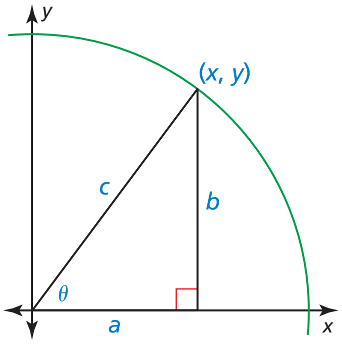
9.7 Using Trigonometric Identities

Essential Question How can you verify a trigonometric identity?

EXPLORATION 1 Writing a Trigonometric Identity

Work with a partner. In the figure, the point (x, y) is on a circle of radius c with center at the origin.



a. Write an equation that relates a, b, and c.

b. Write expressions for the sine and cosine ratios of angle θ.

c. Use the results from parts (a) and (b) to find the sum of $\sin^2\theta$ and $\cos^2\theta$. What do you observe?

d. Complete the table to verify that the identity you wrote in part (c) is valid for angles (of your choice) in each of the four quadrants.

	θ	$\sin^2 \theta$	$\cos^2 \theta$	$\sin^2 \theta + \cos^2 \theta$
QI				
QII				
QIII				
QIV				

EXPLORATION 2 Writing Other Trigonometric Identities

Work with a partner. The trigonometric identity you derived in Exploration 1 is called a Pythagorean identity. There are two other Pythagorean identities. To derive them, recall the four relationships:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta} \qquad \csc \theta = \frac{1}{\sin \theta}$$

a. Divide each side of the Pythagorean identity you derived in Exploration 1 by $\cos^2\theta$ and simplify. What do you observe?

b. Divide each side of the Pythagorean identity you derived in Exploration 1 by $\sin^2\theta$ and simplify. What do you observe?

REASONING ABSTRACTLY

To be proficient in math, you need to know and flexibly use different properties of operations and objects.

Communicate Your Answer

3. How can you verify a trigonometric identity?

4. Is $\sin \theta = \cos \theta$ a trigonometric identity? Explain your reasoning.

5. Give some examples of trigonometric identities that are different than those in Explorations 1 and 2.

9.7 Lesson

Core Vocabulary

trigonometric identity, *p. 514*

Previous

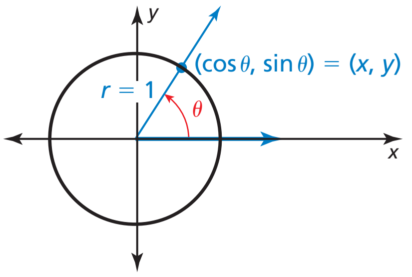
unit circle

What You Will Learn

- Use trigonometric identities to evaluate trigonometric functions and simplify trigonometric expressions.
- Verify trigonometric identities.

Using Trigonometric Identities

Recall that when an angle θ is in standard position with its terminal side intersecting the unit circle at (x, y), then $x = \cos\theta$ and $y = \sin\theta$. Because (x, y) is on a circle centered at the origin with radius 1, it follows that



$$x^2 + y^2 = 1$$

and

$$\cos^2\theta + \sin^2\theta = 1.$$

STUDY TIP

Note that $\sin^2\theta$ represents $(\sin\theta)^2$ and $\cos^2\theta$ represents $(\cos\theta)^2$.

The equation $\cos^2\theta + \sin^2\theta = 1$ is true for any value of θ. A trigonometric equation that is true for all values of the variable for which both sides of the equation are defined is called a **trigonometric identity**. In Section 9.1, you used reciprocal identities to find the values of the cosecant, secant, and cotangent functions. These and other fundamental trigonometric identities are listed below.

Core Concept

Fundamental Trigonometric Identities

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Tangent and Cotangent Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

Negative Angle Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

In this section, you will use trigonometric identities to do the following.

- Evaluate trigonometric functions.
- Simplify trigonometric expressions.
- Verify other trigonometric identities.

EXAMPLE 1 Finding Trigonometric Values

Given that $\sin \theta = \frac{4}{5}$ and $\frac{\pi}{2} < \theta < \pi$, find the values of the other five trigonometric functions of θ.

SOLUTION

Step 1 Find $\cos \theta$.

