9.3 Trigonometric Functions of Any Angle

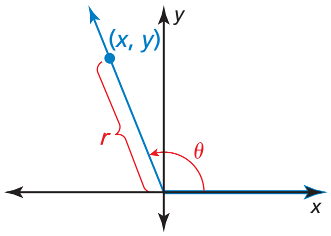
Essential Question How can you use the unit circle to define the trigonometric functions of any angle?

Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$. The six trigonometric functions of θ are defined as shown.

$\sin \theta = \dfrac{y}{r}$ $\qquad$ $\csc \theta = \dfrac{r}{y}, y \neq 0$

$\cos \theta = \dfrac{x}{r}$ $\qquad$ $\sec \theta = \dfrac{r}{x}, x \neq 0$

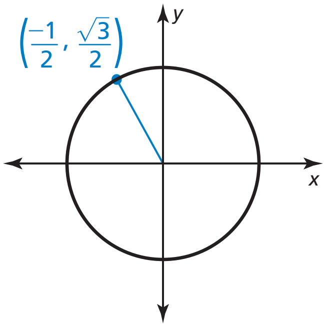
$\tan \theta = \dfrac{y}{x}, x \neq 0$ $\qquad$ $\cot \theta = \dfrac{x}{y}, y \neq 0$



EXPLORATION 1 Writing Trigonometric Functions

Work with a partner. Find the sine, cosine, and tangent of the angle θ in standard position whose terminal side intersects the unit circle at the point (x, y) shown.

a.



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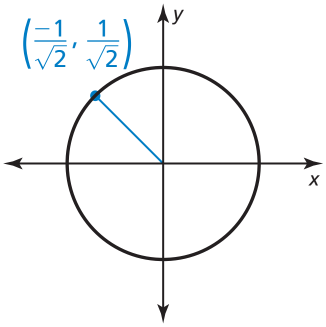

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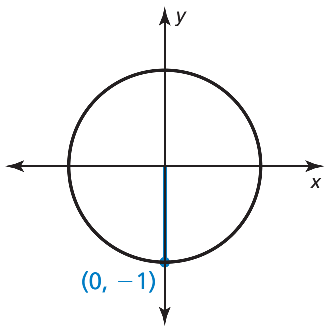

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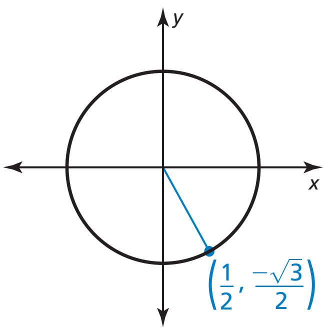
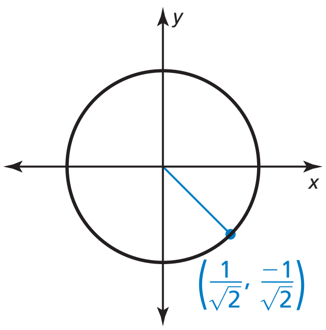


e.



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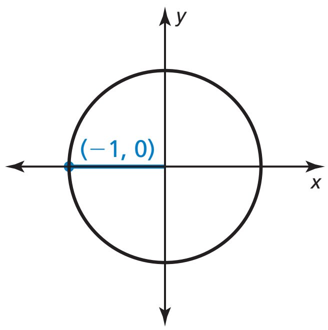


CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results.

Communicate Your Answer

2. How can you use the unit circle to define the trigonometric functions of any angle?

3. For which angles are each function undefined? Explain your reasoning.

   a. tangent $\qquad$ **b.** cotangent $\qquad$ **c.** secant $\qquad$ **d.** cosecant

9.3 Lesson

What You Will Learn

- Evaluate trigonometric functions of any angle.
- Find and use reference angles to evaluate trigonometric functions.

