# Analyzing Geometric Sequences and Series

**Essential Question** How can you recognize a geometric sequence from its graph?

In a **geometric sequence**, the ratio of any term to the previous term, called the *common ratio*, is constant. For example, in the geometric sequence 1, 2, 4, 8, . . . , the common ratio is 2.

**EXPLORATION 1** **Recognizing Graphs of Geometric Sequences**

**Work with a partner.** Determine whether each graph shows a geometric sequence. If it does, then write a rule for the $n$th term of the sequence and use a spreadsheet to find the sum of the first 20 terms. What do you notice about the graph of a geometric sequence?

**a.**



**b.**



**c.**



**d.**
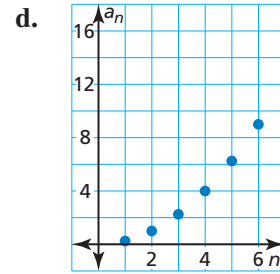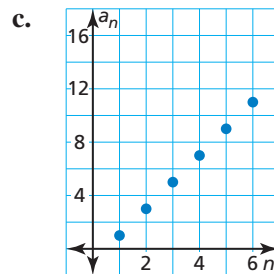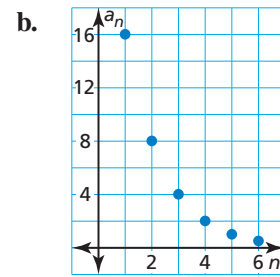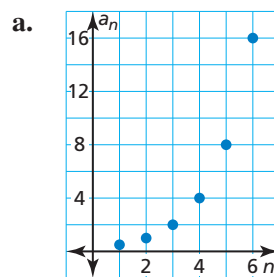


**LOOKING FOR REGULARITY IN REPEATED REASONING**

To be proficient in math, you need to notice when calculations are repeated, and look both for general methods and for shortcuts.

**EXPLORATION 2** **Finding the Sum of a Geometric Sequence**

**Work with a partner.** You can write the $n$th term of a geometric sequence with first term $a_1$ and common ratio $r$ as

$$a_n = a_1 r^{n-1}.$$

So, you can write the sum $S_n$ of the first $n$ terms of a geometric sequence as

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}.$$

Rewrite this formula by finding the difference $S_n - rS_n$ and solving for $S_n$. Then verify your rewritten formula by finding the sums of the first 20 terms of the geometric sequences in Exploration 1. Compare your answers to those you obtained using a spreadsheet.

## Communicate Your Answer

3. How can you recognize a geometric sequence from its graph?

4. Find the sum of the terms of each geometric sequence.

   **a.** 1, 2, 4, 8, . . . , 8192
   
   **b.** 0.1, 0.01, 0.001, 0.0001, . . . , $10^{-10}$

## What You Will Learn

▶  Identify geometric sequences.

▶  Write rules for geometric sequences.

▶  Find sums of finite geometric series.

## Identifying Geometric Sequences

In a **geometric sequence**, the ratio of any term to the previous term is constant. This constant ratio is called the **common ratio** and is denoted by $r$.

**EXAMPLE 1**   **Identifying Geometric Sequences**

Tell whether each sequence is geometric.

**a.** 6, 12, 20, 30, 42, . . .

**b.** 256, 64, 16, 4, 1, . . .

**SOLUTION**

Find the ratios of consecutive terms.

**a.** $\dfrac{a_2}{a_1} = \dfrac{12}{6} = 2$    $\dfrac{a_3}{a_2} = \dfrac{20}{12} = \dfrac{5}{3}$    $\dfrac{a_4}{a_3} = \dfrac{30}{20} = \dfrac{3}{2}$    $\dfrac{a_5}{a_4} = \dfrac{42}{30} = \dfrac{7}{5}$

▶  The ratios are not constant, so the sequence is not geometric.

**b.** $\dfrac{a_2}{a_1} = \dfrac{64}{256} = \dfrac{1}{4}$    $\dfrac{a_3}{a_2} = \dfrac{16}{64} = \dfrac{1}{4}$    $\dfrac{a_4}{a_3} = \dfrac{4}{16} = \dfrac{1}{4}$    $\dfrac{a_5}{a_4} = \dfrac{1}{4}$

▶  Each ratio is $\dfrac{1}{4}$, so the sequence is geometric.

