normal distribution, you can use the table below to find the probability that z is less than or equal to a given value. For example, the table shows that $P(z \leq -0.4) = 0.3446$. You can find the value of $P(z \leq -0.4)$ in the table by finding the value where row -0 and column .4 intersect.

Standard Normal Table

z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
−3	.0013	.0010	.0007	.0005	.0003	.0002	.0002	.0001	.0001	.0000+
−2	.0228	.0179	.0139	.0107	.0082	.0062	.0047	.0035	.0026	.0019
−1	.1587	.1357	.1151	.0968	.0808	.0668	.0548	.0446	.0359	.0287
−0	.5000	.4602	.4207	.3821	.3446	.3085	.2743	.2420	.2119	.1841
0	.5000	.5398	.5793	.6179	.6554	.6915	.7257	.7580	.7881	.8159
1	.8413	.8643	.8849	.9032	.9192	.9332	.9452	.9554	.9641	.9713
2	.9772	.9821	.9861	.9893	.9918	.9938	.9953	.9965	.9974	.9981
3	.9987	.9990	.9993	.9995	.9997	.9998	.9998	.9999	.9999	1.0000−

You can also use the standard normal table to find probabilities for any normal distribution by first converting values from the distribution to z-scores.

EXAMPLE 3 Using a z-Score and the Standard Normal Table

A study finds that the weights of infants at birth are normally distributed with a mean of 3270 grams and a standard deviation of 600 grams. An infant is randomly chosen. What is the probability that the infant weighs 4170 grams or less?

SOLUTION

Step 1 Find the z-score corresponding to an x-value of 4170.

$$z = \frac{x - \mu}{\sigma} = \frac{4170 - 3270}{600} = 1.5$$

Step 2 Use the table to find $P(z \leq 1.5)$. The table shows that $P(z \leq 1.5) = 0.9332$.

Standard Normal Table

z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
−3	.0013	.0010	.0007	.0005	.0003	.0002	.0002	.0001	.0001	.0000+
−2	.0228	.0179	.0139	.0107	.0082	.0062	.0047	.0035	.0026	.0019
−1	.1587	.1357	.1151	.0968	.0808	.0668	.0548	.0446	.0359	.0287
−0	.5000	.4602	.4207	.3821	.3446	.3085	.2743	.2420	.2119	.1841
0	.5000	.5398	.5793	.6179	.6554	.6915	.7257	.7580	.7881	.8159
1	.8413	.8643	.8849	.9032	.9192	.9332	.9452	.9554	.9641	.9713

STUDY TIP

When n% of the data are less than or equal to a certain value, that value is called the nth *percentile*. In Example 3, a weight of 4170 grams is the 93rd percentile.

▶ So, the probability that the infant weighs 4170 grams or less is about 0.9332.

Monitoring Progress

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8. **WHAT IF?** In Example 3, what is the probability that the infant weighs 3990 grams or more?

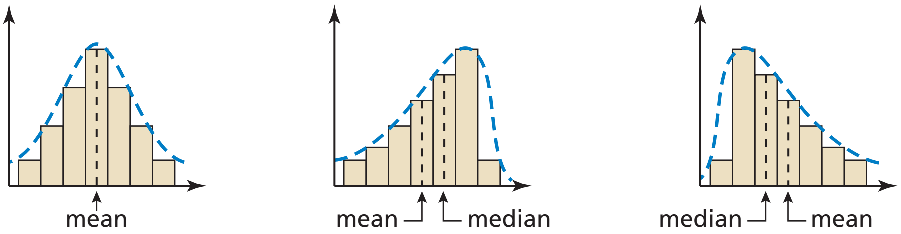
9. Explain why it makes sense that $P(z \leq 0) = 0.5$.

Recognizing Normal Distributions

Not all distributions are normal. For instance, consider the histograms shown below. The first histogram has a normal distribution. Notice that it is bell-shaped and symmetric. Recall that a distribution is symmetric when you can draw a vertical line that divides the histogram into two parts that are mirror images. Some distributions are skewed. The second histogram is *skewed left* and the third histogram is *skewed right*. The second and third histograms do *not* have normal distributions.

UNDERSTANDING MATHEMATICAL TERMS

Be sure you understand that you cannot use a normal distribution to interpret skewed distributions. The areas under a normal curve do not correspond to the areas of a skewed distribution.