Core Vocabulary

unit circle, *p. 479*
quadrantal angle, *p. 479*
reference angle, *p. 480*

Previous
circle
radius
Pythagorean Theorem

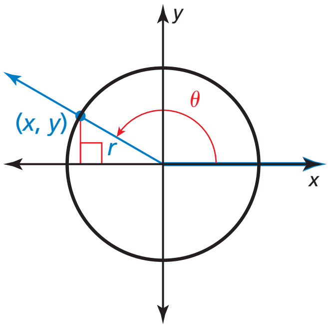
Trigonometric Functions of Any Angle

You can generalize the right-triangle definitions of trigonometric functions so that they apply to any angle in standard position.

Core Concept

General Definitions of Trigonometric Functions

Let θ be an angle in standard position, and let (x, y) be the point where the terminal side of θ intersects the circle $x^2 + y^2 = r^2$. The six trigonometric functions of θ are defined as shown.



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y},\ y \neq 0$$

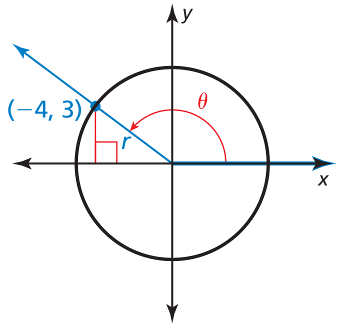
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x},\ x \neq 0$$

$$\tan\theta = \frac{y}{x},\ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$

These functions are sometimes called *circular functions*.

EXAMPLE 1 Evaluating Trigonometric Functions Given a Point

Let $(-4, 3)$ be a point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ.



SOLUTION

Use the Pythagorean Theorem to find the length of r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-4)^2 + 3^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Using $x = -4$, $y = 3$, and $r = 5$, the values of the six trigonometric functions of θ are:

$$\sin\theta = \frac{y}{r} = \frac{3}{5} \qquad \csc\theta = \frac{r}{y} = \frac{5}{3}$$

$$\cos\theta = \frac{x}{r} = -\frac{4}{5} \qquad \sec\theta = \frac{r}{x} = -\frac{5}{4}$$

$$\tan\theta = \frac{y}{x} = -\frac{3}{4} \qquad \cot\theta = \frac{x}{y} = -\frac{4}{3}$$

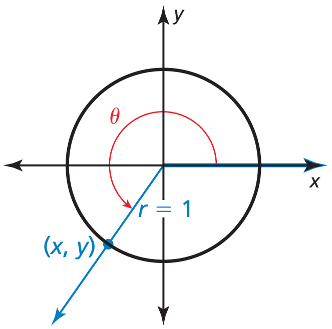
Core Concept

The Unit Circle

The circle $x^2 + y^2 = 1$, which has center (0, 0) and radius 1, is called the **unit circle**. The values of sin θ and cos θ are simply the y-coordinate and x-coordinate, respectively, of the point where the terminal side of θ intersects the unit circle.

$$\sin \theta = \frac{y}{r} = \frac{y}{1} = y$$

$$\cos \theta = \frac{x}{r} = \frac{x}{1} = x$$



ANOTHER WAY

The general circle $x^2 + y^2 = r^2$ can also be used to find the six trigonometric functions of θ. The terminal side of θ intersects the circle at $(0, -r)$. So,

$$\sin \theta = \frac{y}{r} = \frac{-r}{r} = -1.$$

The other functions can be evaluated similarly.

It is convenient to use the unit circle to find trigonometric functions of **quadrantal angles**. A quadrantal angle is an angle in standard position whose terminal side lies on an axis. The measure of a quadrantal angle is always a multiple of 90º, or $\frac{\pi}{2}$ radians.

EXAMPLE 2 Using the Unit Circle

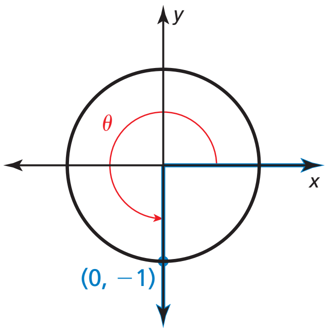
Use the unit circle to evaluate the six trigonometric functions of $\theta = 270º$.