## Monitoring Progress    🔊 Help in English and Spanish at *BigIdeasMath.com*

**Tell whether the sequence is geometric. Explain your reasoning.**

**1.** $27, 9, 3, 1, \dfrac{1}{3}, \ldots$    **2.** $2, 6, 24, 120, 720, \ldots$    **3.** $-1, 2, -4, 8, -16, \ldots$

## Writing Rules for Geometric Sequences

### 🎯 Core Concept

**Rule for a Geometric Sequence**

**Algebra**   The $n$th term of a geometric sequence with first term $a_1$ and common ratio $r$ is given by:

$$a_n = a_1 r^{n-1}$$

**Example**   The $n$th term of a geometric sequence with a first term of 2 and a common ratio of 3 is given by:

$$a_n = 2(3)^{n-1}$$

**EXAMPLE 2**  **Writing a Rule for the *n*th Term**

Write a rule for the *n*th term of each sequence. Then find $a_8$.

**a.** 5, 15, 45, 135, . . .

**b.** 88, −44, 22, −11, . . .

**COMMON ERROR**

In the general rule for a geometric sequence, note that the exponent is $n - 1$, not *n*.

**SOLUTION**

**a.** The sequence is geometric with first term $a_1 = 5$ and common ratio $r = \frac{15}{5} = 3$.

So, a rule for the *n*th term is

$$a_n = a_1 r^{n-1} \qquad \text{Write general rule.}$$

$$= 5(3)^{n-1}. \qquad \text{Substitute 5 for } a_1 \text{ and 3 for } r.$$

▶ A rule is $a_n = 5(3)^{n-1}$, and the 8th term is $a_8 = 5(3)^{8-1} = 10{,}935$.

**b.** The sequence is geometric with first term $a_1 = 88$ and common ratio $r = \dfrac{-44}{88} = -\dfrac{1}{2}$. So, a rule for the *n*th term is

$$a_n = a_1 r^{n-1} \qquad \text{Write general rule.}$$

$$= 88\left(-\frac{1}{2}\right)^{n-1}. \qquad \text{Substitute 88 for } a_1 \text{ and } -\frac{1}{2} \text{ for } r.$$

▶ A rule is $a_n = 88\left(-\dfrac{1}{2}\right)^{n-1}$, and the 8th term is $a_8 = 88\left(-\dfrac{1}{2}\right)^{8-1} = -\dfrac{11}{16}$.

## Monitoring Progress  🔊 Help in English and Spanish at *BigIdeasMath.com*

**4.** Write a rule for the *n*th term of the sequence 3, 15, 75, 375, . . . .. Then find $a_9$.

**EXAMPLE 3**  **Writing a Rule Given a Term and Common Ratio**

One term of a geometric sequence is $a_4 = 12$. The common ratio is $r = 2$. Write a rule for the *n*th term. Then graph the first six terms of the sequence.

**SOLUTION**

**Step 1** Use the general rule to find the first term.

$$a_n = a_1 r^{n-1} \qquad \text{Write general rule.}$$

$$a_4 = a_1 r^{4-1} \qquad \text{Substitute 4 for } n.$$

$$12 = a_1(2)^3 \qquad \text{Substitute 12 for } a_4 \text{ and 2 for } r.$$

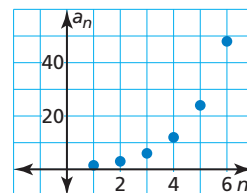$$1.5 = a_1 \qquad \text{Solve for } a_1.$$

**ANALYZING RELATIONSHIPS**

Notice that the points lie on an exponential curve because consecutive terms change by equal factors. So, a geometric sequence in which $r > 0$ and $r \ne 1$ is an exponential function whose domain is a subset of the integers.

**Step 2** Write a rule for the *n*th term.

$$a_n = a_1 r^{n-1} \qquad \text{Write general rule.}$$

$$= 1.5(2)^{n-1} \qquad \text{Substitute 1.5 for } a_1 \text{ and 2 for } r.$$

**Step 3** Use the rule to create a table of values for the sequence. Then plot the points.

| *n* | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|-----|---|---|----|----|----|
| $a_n$ | 1.5 | 3 | 6 | 12 | 24 | 48 |

**EXAMPLE 4**    **Writing a Rule Given Two Terms**

Two terms of a geometric sequence are $a_2 = 12$ and $a_5 = -768$. Write a rule for the $n$th term.