Bell-shaped and symmetric
- histogram has a normal distribution
- mean = median

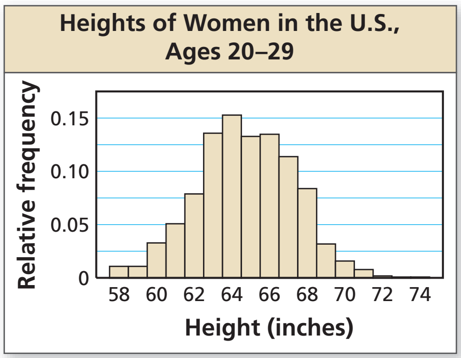
Skewed left
- histogram does not have a normal distribution
- mean < median

Skewed right
- histogram does not have a normal distribution
- mean > median

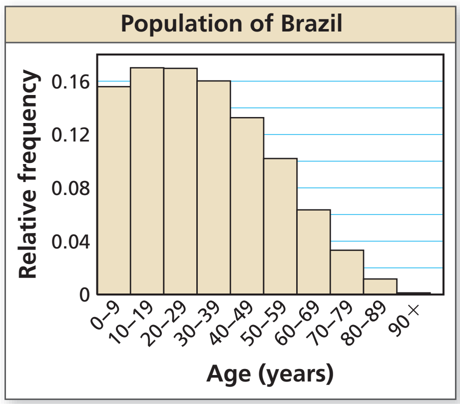
EXAMPLE 4 Recognizing Normal Distributions

Determine whether each histogram has a normal distribution.

a.



b.



SOLUTION

a. The histogram is bell-shaped and fairly symmetric. So, the histogram has an approximately normal distribution.

b. The histogram is skewed right. So, the histogram does not have a normal distribution, and you cannot use a normal distribution to interpret the histogram.

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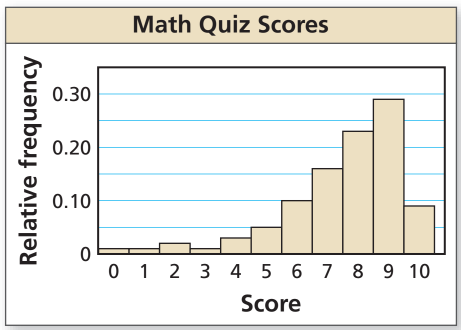
10. Determine whether the histogram has a normal distribution.



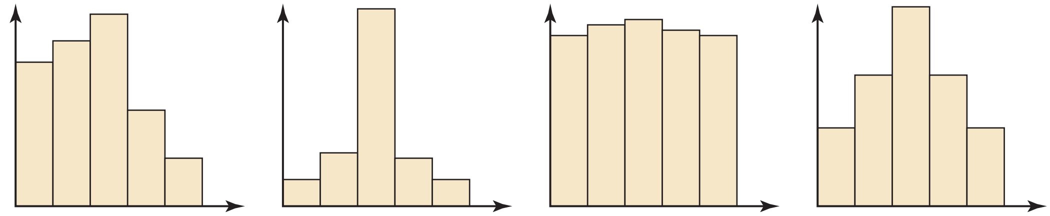
11.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Describe how to use the standard normal table to find $P(z \leq 1.4)$.

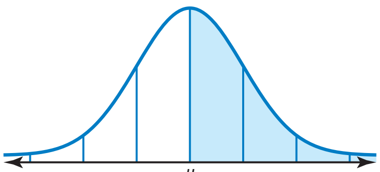
2. **WHICH ONE DOESN'T BELONG?** Which histogram does *not* belong with the other three? Explain your reasoning.



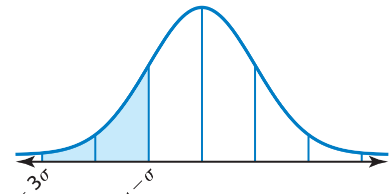
Monitoring Progress and Modeling with Mathematics

ATTENDING TO PRECISION **In Exercises 3–6, give the percent of the area under the normal curve represented by the shaded region(s).**

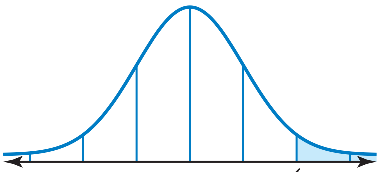
3.



4.



5.



