

8.4 Finding Sums of Infinite Geometric Series

Essential Question How can you find the sum of an infinite geometric series?

EXPLORATION 1 Finding Sums of Infinite Geometric Series

USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools, such as a spreadsheet, to explore and deepen your understanding of concepts.

Work with a partner. Enter each geometric series in a spreadsheet. Then use the spreadsheet to determine whether the infinite geometric series has a finite sum. If it does, find the sum. Explain your reasoning. (The figure shows a partially completed spreadsheet for part (a).)

a. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

b. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$

c. $1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16} + \dots$

d. $1 + \frac{5}{4} + \frac{25}{16} + \frac{125}{64} + \frac{625}{256} + \dots$

e. $1 + \frac{4}{5} + \frac{16}{25} + \frac{64}{125} + \frac{256}{625} + \dots$

f. $1 + \frac{9}{10} + \frac{81}{100} + \frac{729}{1000} + \frac{6561}{10,000} + \dots$

	A	B
1	1	1
2	2	0.5
3	3	0.25
4	4	0.125
5	5	0.0625
6	6	0.03125
7	7	
8	8	
9	9	
10	10	
11	11	
12	12	
13	13	
14	14	
15	15	
16	Sum	

EXPLORATION 2 Writing a Conjecture

Work with a partner. Look back at the infinite geometric series in Exploration 1. Write a conjecture about how you can determine whether the infinite geometric series

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \dots$$

has a finite sum.

EXPLORATION 3 Writing a Formula

Work with a partner. In Lesson 8.3, you learned that the sum of the first n terms of a geometric series with first term a_1 and common ratio $r \neq 1$ is

$$S_n = a_1 \left(\frac{1 - r^n}{1 - r} \right).$$

When an infinite geometric series has a finite sum, what happens to r^n as n increases? Explain your reasoning. Write a formula to find the sum of an infinite geometric series. Then verify your formula by checking the sums you obtained in Exploration 1.

Communicate Your Answer

4. How can you find the sum of an infinite geometric series?

5. Find the sum of each infinite geometric series, if it exists.

a. $1 + 0.1 + 0.01 + 0.001 + 0.0001 + \dots$ b. $2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \frac{32}{81} + \dots$

8.4 Lesson

Core Vocabulary

partial sum, p. 436

Previous

repeating decimal
fraction in simplest form
rational number

What You Will Learn

- ▶ Find partial sums of infinite geometric series.
- ▶ Find sums of infinite geometric series.

Partial Sums of Infinite Geometric Series

The sum S_n of the first n terms of an infinite series is called a **partial sum**. The partial sums of an infinite geometric series may approach a limiting value.

EXAMPLE 1 Finding Partial Sums

Consider the infinite geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

Find and graph the partial sums S_n for $n = 1, 2, 3, 4,$ and 5 . Then describe what happens to S_n as n increases.

SOLUTION

Step 1 Find the partial sums.

$$S_1 = \frac{1}{2} = 0.5$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = 0.75$$

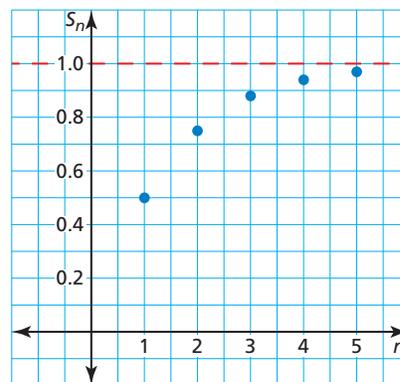
$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \approx 0.88$$

$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \approx 0.94$$

$$S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \approx 0.97$$

Step 2 Plot the points $(1, 0.5)$, $(2, 0.75)$, $(3, 0.88)$, $(4, 0.94)$, and $(5, 0.97)$. The graph is shown at the right.

- ▶ From the graph, S_n appears to approach 1 as n increases.