### SOLUTION

**Step 1**   Write a system of equations using $a_n = a_1 r^{n-1}$. Substitute 2 for $n$ to write Equation 1. Substitute 5 for $n$ to write Equation 2.

$$a_2 = a_1 r^{2-1} \quad \Longrightarrow \quad 12 = a_1 r \qquad \text{Equation 1}$$

$$a_5 = a_1 r^{5-1} \quad \Longrightarrow \quad -768 = a_1 r^4 \qquad \text{Equation 2}$$

**Step 2**   Solve the system.

$$\frac{12}{r} = a_1 \qquad \text{Solve Equation 1 for } a_1.$$

$$-768 = \frac{12}{r}(r^4) \qquad \text{Substitute for } a_1 \text{ in Equation 2.}$$

$$-768 = 12r^3 \qquad \text{Simplify.}$$

$$-4 = r \qquad \text{Solve for } r.$$

$$12 = a_1(-4) \qquad \text{Substitute for } r \text{ in Equation 1.}$$

$$-3 = a_1 \qquad \text{Solve for } a_1.$$

**Step 3**   Write a rule for $a_n$.    $a_n = a_1 r^{n-1}$      Write general rule.

$$= -3(-4)^{n-1} \qquad \text{Substitute for } a_1 \text{ and } r.$$

**Check**

Use the rule to verify that the 2nd term is 12 and the 5th term is $-768$.

$$a_2 = -3(-4)^{2-1}$$
$$= -3(-4) = 12 \ ✓$$
$$a_5 = -3(-4)^{5-1}$$
$$= -3(256) = -768 \ ✓$$

## Monitoring Progress ◀))   Help in English and Spanish at *BigIdeasMath.com*

Write a rule for the $n$th term of the sequence. Then graph the first six terms of the sequence.

**5.** $a_6 = -96, r = -2$            **6.** $a_2 = 12, a_4 = 3$

## Finding Sums of Finite Geometric Series

The expression formed by adding the terms of a geometric sequence is called a **geometric series**. The sum of the first $n$ terms of a geometric series is denoted by $S_n$. You can develop a rule for $S_n$ as follows.

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}$$

$$-rS_n = \quad -a_1 r - a_1 r^2 - a_1 r^3 - \cdots - a_1 r^{n-1} - a_1 r^n$$

$$\overline{S_n - rS_n = a_1 + \ 0 \ + \ 0 \ + \ 0 \ + \cdots + \ 0 \ \quad - a_1 r^n}$$

$$S_n(1 - r) = a_1(1 - r^n)$$

When $r \neq 1$, you can divide each side of this equation by $1 - r$ to obtain the following rule for $S_n$.

### 🅖 Core Concept

**The Sum of a Finite Geometric Series**

The sum of the first $n$ terms of a geometric series with common ratio $r \neq 1$ is

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right).$$

**EXAMPLE 5**  **Finding the Sum of a Geometric Series**

Find the sum $\displaystyle\sum_{k=1}^{10} 4(3)^{k-1}$.

**SOLUTION**

**Step 1** Find the first term and the common ratio.

$$a_1 = 4(3)^{1-1} = 4 \qquad\qquad \text{Identify first term.}$$

$$r = 3 \qquad\qquad\qquad\qquad \text{Identify common ratio.}$$

**Step 2** Find the sum.

$$S_{10} = a_1\left(\frac{1-r^{10}}{1-r}\right) \qquad \text{Write rule for } S_{10}.$$

$$= 4\left(\frac{1-3^{10}}{1-3}\right) \qquad \text{Substitute 4 for } a_1 \text{ and 3 for } r.$$

$$= 118{,}096 \qquad\qquad \text{Simplify.}$$

**Check**

Use a graphing calculator to check the sum.

```
sum(seq(4*3^(X-1
),X,1,10))
           118096
```

**EXAMPLE 6**  **Solving a Real-Life Problem**

You can calculate the monthly payment $M$ (in dollars) for a loan using the formula

$$M = \frac{L}{\displaystyle\sum_{k=1}^{t}\left(\frac{1}{1+i}\right)^k}$$

where $L$ is the loan amount (in dollars), $i$ is the monthly interest rate (in decimal form), and $t$ is the term (in months). Calculate the monthly payment on a 5-year loan for $20,000 with an annual interest rate of 6%.