$\sin^2 \theta + \cos^2 \theta = 1$	Write Pythagorean identity.
$\left(\frac{4}{5}\right)^2 + \cos^2 \theta = 1$	Substitute $\frac{4}{5}$ for $\sin \theta$.
$\cos^2 \theta = 1 - \left(\frac{4}{5}\right)^2$	Subtract $\left(\frac{4}{5}\right)^2$ from each side.
$\cos^2 \theta = \frac{9}{25}$	Simplify.
$\cos \theta = \pm\frac{3}{5}$	Take square root of each side.
$\cos \theta = -\frac{3}{5}$	Because θ is in Quadrant II, $\cos \theta$ is negative.

Step 2 Find the values of the other four trigonometric functions of θ using the values of $\sin \theta$ and $\cos \theta$.

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{-\frac{3}{5}}{\frac{4}{5}} = -\frac{3}{4}$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{1}{\frac{4}{5}} = \frac{5}{4} \qquad \sec \theta = \frac{1}{\cos \theta} = \frac{1}{-\frac{3}{5}} = -\frac{5}{3}$$

EXAMPLE 2 Simplifying Trigonometric Expressions

Simplify (a) $\tan\left(\frac{\pi}{2} - \theta\right)\sin \theta$ and (b) $\sec \theta \tan^2 \theta + \sec \theta$.

SOLUTION

a. $\tan\left(\frac{\pi}{2} - \theta\right)\sin \theta = \cot \theta \sin \theta$	Cofunction identity
$= \left(\frac{\cos \theta}{\sin \theta}\right)(\sin \theta)$	Cotangent identity
$= \cos \theta$	Simplify.
b. $\sec \theta \tan^2 \theta + \sec \theta = \sec \theta(\sec^2 \theta - 1) + \sec \theta$	Pythagorean identity
$= \sec^3 \theta - \sec \theta + \sec \theta$	Distributive Property
$= \sec^3 \theta$	Simplify.

Monitoring Progress

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1. Given that $\cos \theta = \frac{1}{6}$ and $0 < \theta < \frac{\pi}{2}$, find the values of the other five trigonometric functions of θ.

Simplify the expression.

2. $\sin x \cot x \sec x$ **3.** $\cos \theta - \cos \theta \sin^2 \theta$ **4.** $\frac{\tan x \csc x}{\sec x}$

Verifying Trigonometric Identities

You can use the fundamental identities from this chapter to verify new trigonometric identities. When verifying an identity, begin with the expression on one side. Use algebra and trigonometric properties to manipulate the expression until it is identical to the other side.

EXAMPLE 3 Verifying a Trigonometric Identity

Verify the identity $\dfrac{\sec^2 \theta - 1}{\sec^2 \theta} = \sin^2 \theta$.

SOLUTION

$$\frac{\sec^2 \theta - 1}{\sec^2 \theta} = \frac{\sec^2 \theta}{\sec^2 \theta} - \frac{1}{\sec^2 \theta} \qquad \text{Write as separate fractions.}$$

$$= 1 - \left(\frac{1}{\sec \theta}\right)^2 \qquad \text{Simplify.}$$

$$= 1 - \cos^2 \theta \qquad \text{Reciprocal identity}$$

$$= \sin^2 \theta \qquad \text{Pythagorean identity}$$

Notice that verifying an identity is not the same as solving an equation. When verifying an identity, you cannot assume that the two sides of the equation are equal because you are trying to verify that they are equal. So, you cannot use any properties of equality, such as adding the same quantity to each side of the equation.

EXAMPLE 4 Verifying a Trigonometric Identity

Verify the identity $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$.

LOOKING FOR STRUCTURE

To verify the identity, you must introduce $1 - \sin x$ into the denominator. Multiply the numerator and the denominator by $1 - \sin x$ so you get an equivalent expression.

SOLUTION

$$\sec x + \tan x = \frac{1}{\cos x} + \tan x \qquad \text{Reciprocal identity}$$

$$= \frac{1}{\cos x} + \frac{\sin x}{\cos x} \qquad \text{Tangent identity}$$

$$= \frac{1 + \sin x}{\cos x} \qquad \text{Add fractions.}$$

$$= \frac{1 + \sin x}{\cos x} \cdot \frac{1 - \sin x}{1 - \sin x} \qquad \text{Multiply by } \frac{1 - \sin x}{1 - \sin x}.$$

$$= \frac{1 - \sin^2 x}{\cos x(1 - \sin x)} \qquad \text{Simplify numerator.}$$

$$= \frac{\cos^2 x}{\cos x(1 - \sin x)} \qquad \text{Pythagorean identity}$$

$$= \frac{\cos x}{1 - \sin x} \qquad \text{Simplify.}$$

Monitoring Progress

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Verify the identity.