SOLUTION

Step 1 Draw a unit circle with the angle $\theta = 270º$ in standard position.

Step 2 Identify the point where the terminal side of θ intersects the unit circle. The terminal side of θ intersects the unit circle at $(0, -1)$.

Step 3 Find the values of the six trigonometric functions. Let $x = 0$ and $y = -1$ to evaluate the trigonometric functions.



$$\sin \theta = \frac{y}{r} = \frac{-1}{1} = -1 \qquad \csc \theta = \frac{r}{y} = \frac{1}{-1} = -1$$

$$\cos \theta = \frac{x}{r} = \frac{0}{1} = 0 \qquad \sec \theta = \frac{r}{x} = \cancel{\frac{1}{0}} \quad \text{undefined}$$

$$\tan \theta = \frac{y}{x} = \cancel{\frac{-1}{0}} \quad \text{undefined} \qquad \cot \theta = \frac{x}{y} = \frac{0}{-1} = 0$$

Monitoring Progress

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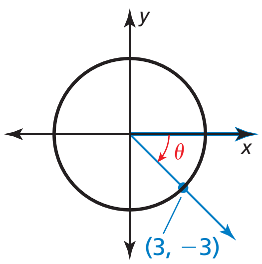
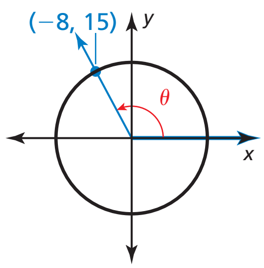
Evaluate the six trigonometric functions of θ.

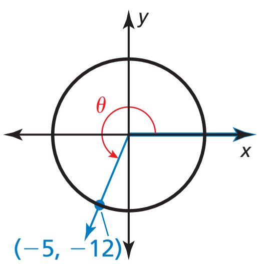
1.



2.



3.



4. Use the unit circle to evaluate the six trigonometric functions of $\theta = 180º$.

Reference Angles

READING

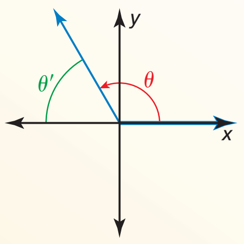
The symbol θ' is read as "theta prime."

Core Concept

Reference Angle Relationships

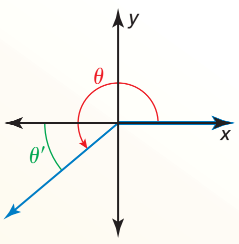
Let θ be an angle in standard position. The **reference angle** for θ is the acute angle θ' formed by the terminal side of θ and the x-axis. The relationship between θ and θ' is shown below for nonquadrantal angles θ such that $90^\circ < \theta < 360^\circ$ or, in radians, $\frac{\pi}{2} < \theta < 2\pi$.

Quadrant II



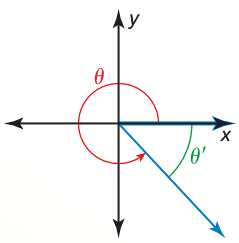
Degrees: $\theta' = 180^\circ - \theta$
Radians: $\theta' = \pi - \theta$

Quadrant III



Degrees: $\theta' = \theta - 180^\circ$
Radians: $\theta' = \theta - \pi$

Quadrant IV



Degrees: $\theta' = 360^\circ - \theta$
Radians: $\theta' = 2\pi - \theta$

EXAMPLE 3 Finding Reference Angles

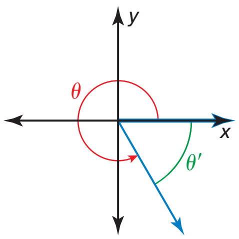
Find the reference angle θ' for (a) $\theta = \frac{5\pi}{3}$ and (b) $\theta = -130^\circ$.