Sums of Infinite Geometric Series

In Example 1, you can understand why S_n approaches 1 as n increases by considering the rule for the sum of a finite geometric series.

$$S_n = a_1 \left(\frac{1 - r^n}{1 - r} \right) = \frac{1}{2} \left(\frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} \right) = 1 - \left(\frac{1}{2}\right)^n$$

As n increases, $\left(\frac{1}{2}\right)^n$ approaches 0, so S_n approaches 1. Therefore, 1 is defined to be the sum of the infinite geometric series in Example 1. More generally, as n increases for *any* infinite geometric series with common ratio r between -1 and 1 , the value of S_n approaches

$$S_n = a_1 \left(\frac{1 - r^n}{1 - r} \right) \approx a_1 \left(\frac{1 - 0}{1 - r} \right) = \frac{a_1}{1 - r}.$$

Core Concept

UNDERSTANDING MATHEMATICAL TERMS

Even though a geometric series with a common ratio of $|r| < 1$ has *infinitely* many terms, the series has a *finite* sum.

The Sum of an Infinite Geometric Series

The sum of an infinite geometric series with first term a_1 and common ratio r is given by

$$S = \frac{a_1}{1 - r}$$

provided $|r| < 1$. If $|r| \geq 1$, then the series has no sum.

EXAMPLE 2 Finding Sums of Infinite Geometric Series

Find the sum of each infinite geometric series.

a. $\sum_{i=1}^{\infty} 3(0.7)^{i-1}$ b. $1 + 3 + 9 + 27 + \dots$ c. $1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \dots$

SOLUTION

a. For this series, $a_1 = 3(0.7)^{1-1} = 3$ and $r = 0.7$. The sum of the series is

$$\begin{aligned} S &= \frac{a_1}{1 - r} && \text{Formula for sum of an infinite geometric series} \\ &= \frac{3}{1 - 0.7} && \text{Substitute 3 for } a_1 \text{ and 0.7 for } r. \\ &= 10. && \text{Simplify.} \end{aligned}$$

b. For this series, $a_1 = 1$ and $a_2 = 3$. So, the common ratio is $r = \frac{3}{1} = 3$. Because $|3| \geq 1$, the sum does not exist.

c. For this series, $a_1 = 1$ and $a_2 = -\frac{3}{4}$. So, the common ratio is

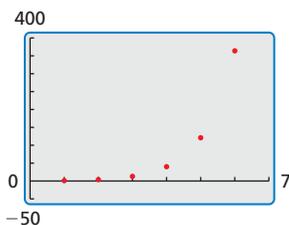
$$r = \frac{-\frac{3}{4}}{1} = -\frac{3}{4}.$$

The sum of the series is

$$\begin{aligned} S &= \frac{a_1}{1 - r} && \text{Formula for sum of an infinite geometric series} \\ &= \frac{1}{1 - \left(-\frac{3}{4}\right)} && \text{Substitute 1 for } a_1 \text{ and } -\frac{3}{4} \text{ for } r. \\ &= \frac{4}{7}. && \text{Simplify.} \end{aligned}$$

STUDY TIP

For the geometric series in part (b), the graph of the partial sums S_n for $n = 1, 2, 3, 4, 5$, and 6 are shown. From the graph, it appears that as n increases, the partial sums do not approach a fixed number.



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1. Consider the infinite geometric series

$$\frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \frac{16}{1625} + \frac{32}{3125} + \dots$$

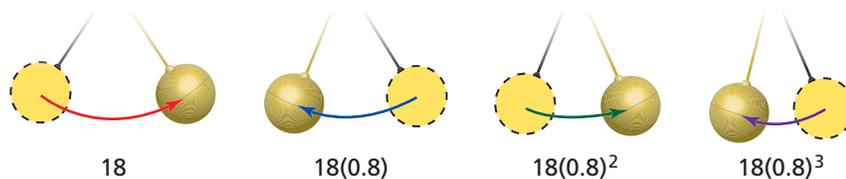
Find and graph the partial sums S_n for $n = 1, 2, 3, 4$, and 5. Then describe what happens to S_n as n increases.

Find the sum of the infinite geometric series, if it exists.

2. $\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n-1}$ 3. $\sum_{n=1}^{\infty} 3\left(\frac{5}{4}\right)^{n-1}$ 4. $3 + \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \dots$

EXAMPLE 3 Solving a Real-Life Problem

A pendulum that is released to swing freely travels 18 inches on the first swing. On each successive swing, the pendulum travels 80% of the distance of the previous swing. What is the total distance the pendulum swings?