**SOLUTION**

**Step 1** Substitute for $L$, $i$, and $t$. The loan amount is $L = 20{,}000$, the monthly interest rate is $i = \dfrac{0.06}{12} = 0.005$, and the term is $t = 5(12) = 60$.

$$M = \frac{20{,}000}{\displaystyle\sum_{k=1}^{60}\left(\frac{1}{1+0.005}\right)^k}$$

**Step 2** Notice that the denominator is a geometric series with first term $\dfrac{1}{1.005}$ and common ratio $\dfrac{1}{1.005}$. Use a calculator to find the monthly payment.

```
1/1.005→R
          .9950248756
R((1-R^60)/(1-R)
)
          51.72556075
20000/Ans
          386.6560306
```

▶ So, the monthly payment is $386.66.

**USING TECHNOLOGY**

Storing the value of $\dfrac{1}{1.005}$ helps minimize mistakes and also assures an accurate answer. Rounding this value to 0.995 results in a monthly payment of $386.94.

**Monitoring Progress** 🔊 Help in English and Spanish at *BigIdeasMath.com*

Find the sum.

**7.** $\displaystyle\sum_{k=1}^{8} 5^{k-1}$

**8.** $\displaystyle\sum_{i=1}^{12} 6(-2)^{i-1}$

**9.** $\displaystyle\sum_{t=1}^{7} -16(0.5)^{t-1}$

**10. WHAT IF?** In Example 6, how does the monthly payment change when the annual interest rate is 5%?

## Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The constant ratio of consecutive terms in a geometric sequence is called the _____.

2. **WRITING** How can you determine whether a sequence is geometric from its graph?

3. **COMPLETE THE SENTENCE** The $n$th term of a geometric sequence has the form $a_n$ = _____.

4. **VOCABULARY** State the rule for the sum of the first $n$ terms of a geometric series.

## Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, tell whether the sequence is geometric. Explain your reasoning. *(See Example 1.)*

5. $96, 48, 24, 12, 6, \ldots$

6. $729, 243, 81, 27, 9, \ldots$

7. $2, 4, 6, 8, 10, \ldots$

8. $5, 20, 35, 50, 65, \ldots$

9. $0.2, 3.2, -12.8, 51.2, -204.8, \ldots$

10. $0.3, -1.5, 7.5, -37.5, 187.5, \ldots$

11. $\dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{18}, \dfrac{1}{54}, \dfrac{1}{162}, \ldots$

12. $\dfrac{1}{4}, \dfrac{1}{16}, \dfrac{1}{64}, \dfrac{1}{256}, \dfrac{1}{1024}, \ldots$

13. **WRITING EQUATIONS** Write a rule for the geometric sequence with the given description.

  a. The first term is $-3$, and each term is 5 times the previous term.

  b. The first term is 72, and each term is $\frac{1}{3}$ times the previous term.

14. **WRITING** Compare the terms of a geometric sequence when $r > 1$ to when $0 < r < 1$.

In Exercises 15–22, write a rule for the $n$th term of the sequence. Then find $a_7$. *(See Example 2.)*

15. $4, 20, 100, 500, \ldots$

16. $6, 24, 96, 384, \ldots$

17. $112, 56, 28, 14, \ldots$

18. $375, 75, 15, 3, \ldots$

19. $4, 6, 9, \dfrac{27}{2}, \ldots$

20. $2, \dfrac{3}{2}, \dfrac{9}{8}, \dfrac{27}{32}, \ldots$

21. $1.3, -3.9, 11.7, -35.1, \ldots$

22. $1.5, -7.5, 37.5, -187.5, \ldots$

In Exercises 23–30, write a rule for the $n$th term of the sequence. Then graph the first six terms of the sequence. *(See Example 3.)*

23. $a_3 = 4, r = 2$

24. $a_3 = 27, r = 3$

25. $a_2 = 30, r = \frac{1}{2}$

26. $a_2 = 64, r = \frac{1}{4}$

27. $a_4 = -192, r = 4$

28. $a_4 = -500, r = 5$

29. $a_5 = 3, r = -\frac{1}{3}$

30. $a_5 = 1, r = -\frac{1}{5}$

**ERROR ANALYSIS** In Exercises 31 and 32, describe and correct the error in writing a rule for the $n$th term of the geometric sequence for which $a_2 = 48$ and $r = 6$.
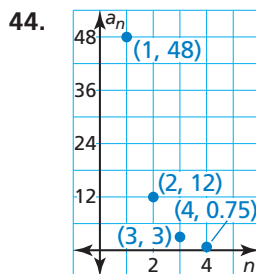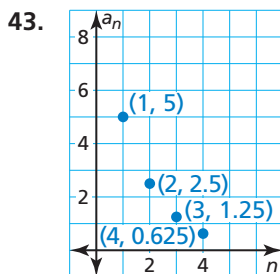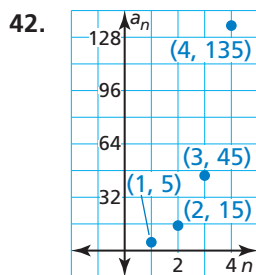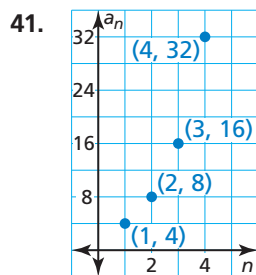
31.