SOLUTION

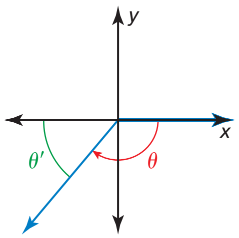
a. The terminal side of θ lies in Quadrant IV. So,

$\theta' = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$. The figure at the right shows

$\theta = \frac{5\pi}{3}$ and $\theta' = \frac{\pi}{3}$.



b. Note that θ is coterminal with 230°, whose terminal side lies in Quadrant III. So, $\theta' = 230^\circ - 180^\circ = 50^\circ$. The figure at the left shows $\theta = -130^\circ$ and $\theta' = 50^\circ$.



Reference angles allow you to evaluate a trigonometric function for any angle θ. The sign of the trigonometric function value depends on the quadrant in which θ lies.

Core Concept

Evaluating Trigonometric Functions

Use these steps to evaluate a trigonometric function for any angle θ:

Step 1 Find the reference angle θ'.

Step 2 Evaluate the trigonometric function for θ'.

Step 3 Determine the sign of the trigonometric function value from the quadrant in which θ lies.

Signs of Function Values

Quadrant II	Quadrant I
$\sin\theta$, $\csc\theta$: +	$\sin\theta$, $\csc\theta$: +
$\cos\theta$, $\sec\theta$: −	$\cos\theta$, $\sec\theta$: +
$\tan\theta$, $\cot\theta$: −	$\tan\theta$, $\cot\theta$: +
Quadrant III	**Quadrant IV**
$\sin\theta$, $\csc\theta$: −	$\sin\theta$, $\csc\theta$: −
$\cos\theta$, $\sec\theta$: −	$\cos\theta$, $\sec\theta$: +
$\tan\theta$, $\cot\theta$: +	$\tan\theta$, $\cot\theta$: −

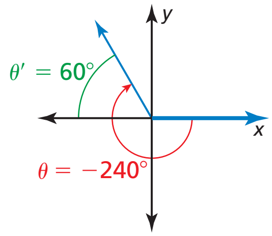
EXAMPLE 4 Using Reference Angles to Evaluate Functions

Evaluate (a) $\tan(-240°)$ and (b) $\csc \frac{17\pi}{6}$.

SOLUTION

a. The angle $-240°$ is coterminal with $120°$. The reference angle is $\theta' = 180° - 120° = 60°$. The tangent function is negative in Quadrant II, so

$$\tan(-240°) = -\tan 60° = -\sqrt{3}.$$

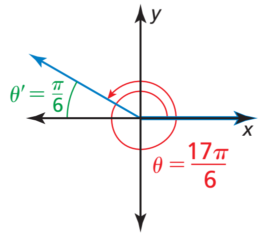


b. The angle $\frac{17\pi}{6}$ is coterminal with $\frac{5\pi}{6}$. The reference angle is

$$\theta' = \pi - \frac{5\pi}{6} = \frac{\pi}{6}.$$

The cosecant function is positive in Quadrant II, so

$$\csc \frac{17\pi}{6} = \csc \frac{\pi}{6} = 2.$$



INTERPRETING MODELS

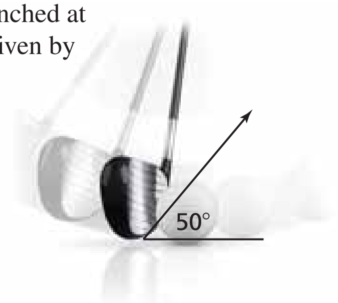
This model neglects air resistance and assumes that the projectile's starting and ending heights are the same.

EXAMPLE 5 Solving a Real-Life Problem

The horizontal distance d (in feet) traveled by a projectile launched at an angle θ and with an initial speed v (in feet per second) is given by

$$d = \frac{v^2}{32} \sin 2\theta. \qquad \text{Model for horizontal distance}$$

Estimate the horizontal distance traveled by a golf ball that is hit at an angle of $50°$ with an initial speed of 105 feet per second.