SOLUTION

The total distance traveled by the pendulum is given by the infinite geometric series

$$18 + 18(0.8) + 18(0.8)^2 + 18(0.8)^3 + \cdots$$

For this series, $a_1 = 18$ and $r = 0.8$. The sum of the series is

$$\begin{aligned} S &= \frac{a_1}{1 - r} && \text{Formula for sum of an infinite geometric series} \\ &= \frac{18}{1 - 0.8} && \text{Substitute 18 for } a_1 \text{ and 0.8 for } r. \\ &= 90. && \text{Simplify.} \end{aligned}$$

REMEMBER

Because a repeating decimal is a rational number, it can be written as $\frac{a}{b}$, where a and b are integers and $b \neq 0$.

► The pendulum travels a total distance of 90 inches, or 7.5 feet.

EXAMPLE 4 Writing a Repeating Decimal as a Fraction

Write $0.242424 \dots$ as a fraction in simplest form.

SOLUTION

Write the repeating decimal as an infinite geometric series.

$$0.242424 \dots = 0.24 + 0.0024 + 0.000024 + 0.00000024 + \cdots$$

For this series, $a_1 = 0.24$ and $r = \frac{0.0024}{0.24} = 0.01$. Next, write the sum of the series.

$$\begin{aligned} S &= \frac{a_1}{1 - r} && \text{Formula for sum of an infinite geometric series} \\ &= \frac{0.24}{1 - 0.01} && \text{Substitute 0.24 for } a_1 \text{ and 0.01 for } r. \\ &= \frac{0.24}{0.99} && \text{Simplify.} \\ &= \frac{24}{99} && \text{Write as a quotient of integers.} \\ &= \frac{8}{33} && \text{Simplify.} \end{aligned}$$

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5. **WHAT IF?** In Example 3, suppose the pendulum travels 10 inches on its first swing. What is the total distance the pendulum swings?

Write the repeating decimal as a fraction in simplest form.

6. $0.555 \dots$

7. $0.727272 \dots$

8. $0.131313 \dots$

Vocabulary and Core Concept Check

- COMPLETE THE SENTENCE** The sum S_n of the first n terms of an infinite series is called $a(n)$ _____.
- WRITING** Explain how to tell whether the series $\sum_{i=1}^{\infty} a_1 r^{i-1}$ has a sum.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, consider the infinite geometric series. Find and graph the partial sums S_n for $n = 1, 2, 3, 4$, and 5. Then describe what happens to S_n as n increases. (See Example 1.)

- $\frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \frac{1}{162} + \dots$
- $\frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \dots$
- $4 + \frac{12}{5} + \frac{36}{25} + \frac{108}{125} + \frac{324}{625} + \dots$
- $2 + \frac{2}{6} + \frac{2}{36} + \frac{2}{216} + \frac{2}{1296} + \dots$

In Exercises 7–14, find the sum of the infinite geometric series, if it exists. (See Example 2.)

- $\sum_{n=1}^{\infty} 8\left(\frac{1}{5}\right)^{n-1}$
- $\sum_{k=1}^{\infty} -6\left(\frac{3}{2}\right)^{k-1}$
- $\sum_{k=1}^{\infty} \frac{11}{3}\left(\frac{3}{8}\right)^{k-1}$
- $\sum_{i=1}^{\infty} \frac{2}{5}\left(\frac{5}{3}\right)^{i-1}$
- $2 + \frac{6}{4} + \frac{18}{16} + \frac{54}{64} + \dots$
- $-5 - 2 - \frac{4}{5} - \frac{8}{25} - \dots$
- $3 + \frac{5}{2} + \frac{25}{12} + \frac{125}{72} + \dots$
- $\frac{1}{2} - \frac{5}{3} + \frac{50}{9} - \frac{500}{27} + \dots$

ERROR ANALYSIS In Exercises 15 and 16, describe and correct the error in finding the sum of the infinite geometric series.

$$15. \sum_{n=1}^{\infty} \left(\frac{7}{2}\right)^{n-1}$$



For this series, $a_1 = 1$ and $r = \frac{7}{2}$.

$$S = \frac{a_1}{1-r} = \frac{1}{1-\frac{7}{2}} = \frac{1}{-\frac{5}{2}} = -\frac{2}{5}$$

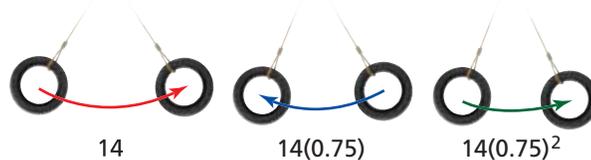
$$16. 4 + \frac{8}{3} + \frac{16}{9} + \frac{32}{27} + \dots$$



For this series, $a_1 = 4$ and $r = \frac{4}{\frac{8}{3}} = \frac{3}{2}$.