✗
$$a_n = a_1 r^n$$
$$48 = a_1 6^2$$
$$\frac{4}{3} = a_1$$
$$a_n = \frac{4}{3}(6)^n$$

32.

✗
$$a_n = r(a_1)^{n-1}$$
$$48 = 6(a_1)^{2-1}$$
$$8 = a_1$$
$$a_n = 6(8)^{n-1}$$

In Exercises 33–40, write a rule for the $n$th term of the geometric sequence. *(See Example 4.)*

33. $a_2 = 28, a_5 = 1792$

34. $a_1 = 11, a_4 = 88$

35. $a_1 = -6, a_5 = -486$

36. $a_2 = -10, a_6 = -6250$

37. $a_2 = 64, a_4 = 1$

38. $a_1 = 1, a_2 = 49$

39. $a_2 = -72, a_6 = -\frac{1}{18}$

40. $a_2 = -48, a_5 = \frac{3}{4}$

**41.**



**42.**



**43.**



**44.**



**45.**

| $n$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $a_n$ | $-12$ | 24 | $-48$ | 96 | $-192$ |

**46.**

| $n$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $a_n$ | $-21$ | 63 | $-189$ | 567 | $-1701$ |

In Exercises 47–52, find the sum. *(See Example 5.)*

**47.** $\displaystyle\sum_{i=1}^{9} 6(7)^{i-1}$

**48.** $\displaystyle\sum_{i=1}^{10} 7(4)^{i-1}$

**49.** $\displaystyle\sum_{i=1}^{10} 4\left(\frac{3}{4}\right)^{i-1}$

**50.** $\displaystyle\sum_{i=1}^{8} 5\left(\frac{1}{3}\right)^{i-1}$

**51.** $\displaystyle\sum_{i=0}^{8} 8\left(-\frac{2}{3}\right)^{i}$

**52.** $\displaystyle\sum_{i=0}^{9} 9\left(-\frac{3}{4}\right)^{i}$

**NUMBER SENSE** In Exercises 53 and 54, find the sum.

**53.** The first 8 terms of the geometric sequence
$-12, -48, -192, -768, \ldots$.

**54.** The first 9 terms of the geometric sequence
$-14, -42, -126, -378, \ldots$.

**55.** **WRITING** Compare the graph of $a_n = 5(3)^{n-1}$, where
$n$ is a positive integer, to the graph of $f(x) = 5 \cdot 3^{x-1}$,
where $x$ is a real number.

**56.** **ABSTRACT REASONING** Use the rule for the sum of a
finite geometric series to write each polynomial as a
rational expression.

**a.** $1 + x + x^2 + x^3 + x^4$

**b.** $3x + 6x^3 + 12x^5 + 24x^7$

**MODELING WITH MATHEMATICS** In Exercises 57 and 58,
use the monthly payment formula given in Example 6.

**57.** You are buying a new car. You take out a 5-year loan
for $15,000. The annual interest rate of the loan is 4%.
Calculate the monthly payment. *(See Example 6.)*

**58.** You are buying a new house. You take out a 30-year
mortgage for $200,000. The annual interest rate of the
loan is 4.5%. Calculate the monthly payment.



**59.** **MODELING WITH MATHEMATICS** A regional soccer
tournament has 64 participating teams. In the first
round of the tournament, 32 games are played. In each
successive round, the number of games decreases by
a factor of $\frac{1}{2}$.

**a.** Write a rule for the number of games played in the
$n$th round. For what values of $n$ does the rule make
sense? Explain.

**b.** Find the total number of games played in the
regional soccer tournament.

**60.** **MODELING WITH MATHEMATICS** In a skydiving
formation with $R$ rings, each ring after the first has
twice as many skydivers as the preceding ring. The
formation for $R = 2$ is shown.