SOLUTION

Note that the golf ball is launched at an angle of $\theta = 50°$ with initial speed of $v = 105$ feet per second.

$$d = \frac{v^2}{32} \sin 2\theta \qquad \text{Write model for horizontal distance.}$$

$$= \frac{105^2}{32} \sin(2 \cdot 50°) \qquad \text{Substitute 105 for } v \text{ and } 50° \text{ for } \theta.$$

$$\approx 339 \qquad \text{Use a calculator.}$$

▶ The golf ball travels a horizontal distance of about 339 feet.

Monitoring Progress

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Sketch the angle. Then find its reference angle.

5. $210°$ **6.** $-260°$ **7.** $\frac{-7\pi}{9}$ **8.** $\frac{15\pi}{4}$

Evaluate the function without using a calculator.

9. $\cos(-210°)$ **10.** $\sec \frac{11\pi}{4}$

11. Use the model given in Example 5 to estimate the horizontal distance traveled by a track and field long jumper who jumps at an angle of $20°$ and with an initial speed of 27 feet per second.

9.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

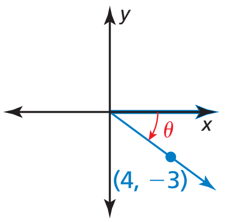
1. COMPLETE THE SENTENCE A(n) ____________ is an angle in standard position whose terminal side lies on an axis.

2. WRITING Given an angle θ in standard position with its terminal side in Quadrant III, explain how you can use a reference angle to find $\cos\theta$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, evaluate the six trigonometric functions of θ. *(See Example 1.)*

3.



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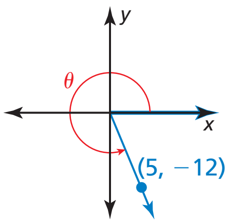

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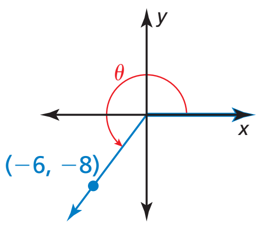

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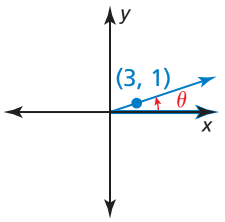

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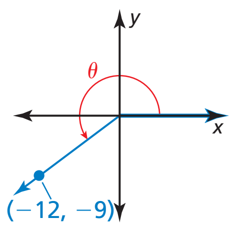

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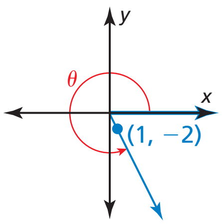


In Exercises 9–14, use the unit circle to evaluate the six trigonometric functions of θ. *(See Example 2.)*

9. $\theta = 0^\circ$

10. $\theta = 540^\circ$

11. $\theta = \dfrac{\pi}{2}$

12. $\theta = \dfrac{7\pi}{2}$

13. $\theta = -270^\circ$

14. $\theta = -2\pi$

In Exercises 15–22, sketch the angle. Then find its reference angle. *(See Example 3.)*

15. -100°

16. 150°

17. 320°

18. -370°

19. $\dfrac{15\pi}{4}$

20. $\dfrac{8\pi}{3}$

21. $-\dfrac{5\pi}{6}$

22. $-\dfrac{13\pi}{6}$

23. ERROR ANALYSIS Let $(-3, 2)$ be a point on the terminal side of an angle θ in standard position. Describe and correct the error in finding $\tan\theta$.

✗ $\tan\theta = \dfrac{x}{y} = -\dfrac{3}{2}$

24. ERROR ANALYSIS Describe and correct the error in finding a reference angle θ' for $\theta = 650^\circ$.

✗ θ is coterminal with 290°, whose terminal side lies in Quadrant IV.

So, $\theta' = 290^\circ - 270^\circ = 20^\circ$.