5. $\cot(-\theta) = -\cot \theta$

6. $\csc^2 x(1 - \sin^2 x) = \cot^2 x$

7. $\cos x \csc x \tan x = 1$

8. $(\tan^2 x + 1)(\cos^2 x - 1) = -\tan^2 x$

9.7 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Describe the difference between a trigonometric identity and a trigonometric equation.

2. **WRITING** Explain how to use trigonometric identities to determine whether $\sec(-\theta) = \sec\theta$ or $\sec(-\theta) = -\sec\theta$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, find the values of the other five trigonometric functions of θ. *(See Example 1.)*

3. $\sin\theta = \frac{1}{3},\ 0 < \theta < \frac{\pi}{2}$

4. $\sin\theta = -\frac{7}{10},\ \pi < \theta < \frac{3\pi}{2}$

5. $\tan\theta = -\frac{3}{7},\ \frac{\pi}{2} < \theta < \pi$

6. $\cot\theta = -\frac{2}{5},\ \frac{\pi}{2} < \theta < \pi$

7. $\cos\theta = -\frac{5}{6},\ \pi < \theta < \frac{3\pi}{2}$

8. $\sec\theta = \frac{9}{4},\ \frac{3\pi}{2} < \theta < 2\pi$

9. $\cot\theta = -3,\ \frac{3\pi}{2} < \theta < 2\pi$

10. $\csc\theta = -\frac{5}{3},\ \pi < \theta < \frac{3\pi}{2}$

In Exercises 11–20, simplify the expression. *(See Example 2.)*

11. $\sin x\cot x$

12. $\cos\theta(1 + \tan^2\theta)$

13. $\dfrac{\sin(-\theta)}{\cos(-\theta)}$

14. $\dfrac{\cos^2 x}{\cot^2 x}$

15. $\dfrac{\cos\left(\frac{\pi}{2} - x\right)}{\csc x}$

16. $\sin\left(\frac{\pi}{2} - \theta\right)\sec\theta$

17. $\dfrac{\csc^2 x - \cot^2 x}{\sin(-x)\cot x}$

18. $\dfrac{\cos^2 x\tan^2(-x) - 1}{\cos^2 x}$

19. $\dfrac{\cos\left(\frac{\pi}{2} - \theta\right)}{\csc\theta} + \cos^2\theta$

20. $\dfrac{\sec x\sin x + \cos\left(\frac{\pi}{2} - x\right)}{1 + \sec x}$

ERROR ANALYSIS In Exercises 21 and 22, describe and correct the error in simplifying the expression.

21. ✗
$$\begin{aligned} 1 - \sin^2\theta &= 1 - (1 + \cos^2\theta) \\ &= 1 - 1 - \cos^2\theta \\ &= -\cos^2\theta \end{aligned}$$

22. ✗
$$\begin{aligned} \tan x\csc x &= \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} \\ &= \frac{\cos x}{\sin^2 x} \end{aligned}$$

In Exercises 23–30, verify the identity. *(See Examples 3 and 4.)*

23. $\sin x\csc x = 1$

24. $\tan\theta\csc\theta\cos\theta = 1$

25. $\cos\left(\frac{\pi}{2} - x\right)\cot x = \cos x$

26. $\sin\left(\frac{\pi}{2} - x\right)\tan x = \sin x$

27. $\dfrac{\cos\left(\frac{\pi}{2} - \theta\right) + 1}{1 - \sin(-\theta)} = 1$

28. $\dfrac{\sin^2(-x)}{\tan^2 x} = \cos^2 x$

29. $\dfrac{1 + \cos x}{\sin x} + \dfrac{\sin x}{1 + \cos x} = 2\csc x$

30. $\dfrac{\sin x}{1 - \cos(-x)} = \csc x + \cot x$

31. **USING STRUCTURE** A function f is *odd* when $f(-x) = -f(x)$. A function f is *even* when $f(-x) = f(x)$. Which of the six trigonometric functions are odd? Which are even? Justify your answers using identities and graphs.