**a.** Let $a_n$ be the number of skydivers in the $n$th ring.
Write a rule for $a_n$.

**b.** Find the total number of skydivers when there are
four rings.

**61. PROBLEM SOLVING** The *Sierpinski carpet* is a fractal created using squares. The process involves removing smaller squares from larger squares. First, divide a large square into nine congruent squares. Then remove the center square. Repeat these steps for each smaller square, as shown below. Assume that each side of the initial square is 1 unit long.



Stage 1          Stage 2          Stage 3
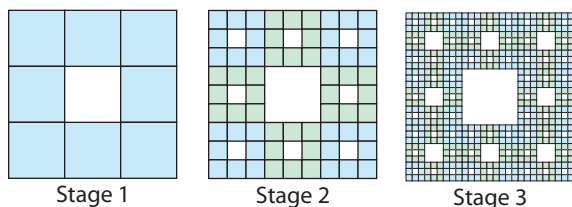
**a.** Let $a_n$ be the total number of squares removed at the $n$th stage. Write a rule for $a_n$. Then find the total number of squares removed through Stage 8.

**b.** Let $b_n$ be the remaining area of the original square after the $n$th stage. Write a rule for $b_n$. Then find the remaining area of the original square after Stage 12.

**62. HOW DO YOU SEE IT?** Match each sequence with its graph. Explain your reasoning.

**a.** $a_n = 10\left(\frac{1}{2}\right)^{n-1}$     **b.** $a_n = 10(2)^{n-1}$

**A.**


**B.**
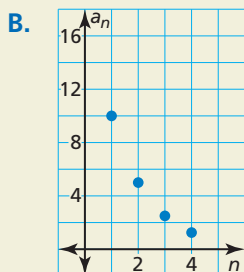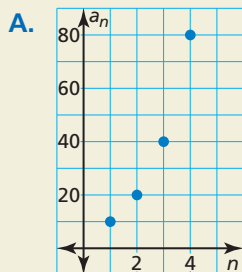

**63. CRITICAL THINKING** On January 1, you deposit $2000 in a retirement account that pays 5% annual interest. You make this deposit each January 1 for the next 30 years. How much money do you have in your account immediately after you make your last deposit?

**64. THOUGHT PROVOKING** The first four iterations of the fractal called the *Koch snowflake* are shown below. Find the perimeter and area of each iteration. Do the perimeters and areas form geometric sequences? Explain your reasoning.
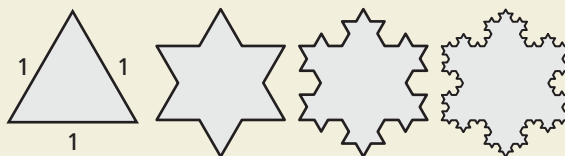


**65. MAKING AN ARGUMENT** You and your friend are comparing two loan options for a $165,000 house. Loan 1 is a 15-year loan with an annual interest rate of 3%. Loan 2 is a 30-year loan with an annual interest rate of 4%. Your friend claims the total amount repaid over the loan will be less for Loan 2. Is your friend correct? Justify your answer.

**66. CRITICAL THINKING** Let $L$ be the amount of a loan (in dollars), $i$ be the monthly interest rate (in decimal form), $t$ be the term (in months), and $M$ be the monthly payment (in dollars).

**a.** When making monthly payments, you are paying the loan amount plus the interest the loan gathers each month. For a 1-month loan, $t = 1$, the equation for repayment is $L(1 + i) - M = 0$. For a 2-month loan, $t = 2$, the equation is $[L(1 + i) - M](1 + i) - M = 0$. Solve both of these repayment equations for $L$.

**b.** Use the pattern in the equations you solved in part (a) to write a repayment equation for a $t$-month loan. (*Hint*: $L$ is equal to $M$ times a geometric series.) Then solve the equation for $M$.

**c.** Use the rule for the sum of a finite geometric series to show that the formula in part (b) is equivalent to

$$M = L\left(\frac{i}{1 - (1 + i)^{-t}}\right).$$

Use this formula to check your answers in Exercises 57 and 58.

---

## Maintaining Mathematical Proficiency
Reviewing what you learned in previous grades and lessons

**Graph the function. State the domain and range.** *(Section 7.2)*

**67.** $f(x) = \dfrac{1}{x - 3}$

**68.** $g(x) = \dfrac{2}{x} + 3$

**69.** $h(x) = \dfrac{1}{x - 2} + 1$

**70.** $p(x) = \dfrac{3}{x + 1} - 2$