In Exercises 25–32, evaluate the function without using a calculator. *(See Example 4.)*

25. $\sec 135^\circ$

26. $\tan 240^\circ$

27. $\sin(-150^\circ)$

28. $\csc(-420^\circ)$

29. $\tan\left(-\dfrac{3\pi}{4}\right)$

30. $\cot\left(\dfrac{-8\pi}{3}\right)$

31. $\cos\dfrac{7\pi}{4}$

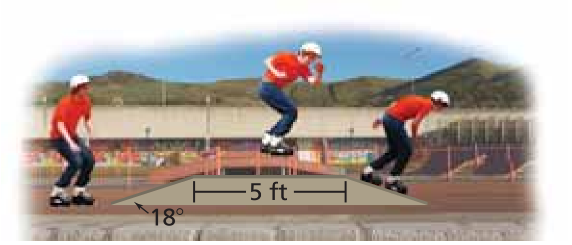
32. $\sec\dfrac{11\pi}{6}$

In Exercises 33–36, use the model for horizontal distance given in Example 5.

33. You kick a football at an angle of 60° with an initial speed of 49 feet per second. Estimate the horizontal distance traveled by the football. *(See Example 5.)*

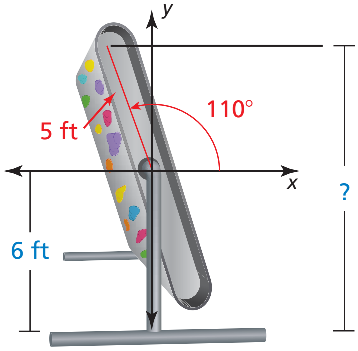
34. The "frogbot" is a robot designed for exploring rough terrain on other planets. It can jump at a 45° angle with an initial speed of 14 feet per second. Estimate the horizontal distance the frogbot can jump on Earth.

35. At what speed must the in-line skater launch himself off the ramp in order to land on the other side of the ramp?



36. To win a javelin throwing competition, your last throw must travel a horizontal distance of at least 100 feet. You release the javelin at a 40° angle with an initial speed of 71 feet per second. Do you win the competition? Justify your answer.

37. **MODELING WITH MATHEMATICS** A rock climber is using a rock climbing treadmill that is 10 feet long. The climber begins by lying horizontally on the treadmill, which is then rotated about its midpoint by 110° so that the rock climber is climbing toward the top. If the midpoint of the treadmill is 6 feet above the ground, how high above the ground is the top of the treadmill?



38. **REASONING** A Ferris wheel has a radius of 75 feet. You board a car at the bottom of the Ferris wheel, which is 10 feet above the ground, and rotate 255° counterclockwise before the ride temporarily stops. How high above the ground are you when the ride stops? If the radius of the Ferris wheel is doubled, is your height above the ground doubled? Explain your reasoning.

39. **DRAWING CONCLUSIONS** A sprinkler at ground level is used to water a garden. The water leaving the sprinkler has an initial speed of 25 feet per second.

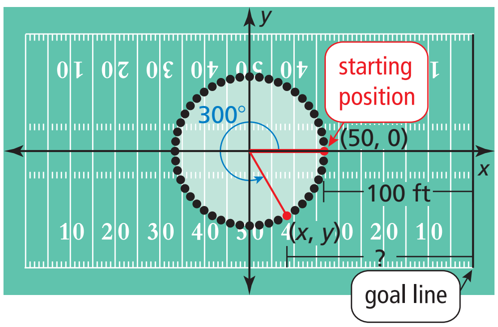
a. Use the model for horizontal distance given in Example 5 to complete the table.

Angle of sprinkler, θ	Horizontal distance water travels, d
30°	
35°	
40°	
45°	
50°	
55°	
60°	

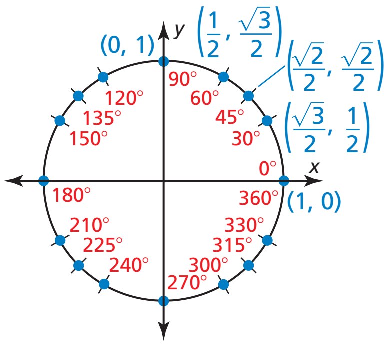
b. Which value of θ appears to maximize the horizontal distance traveled by the water? Use the model for horizontal distance and the unit circle to explain why your answer makes sense.

c. Compare the horizontal distance traveled by the water when $\theta = (45 - k)°$ with the distance when $\theta = (45 + k)°$, for $0 < k < 45$.

