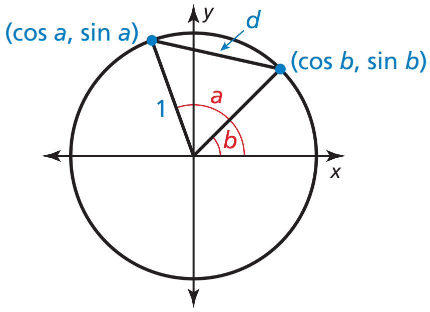
9.8 Using Sum and Difference Formulas

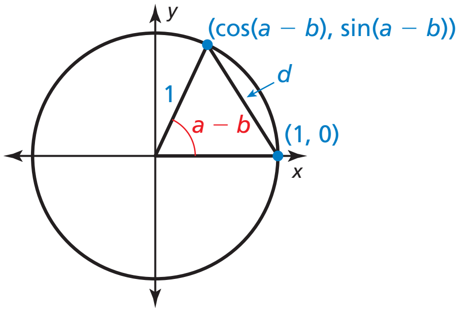
Essential Question How can you evaluate trigonometric functions of the sum or difference of two angles?

EXPLORATION 1 Deriving a Difference Formula

Work with a partner.

a. Explain why the two triangles shown are congruent.





b. Use the Distance Formula to write an expression for d in the first unit circle.

c. Use the Distance Formula to write an expression for d in the second unit circle.

d. Write an equation that relates the expressions in parts (b) and (c). Then simplify this equation to obtain a formula for $\cos(a - b)$.

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results.

EXPLORATION 2 Deriving a Sum Formula

Work with a partner. Use the difference formula you derived in Exploration 1 to write a formula for $\cos(a + b)$ in terms of sine and cosine of a and b. *Hint*: Use the fact that

$$\cos(a + b) = \cos[a - (-b)].$$

EXPLORATION 3 Deriving Difference and Sum Formulas

Work with a partner. Use the formulas you derived in Explorations 1 and 2 to write formulas for $\sin(a - b)$ and $\sin(a + b)$ in terms of sine and cosine of a and b. *Hint*: Use the cofunction identities

$$\sin\left(\frac{\pi}{2} - a\right) = \cos a \text{ and } \cos\left(\frac{\pi}{2} - a\right) = \sin a$$

and the fact that

$$\cos\left[\left(\frac{\pi}{2} - a\right) + b\right] = \sin(a - b) \text{ and } \sin(a + b) = \sin[a - (-b)].$$

Communicate Your Answer

4. How can you evaluate trigonometric functions of the sum or difference of two angles?

5. a. Find the exact values of $\sin 75^\circ$ and $\cos 75^\circ$ using sum formulas. Explain your reasoning.

b. Find the exact values of $\sin 75^\circ$ and $\cos 75^\circ$ using difference formulas. Compare your answers to those in part (a).

9.8 Lesson

Core Vocabulary

Previous
ratio

What You Will Learn

- Use sum and difference formulas to evaluate and simplify trigonometric expressions.
- Use sum and difference formulas to solve trigonometric equations and rewrite real-life formulas.

Using Sum and Difference Formulas

In this lesson, you will study formulas that allow you to evaluate trigonometric functions of the sum or difference of two angles.

Core Concept

Sum and Difference Formulas

Sum Formulas	Difference Formulas
$\sin(a + b) = \sin a \cos b + \cos a \sin b$	$\sin(a - b) = \sin a \cos b - \cos a \sin b$
$\cos(a + b) = \cos a \cos b - \sin a \sin b$	$\cos(a - b) = \cos a \cos b + \sin a \sin b$
$\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$	$\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$

In general, $\sin(a + b) \neq \sin a + \sin b$. Similar statements can be made for the other trigonometric functions of sums and differences.

EXAMPLE 1 Evaluating Trigonometric Expressions

Find the exact value of (a) $\sin 15°$ and (b) $\tan \frac{7\pi}{12}$.

SOLUTION

a. $\sin 15° = \sin(60° - 45°)$ — Substitute $60° - 45°$ for $15°$.

$= \sin 60° \cos 45° - \cos 60° \sin 45°$ — Difference formula for sine

$= \dfrac{\sqrt{3}}{2}\left(\dfrac{\sqrt{2}}{2}\right) - \dfrac{1}{2}\left(\dfrac{\sqrt{2}}{2}\right)$ — Evaluate.

$= \dfrac{\sqrt{6} - \sqrt{2}}{4}$ — Simplify.

▶ The exact value of $\sin 15°$ is $\dfrac{\sqrt{6} - \sqrt{2}}{4}$. Check this with a calculator.