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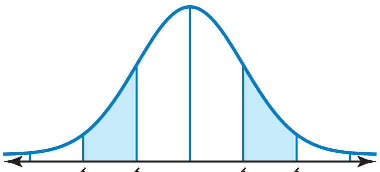


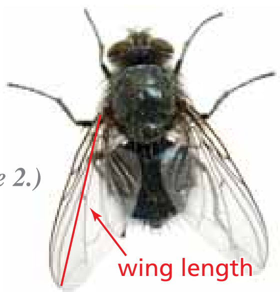
In Exercises 7–12, a normal distribution has mean μ and standard deviation σ. Find the indicated probability for a randomly selected x-value from the distribution. *(See Example 1.)*

7. $P(x \leq \mu - \sigma)$
8. $P(x \geq \mu - \sigma)$
9. $P(x \geq \mu + 2\sigma)$
10. $P(x \leq \mu + \sigma)$
11. $P(\mu - \sigma \leq x \leq \mu + \sigma)$
12. $P(\mu - 3\sigma \leq x \leq \mu)$

In Exercises 13–18, a normal distribution has a mean of 33 and a standard deviation of 4. Find the probability that a randomly selected x-value from the distribution is in the given interval.

13. between 29 and 37
14. between 33 and 45
15. at least 25
16. at least 29
17. at most 37
18. at most 21

19. **PROBLEM SOLVING** The wing lengths of houseflies are normally distributed with a mean of 4.6 millimeters and a standard deviation of 0.4 millimeter. *(See Example 2.)*



   a. About what percent of houseflies have wing lengths between 3.8 millimeters and 5.0 millimeters?

   b. About what percent of houseflies have wing lengths longer than 5.8 millimeters?

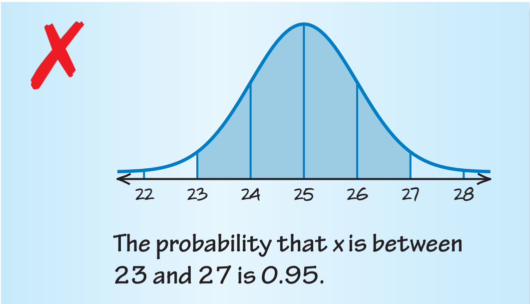
20. PROBLEM SOLVING The times a fire department takes to arrive at the scene of an emergency are normally distributed with a mean of 6 minutes and a standard deviation of 1 minute.

a. For about what percent of emergencies does the fire department arrive at the scene in 8 minutes or less?

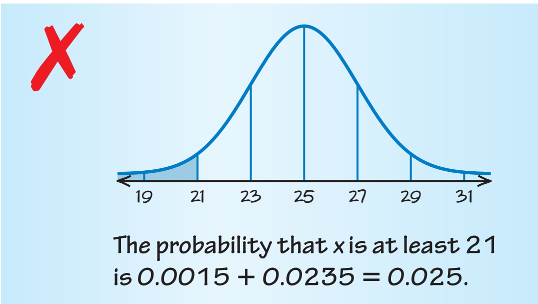
b. The goal of the fire department is to reach the scene of an emergency in 5 minutes or less. About what percent of the time does the fire department achieve its goal?

ERROR ANALYSIS In Exercises 21 and 22, a normal distribution has a mean of 25 and a standard deviation of 2. Describe and correct the error in finding the probability that a randomly selected *x*-value is in the given interval.

21. between 23 and 27



22. at least 21

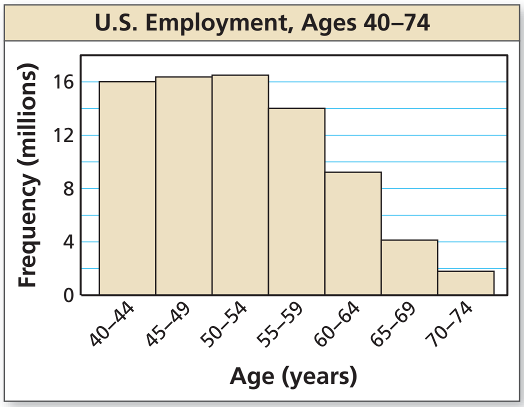


23. PROBLEM SOLVING A busy time to visit a bank is during its Friday evening rush hours. For these hours, the waiting times at the drive-through window are normally distributed with a mean of 8 minutes and a standard deviation of 2 minutes. You have no more than 11 minutes to do your banking and still make it to your meeting on time. What is the probability that you will be late for the meeting? *(See Example 3.)*