40. **MODELING WITH MATHEMATICS** Your school's marching band is performing at halftime during a football game. In the last formation, the band members form a circle 100 feet wide in the center of the field. You start at a point on the circle 100 feet from the goal line, march 300° around the circle, and then walk toward the goal line to exit the field. How far from the goal line are you at the point where you leave the circle?



41. **ANALYZING RELATIONSHIPS** Use symmetry and the given information to label the coordinates of the other points corresponding to special angles on the unit circle.



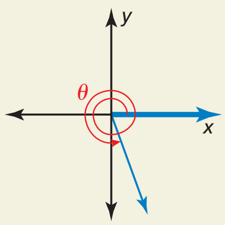
42. **THOUGHT PROVOKING** Use the interactive unit circle tool at *BigIdeasMath.com* to describe all values of θ for each situation.

a. $\sin\theta > 0$, $\cos\theta < 0$, and $\tan\theta > 0$

b. $\sin\theta > 0$, $\cos\theta < 0$, and $\tan\theta < 0$

43. **CRITICAL THINKING** Write $\tan\theta$ as the ratio of two other trigonometric functions. Use this ratio to explain why $\tan 90°$ is undefined but $\cot 90° = 0$.

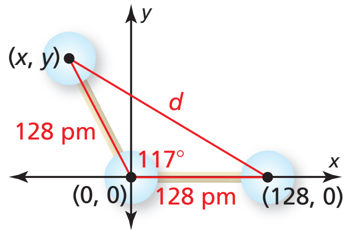
44. **HOW DO YOU SEE IT?** Determine whether each of the six trigonometric functions of θ is *positive*, *negative*, or *zero*. Explain your reasoning.



45. **USING STRUCTURE** A line with slope m passes through the origin. An angle θ in standard position has a terminal side that coincides with the line. Use a trigonometric function to relate the slope of the line to the angle.

46. **MAKING AN ARGUMENT** Your friend claims that the only solution to the trigonometric equation $\tan\theta = \sqrt{3}$ is $\theta = 60°$. Is your friend correct? Explain your reasoning.

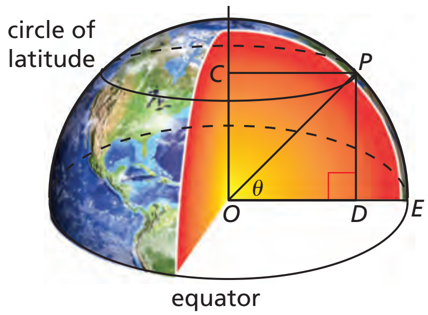
47. **PROBLEM SOLVING** When two atoms in a molecule are bonded to a common atom, chemists are interested in both the bond angle and the lengths of the bonds. An ozone molecule is made up of two oxygen atoms bonded to a third oxygen atom, as shown.



a. In the diagram, coordinates are given in picometers (pm). (*Note*: 1 pm $= 10^{-12}$ m) Find the coordinates (x, y) of the center of the oxygen atom in Quadrant II.

b. Find the distance d (in picometers) between the centers of the two unbonded oxygen atoms.

48. **MATHEMATICAL CONNECTIONS** The latitude of a point on Earth is the degree measure of the shortest arc from that point to the equator. For example, the latitude of point P in the diagram equals the degree measure of arc PE. At what latitude θ is the circumference of the circle of latitude at P half the distance around the equator?



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find all real zeros of the polynomial function. *(Section 4.6)*

49. $f(x) = x^4 + 2x^3 + x^2 + 8x - 12$

50. $f(x) = x^5 + 4x^4 - 14x^3 - 14x^2 - 15x - 18$

Graph the function. *(Section 4.8)*

51. $f(x) = 2(x + 3)^2(x - 1)$

52. $f(x) = \frac{1}{3}(x - 4)(x + 5)(x + 9)$

53. $f(x) = x^2(x + 1)^3(x - 2)$