Check

```
sin(15°)
       .2588190451
(√(6)-√(2))/4
       .2588190451
```

b. $\tan \dfrac{7\pi}{12} = \tan\left(\dfrac{\pi}{3} + \dfrac{\pi}{4}\right)$ — Substitute $\frac{\pi}{3} + \frac{\pi}{4}$ for $\frac{7\pi}{12}$.

$= \dfrac{\tan \frac{\pi}{3} + \tan \frac{\pi}{4}}{1 - \tan \frac{\pi}{3} \tan \frac{\pi}{4}}$ — Sum formula for tangent

$= \dfrac{\sqrt{3} + 1}{1 - \sqrt{3} \cdot 1}$ — Evaluate.

$= -2 - \sqrt{3}$ — Simplify.

▶ The exact value of $\tan \dfrac{7\pi}{12}$ is $-2 - \sqrt{3}$. Check this with a calculator.

Check

```
tan(7π/12)
       -3.732050808
-2-√(3)
       -3.732050808
```

EXAMPLE 2 Using a Difference Formula

ANOTHER WAY

You can also use a Pythagorean identity and quadrant signs to find sin *a* and cos *b*.

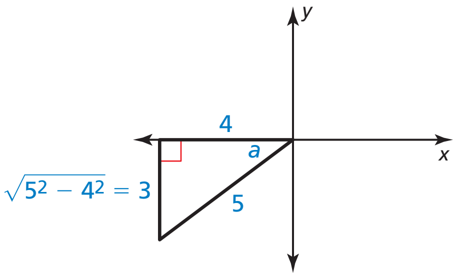
Find $\cos(a - b)$ given that $\cos a = -\frac{4}{5}$ with $\pi < a < \frac{3\pi}{2}$ and $\sin b = \frac{5}{13}$ with $0 < b < \frac{\pi}{2}$.

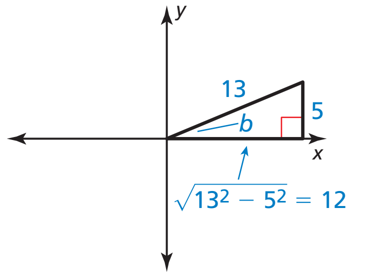
SOLUTION

Step 1 Find $\sin a$ and $\cos b$.

Because $\cos a = -\frac{4}{5}$ and a is in Quadrant III, $\sin a = -\frac{3}{5}$, as shown in the figure.

Because $\sin b = \frac{5}{13}$ and b is in Quadrant I, $\cos b = \frac{12}{13}$, as shown in the figure.





Step 2 Use the difference formula for cosine to find $\cos(a - b)$.

$$\cos(a - b) = \cos a \cos b + \sin a \sin b \qquad \text{Difference formula for cosine}$$

$$= -\frac{4}{5}\left(\frac{12}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) \qquad \text{Evaluate.}$$

$$= -\frac{63}{65} \qquad \text{Simplify.}$$

▶ The value of $\cos(a - b)$ is $-\frac{63}{65}$.

EXAMPLE 3 Simplifying an Expression

Simplify the expression $\cos(x + \pi)$.

SOLUTION

$$\cos(x + \pi) = \cos x \cos \pi - \sin x \sin \pi \qquad \text{Sum formula for cosine}$$

$$= (\cos x)(-1) - (\sin x)(0) \qquad \text{Evaluate.}$$

$$= -\cos x \qquad \text{Simplify.}$$

Monitoring Progress

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Find the exact value of the expression.

1. $\sin 105°$ **2.** $\cos 15°$ **3.** $\tan \frac{5\pi}{12}$ **4.** $\cos \frac{\pi}{12}$

5. Find $\sin(a - b)$ given that $\sin a = \frac{8}{17}$ with $0 < a < \frac{\pi}{2}$ and $\cos b = -\frac{24}{25}$ with $\pi < b < \frac{3\pi}{2}$.

Simplify the expression.

6. $\sin(x + \pi)$ **7.** $\cos(x - 2\pi)$ **8.** $\tan(x - \pi)$

Solving Equations and Rewriting Formulas

EXAMPLE 4 Solving a Trigonometric Equation

Solve $\sin\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{3}\right) = 1$ for $0 \le x < 2\pi$.

SOLUTION

$$\sin\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{3}\right) = 1 \quad \text{Write equation.}$$

$$\sin x \cos \frac{\pi}{3} + \cos x \sin \frac{\pi}{3} + \sin x \cos \frac{\pi}{3} - \cos x \sin \frac{\pi}{3} = 1 \quad \text{Use formulas.}$$

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x = 1 \quad \text{Evaluate.}$$

$$\sin x = 1 \quad \text{Simplify.}$$

▶ In the interval $0 \le x < 2\pi$, the solution is $x = \frac{\pi}{2}$.