Because $\left|\frac{3}{2}\right| > 1$, the series has no sum.

- MODELING WITH MATHEMATICS** You push your younger cousin on a tire swing one time and then allow your cousin to swing freely. On the first swing, your cousin travels a distance of 14 feet. On each successive swing, your cousin travels 75% of the distance of the previous swing. What is the total distance your cousin swings? (See Example 3.)



- MODELING WITH MATHEMATICS** A company had a profit of \$350,000 in its first year. Since then, the company's profit has decreased by 12% per year. Assuming this trend continues, what is the total profit the company can make over the course of its lifetime? Justify your answer.

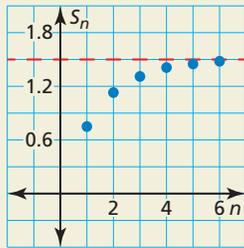
In Exercises 19–24, write the repeating decimal as a fraction in simplest form. (See Example 4.)

- 0.222...
- 0.444...
- 0.161616...
- 0.625625625...
- 32.323232...
- 130.130130130...
- PROBLEM SOLVING** Find two infinite geometric series whose sums are each 6. Justify your answers.

26. HOW DO YOU SEE IT?

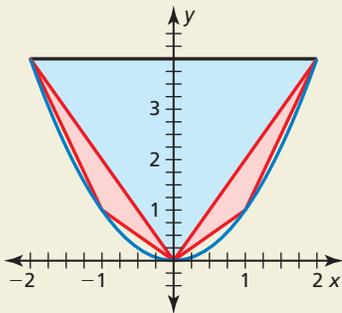
The graph shows the partial sums of the geometric series $a_1 + a_2 + a_3 + a_4 + \dots$.

What is the value of $\sum_{n=1}^{\infty} a_n$? Explain.



27. MODELING WITH MATHEMATICS A radio station has a daily contest in which a random listener is asked a trivia question. On the first day, the station gives \$500 to the first listener who answers correctly. On each successive day, the winner receives 90% of the winnings from the previous day. What is the total amount of prize money the radio station gives away during the contest?

28. THOUGHT PROVOKING Archimedes used the sum of a geometric series to compute the area enclosed by a parabola and a straight line. In “Quadrature of the Parabola,” he proved that the area of the region is $\frac{4}{3}$ the area of the inscribed triangle. The first term of the series for the parabola below is represented by the area of the blue triangle and the second term is represented by the area of the red triangles. Use Archimedes’ result to find the area of the region. Then write the area as the sum of an infinite geometric series.



29. DRAWING CONCLUSIONS Can a person running at 20 feet per second ever catch up to a tortoise that runs 10 feet per second when the tortoise has a 20-foot head start? The Greek mathematician Zeno said no. He reasoned as follows:



The person will keep halving the distance but will never catch up to the tortoise.

Looking at the race as Zeno did, the distances and the times it takes the person to run those distances both form infinite geometric series. Using the table, show that both series have finite sums. Does the person catch up to the tortoise? Justify your answer.

Distance (ft)	20	10	5	2.5	...
Time (sec)	1	0.5	0.25	0.125	...

30. MAKING AN ARGUMENT Your friend claims that $0.999\dots$ is equal to 1. Is your friend correct? Justify your answer.

31. CRITICAL THINKING The *Sierpinski triangle* is a fractal created using equilateral triangles. The process involves removing smaller triangles from larger triangles by joining the midpoints of the sides of the larger triangles as shown. Assume that the initial triangle has an area of 1 square foot.



- Let a_n be the total area of all the triangles that are removed at Stage n . Write a rule for a_n .
- Find $\sum_{n=1}^{\infty} a_n$. Interpret your answer in the context of this situation.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Determine the type of function represented by the table. (Section 6.7)

32.

x	-3	-2	-1	0	1
y	0.5	1.5	4.5	13.5	40.5

33.

x	0	4	8	12	16
y	-7	-1	2	2	-1

Determine whether the sequence is *arithmetic*, *geometric*, or *neither*. (Sections 8.2 and 8.3)

34. $-7, -1, 5, 11, 17, \dots$

35. $0, -1, -3, -7, -15, \dots$

36. $13.5, 40.5, 121.5, 364.5, \dots$