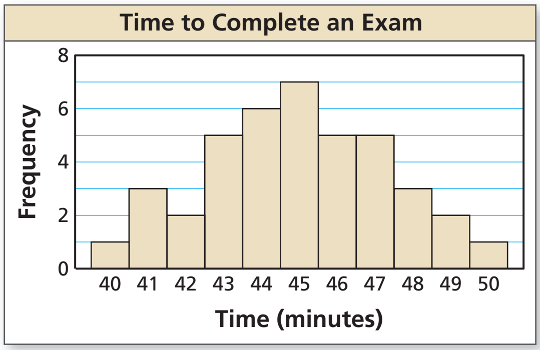
24. PROBLEM SOLVING Scientists conducted aerial surveys of a seal sanctuary and recorded the number *x* of seals they observed during each survey. The numbers of seals observed were normally distributed with a mean of 73 seals and a standard deviation of 14.1 seals. Find the probability that at most 50 seals were observed during a randomly chosen survey.

In Exercises 25 and 26, determine whether the histogram has a normal distribution. *(See Example 4.)*

25.



26.



27. ANALYZING RELATIONSHIPS The table shows the numbers of tickets that are sold for various baseball games in a league over an entire season. Display the data in a histogram. Do the data fit a normal distribution? Explain.

Tickets sold	Frequency
150–189	1
190–229	2
230–269	4
270–309	8
310–349	8
350–389	7

28. **PROBLEM SOLVING** The guayule plant, which grows in the southwestern United States and in Mexico, is one of several plants that can be used as a source of rubber. In a large group of guayule plants, the heights of the plants are normally distributed with a mean of 12 inches and a standard deviation of 2 inches.

a. What percent of the plants are taller than 16 inches?

b. What percent of the plants are at most 13 inches?

c. What percent of the plants are between 7 inches and 14 inches?

d. What percent of the plants are at least 3 inches taller than or at least 3 inches shorter than the mean height?

29. **REASONING** Boxes of cereal are filled by a machine. Tests show that the amount of cereal in each box varies. The weights are normally distributed with a mean of 20 ounces and a standard deviation of 0.25 ounce. Four boxes of cereal are randomly chosen.

a. What is the probability that all four boxes contain no more than 19.4 ounces of cereal?

b. Do you think the machine is functioning properly? Explain.

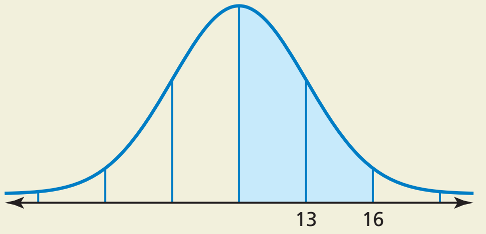
30. **THOUGHT PROVOKING** Sketch the graph of the standard normal distribution function, given by

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

Estimate the area of the region bounded by the x-axis, the graph of f, and the vertical lines $x = -3$ and $x = 3$.

31. **REASONING** For normally distributed data, describe the value that represents the 84th percentile in terms of the mean and standard deviation.

32. **HOW DO YOU SEE IT?** In the figure, the shaded region represents 47.5% of the area under a normal curve. What are the mean and standard deviation of the normal distribution?



33. **DRAWING CONCLUSIONS** You take both the SAT (Scholastic Aptitude Test) and the ACT (American College Test). You score 650 on the mathematics section of the SAT and 29 on the mathematics section of the ACT. The SAT test scores and the ACT test scores are each normally distributed. For the SAT, the mean is 514 and the standard deviation is 118. For the ACT, the mean is 21.0 and the standard deviation is 5.3.

a. What percentile is your SAT math score?

b. What percentile is your ACT math score?

c. On which test did you perform better? Explain your reasoning.

34. **WRITING** Explain how you can convert ACT scores into corresponding SAT scores when you know the mean and standard deviation of each distribution.

35. **MAKING AN ARGUMENT** A data set has a median of 80 and a mean of 90. Your friend claims that the distribution of the data is skewed left. Is your friend correct? Explain your reasoning.

36. **CRITICAL THINKING** The average scores on a statistics test are normally distributed with a mean of 75 and a standard deviation of 10. You randomly select a test score x. Find $P(|x - \mu| \geq 15)$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Graph the function. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing. *(Section 4.8)*

37. $f(x) = x^3 - 4x^2 + 5$

38. $g(x) = \frac{1}{4}x^4 - 2x^2 - x - 3$

39. $h(x) = -0.5x^2 + 3x + 7$

40. $f(x) = -x^4 + 6x^2 - 13$