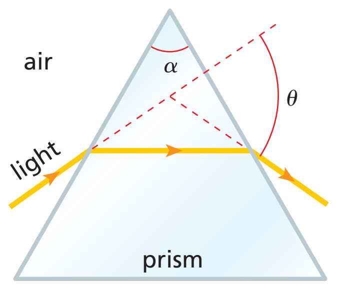
ANOTHER WAY

You can also solve the equation by using a graphing calculator. First, graph each side of the original equation. Then use the *intersect* feature to find the *x*-value(s) where the expressions are equal.

EXAMPLE 5 Rewriting a Real-Life Formula

The *index of refraction* of a transparent material is the ratio of the speed of light in a vacuum to the speed of light in the material. A triangular prism, like the one shown, can be used to measure the index of refraction using the formula

$$n = \frac{\sin\left(\frac{\theta}{2} + \frac{\alpha}{2}\right)}{\sin \frac{\theta}{2}}.$$



For $\alpha = 60°$, show that the formula can be rewritten as $n = \frac{\sqrt{3}}{2} + \frac{1}{2}\cot \frac{\theta}{2}$.

SOLUTION

$$n = \frac{\sin\left(\frac{\theta}{2} + 30°\right)}{\sin \frac{\theta}{2}} \quad \text{Write formula with } \frac{\alpha}{2} = \frac{60°}{2} = 30°.$$

$$= \frac{\sin \frac{\theta}{2} \cos 30° + \cos \frac{\theta}{2} \sin 30°}{\sin \frac{\theta}{2}} \quad \text{Sum formula for sine}$$

$$= \frac{\left(\sin \frac{\theta}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\cos \frac{\theta}{2}\right)\left(\frac{1}{2}\right)}{\sin \frac{\theta}{2}} \quad \text{Evaluate.}$$

$$= \frac{\frac{\sqrt{3}}{2}\sin \frac{\theta}{2}}{\sin \frac{\theta}{2}} + \frac{\frac{1}{2}\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \quad \text{Write as separate fractions.}$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\cot \frac{\theta}{2} \quad \text{Simplify.}$$

Monitoring Progress

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9. Solve $\sin\left(\frac{\pi}{4} - x\right) - \sin\left(x + \frac{\pi}{4}\right) = 1$ for $0 \le x < 2\pi$.

9.8 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** Write the expression $\cos 130° \cos 40° - \sin 130° \sin 40°$ as the cosine of an angle.

2. **WRITING** Explain how to evaluate $\tan 75°$ using either the sum or difference formula for tangent.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, find the exact value of the expression. *(See Example 1.)*

3. $\tan(-15°)$
4. $\tan 195°$
5. $\sin \frac{23\pi}{12}$
6. $\sin(-165°)$
7. $\cos 105°$
8. $\cos \frac{11\pi}{12}$
9. $\tan \frac{17\pi}{12}$
10. $\sin\left(-\frac{7\pi}{12}\right)$

In Exercises 11–16, evaluate the expression given that $\cos a = \frac{4}{5}$ with $0 < a < \frac{\pi}{2}$ and $\sin b = -\frac{15}{17}$ with $\frac{3\pi}{2} < b < 2\pi$. *(See Example 2.)*

11. $\sin(a + b)$
12. $\sin(a - b)$
13. $\cos(a - b)$
14. $\cos(a + b)$
15. $\tan(a + b)$
16. $\tan(a - b)$

In Exercises 17–22, simplify the expression. *(See Example 3.)*

17. $\tan(x + \pi)$
18. $\cos\left(x - \frac{\pi}{2}\right)$
19. $\cos(x + 2\pi)$
20. $\tan(x - 2\pi)$
21. $\sin\left(x - \frac{3\pi}{2}\right)$
22. $\tan\left(x + \frac{\pi}{2}\right)$

ERROR ANALYSIS **In Exercises 23 and 24, describe and correct the error in simplifying the expression.**

23.
$$\tan\left(x + \frac{\pi}{4}\right) = \frac{\tan x + \tan\frac{\pi}{4}}{1 + \tan x \tan\frac{\pi}{4}} = \frac{\tan x + 1}{1 + \tan x} = 1$$

24.
$$\sin\left(x - \frac{\pi}{4}\right) = \sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x = \frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x = \frac{\sqrt{2}}{2}(\cos x - \sin x)$$

25. What are the solutions of the equation $2 \sin x - 1 = 0$ for $0 \le x < 2\pi$?

   (A) $\frac{\pi}{3}$ (B) $\frac{\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

26. What are the solutions of the equation $\tan x + 1 = 0$ for $0 \le x < 2\pi$?

   (A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$ (C) $\frac{5\pi}{4}$ (D) $\frac{7\pi}{4}$