32. **ANALYZING RELATIONSHIPS** As the value of $\cos\theta$ increases, what happens to the value of $\sec\theta$? Explain your reasoning.

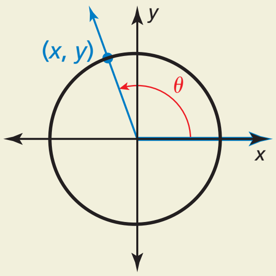
33. MAKING AN ARGUMENT Your friend simplifies an expression and obtains $\sec x \tan x - \sin x$. You simplify the same expression and obtain $\sin x \tan^2 x$. Are your answers equivalent? Justify your answer.

34. HOW DO YOU SEE IT? The figure shows the unit circle and the angle θ.

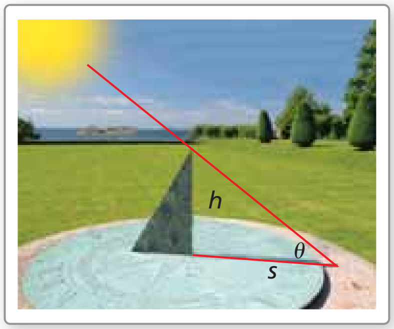
a. Is $\sin \theta$ positive or negative? $\cos \theta$? $\tan \theta$?

b. In what quadrant does the terminal side of $-\theta$ lie?

c. Is $\sin(-\theta)$ positive or negative? $\cos(-\theta)$? $\tan(-\theta)$?



35. MODELING WITH MATHEMATICS A vertical *gnomon* (the part of a sundial that projects a shadow) has height h. The length s of the shadow cast by the gnomon when the angle of the Sun above the horizon is θ can be modeled by the equation below. Show that the equation below is equivalent to $s = h \cot \theta$.



$$s = \frac{h \sin(90^\circ - \theta)}{\sin \theta}$$

36. THOUGHT PROVOKING Explain how you can use a trigonometric identity to find all the values of x for which $\sin x = \cos x$.

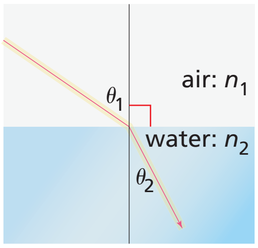
37. DRAWING CONCLUSIONS *Static friction* is the amount of force necessary to keep a stationary object on a flat surface from moving. Suppose a book weighing W pounds is lying on a ramp inclined at an angle θ. The coefficient of static friction u for the book can be found using the equation $uW \cos \theta = W \sin \theta$.

a. Solve the equation for u and simplify the result.

b. Use the equation from part (a) to determine what happens to the value of u as the angle θ increases from 0° to 90°.

38. PROBLEM SOLVING When light traveling in a medium (such as air) strikes the surface of a second medium (such as water) at an angle θ_1, the light begins to travel at a different angle θ_2. This change of direction is defined by Snell's law, $n_1 \sin \theta_1 = n_2 \sin \theta_2$, where n_1 and n_2 are the *indices of refraction* for the two mediums. Snell's law can be derived from the equation

$$\frac{n_1}{\sqrt{\cot^2 \theta_1 + 1}} = \frac{n_2}{\sqrt{\cot^2 \theta_2 + 1}}.$$



a. Simplify the equation to derive Snell's law.

b. What is the value of n_1 when $\theta_1 = 55^\circ$, $\theta_2 = 35^\circ$, and $n_2 = 2$?

c. If $\theta_1 = \theta_2$, then what must be true about the values of n_1 and n_2? Explain when this situation would occur.

39. WRITING Explain how transformations of the graph of the parent function $f(x) = \sin x$ support the cofunction identity $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$.

40. USING STRUCTURE Verify each identity.

a. $\ln|\sec \theta| = -\ln|\cos \theta|$

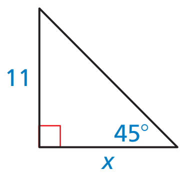
b. $\ln|\tan \theta| = \ln|\sin \theta| - \ln|\cos \theta|$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the value of x for the right triangle. *(Section 9.1)*

41.



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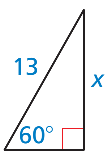

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