In Exercises 27–32, solve the equation for $0 \le x < 2\pi$. *(See Example 4.)*

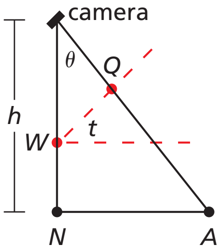
27. $\sin\left(x + \frac{\pi}{2}\right) = \frac{1}{2}$
28. $\tan\left(x - \frac{\pi}{4}\right) = 0$
29. $\cos\left(x + \frac{\pi}{6}\right) - \cos\left(x - \frac{\pi}{6}\right) = 1$
30. $\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) = 0$
31. $\tan(x + \pi) - \tan(\pi - x) = 0$
32. $\sin(x + \pi) + \cos(x + \pi) = 0$

33. **USING EQUATIONS** Derive the cofunction identity $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$ using the difference formula for sine.

34. **MAKING AN ARGUMENT** Your friend claims it is possible to use the difference formula for tangent to derive the cofunction identity $\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$. Is your friend correct? Explain your reasoning.

35. **MODELING WITH MATHEMATICS** A photographer is at a height h taking aerial photographs with a 35-millimeter camera. The ratio of the image length WQ to the length NA of the actual object is given by the formula

$$\frac{WQ}{NA} = \frac{35 \tan(\theta - t) + 35 \tan t}{h \tan \theta}$$



where θ is the angle between the vertical line perpendicular to the ground and the line from the camera to point A and t is the tilt angle of the film. When $t = 45°$, show that the formula can be rewritten as $\frac{WQ}{NA} = \frac{70}{h(1 + \tan \theta)}$. *(See Example 5.)*

36. **MODELING WITH MATHEMATICS** When a wave travels through a taut string, the displacement y of each point on the string depends on the time t and the point's position x. The equation of a *standing wave* can be obtained by adding the displacements of two waves traveling in opposite directions. Suppose a standing wave can be modeled by the formula
$y = A \cos\left(\frac{2\pi t}{3} - \frac{2\pi x}{5}\right) + A \cos\left(\frac{2\pi t}{3} + \frac{2\pi x}{5}\right)$.
When $t = 1$, show that the formula can be rewritten as
$y = -A \cos \frac{2\pi x}{5}$.

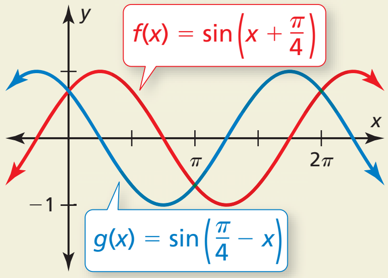
37. **MODELING WITH MATHEMATICS** The busy signal on a touch-tone phone is a combination of two tones with frequencies of 480 hertz and 620 hertz. The individual tones can be modeled by the equations:

480 hertz: $y_1 = \cos 960\pi t$

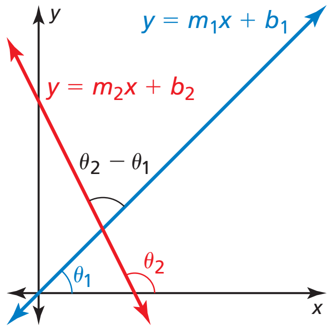
620 hertz: $y_2 = \cos 1240\pi t$

The sound of the busy signal can be modeled by $y_1 + y_2$. Show that $y_1 + y_2 = 2 \cos 1100\pi t \cos 140\pi t$.

38. **HOW DO YOU SEE IT?** Explain how to use the figure to solve the equation $\sin\left(x + \frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4} - x\right) = 0$ for $0 \le x < 2\pi$.



39. **MATHEMATICAL CONNECTIONS** The figure shows the acute angle of intersection, $\theta_2 - \theta_1$, of two lines with slopes m_1 and m_2.



a. Use the difference formula for tangent to write an equation for $\tan(\theta_2 - \theta_1)$ in terms of m_1 and m_2.

b. Use the equation from part (a) to find the acute angle of intersection of the lines $y = x - 1$ and $y = \left(\frac{1}{\sqrt{3} - 2}\right)x + \frac{4 - \sqrt{3}}{2 - \sqrt{3}}$.

40. **THOUGHT PROVOKING** Rewrite each function. Justify your answers.

a. Write $\sin 3x$ as a function of $\sin x$.

b. Write $\cos 3x$ as a function of $\cos x$.

c. Write $\tan 3x$ as a function of $\tan x$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution(s). *(Section 7.5)*

41. $1 - \frac{9}{x - 2} = -\frac{7}{2}$

42. $\frac{12}{x} + \frac{3}{4} = \frac{8}{x}$

43. $\frac{2x - 3}{x + 1} = \frac{10}{x^2 - 1} + 5$