8.1 Defining and Using Sequences and Series

Essential Question How can you write a rule for the *n*th term of a sequence?

A **sequence** is an ordered list of numbers. There can be a limited number or an infinite number of *terms* of a sequence.

$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ Terms of a sequence

Here is an example.

$1, 4, 7, 10, \ldots, 3n - 2, \ldots$

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to reason inductively about data.

EXPLORATION 1 Writing Rules for Sequences

Work with a partner. Match each sequence with its graph. The horizontal axes represent n, the position of each term in the sequence. Then write a rule for the nth term of the sequence, and use the rule to find a_{10}.

a. $1, 2.5, 4, 5.5, 7, \ldots$ **b.** $8, 6.5, 5, 3.5, 2, \ldots$ **c.** $\frac{1}{4}, \frac{4}{4}, \frac{9}{4}, \frac{16}{4}, \frac{25}{4}, \ldots$

d. $\frac{25}{4}, \frac{16}{4}, \frac{9}{4}, \frac{4}{4}, \frac{1}{4}, \ldots$ **e.** $\frac{1}{2}, 1, 2, 4, 8, \ldots$ **f.** $8, 4, 2, 1, \frac{1}{2}, \ldots$

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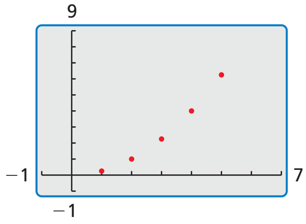

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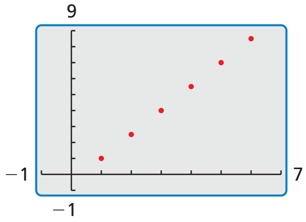

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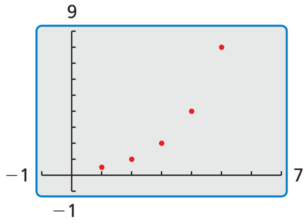

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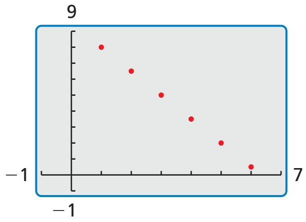

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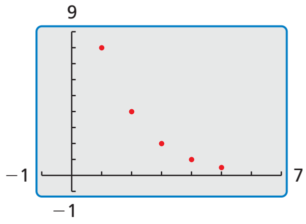

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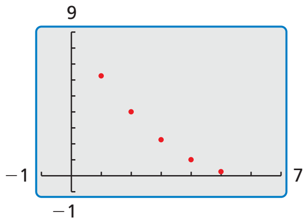

Communicate Your Answer

2. How can you write a rule for the *n*th term of a sequence?

3. What do you notice about the relationship between the terms in (a) an arithmetic sequence and (b) a geometric sequence? Justify your answers.

8.1 Lesson

Core Vocabulary

sequence, *p. 410*
terms of a sequence, *p. 410*
series, *p. 412*
summation notation, *p. 412*
sigma notation, *p. 412*

Previous
domain
range

What You Will Learn

- Use sequence notation to write terms of sequences.
- Write a rule for the *n*th term of a sequence.
- Sum the terms of a sequence to obtain a series and use summation notation.

Writing Terms of Sequences

Core Concept

Sequences

A **sequence** is an ordered list of numbers. A *finite sequence* is a function that has a limited number of terms and whose domain is the finite set $\{1, 2, 3, \ldots, n\}$. The values in the range are called the **terms** of the sequence.

Domain:	1	2	3	4	. . .	n	Relative position of each term
	↓	↓	↓	↓		↓	
Range:	a_1	a_2	a_3	a_4	. . .	a_n	Terms of the sequence

An *infinite sequence* is a function that continues without stopping and whose domain is the set of positive integers. Here are examples of a finite sequence and an infinite sequence.

Finite sequence: 2, 4, 6, 8 **Infinite sequence:** 2, 4, 6, 8, . . .

A sequence can be specified by an equation, or *rule*. For example, both sequences above can be described by the rule $a_n = 2n$ or $f(n) = 2n$.

The domain of a sequence may begin with 0 instead of 1. When this is the case, the domain of a finite sequence is the set $\{0, 1, 2, 3, \ldots, n\}$ and the domain of an infinite sequence becomes the set of nonnegative integers. Unless otherwise indicated, assume the domain of a sequence begins with 1.

EXAMPLE 1 Writing the Terms of Sequences

Write the first six terms of (a) $a_n = 2n + 5$ and (b) $f(n) = (-3)^{n-1}$.

SOLUTION

a.

$a_1 = 2(1) + 5 = 7$ 1st term
$a_2 = 2(2) + 5 = 9$ 2nd term
$a_3 = 2(3) + 5 = 11$ 3rd term
$a_4 = 2(4) + 5 = 13$ 4th term
$a_5 = 2(5) + 5 = 15$ 5th term
$a_6 = 2(6) + 5 = 17$ 6th term

b.

$f(1) = (-3)^{1-1} = 1$
$f(2) = (-3)^{2-1} = -3$
$f(3) = (-3)^{3-1} = 9$
$f(4) = (-3)^{4-1} = -27$
$f(5) = (-3)^{5-1} = 81$
$f(6) = (-3)^{6-1} = -243$

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Write the first six terms of the sequence.

1. $a_n = n + 4$

2. $f(n) = (-2)^{n-1}$

3. $a_n = \dfrac{n}{n+1}$

Writing Rules for Sequences

When the terms of a sequence have a recognizable pattern, you may be able to write a rule for the nth term of the sequence.

EXAMPLE 2 Writing Rules for Sequences

Describe the pattern, write the next term, and write a rule for the nth term of the sequences (a) $-1, -8, -27, -64, \ldots$ and (b) $0, 2, 6, 12, \ldots$.

SOLUTION

a. You can write the terms as $(-1)^3, (-2)^3, (-3)^3, (-4)^3, \ldots$. The next term is $a_5 = (-5)^3 = -125$. A rule for the nth term is $a_n = (-n)^3$.

b. You can write the terms as $0(1), 1(2), 2(3), 3(4), \ldots$. The next term is $f(5) = 4(5) = 20$. A rule for the nth term is $f(n) = (n - 1)n$.

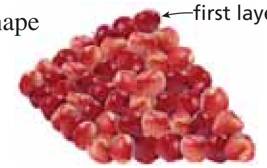
To graph a sequence, let the horizontal axis represent the position numbers (the domain) and the vertical axis represent the terms (the range).

STUDY TIP

When you are given only the first several terms of a sequence, there may be more than one rule for the nth term. For instance, the sequence $2, 4, 8, \ldots$ can be given by $a_n = 2^n$ or $a_n = n^2 - n + 2$.

EXAMPLE 3 Solving a Real-Life Problem

You work in a grocery store and are stacking apples in the shape of a square pyramid with seven layers. Write a rule for the number of apples in each layer. Then graph the sequence.



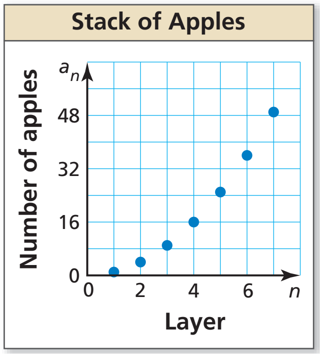
SOLUTION

Step 1 Make a table showing the number of fruit in the first three layers. Let a_n represent the number of apples in layer n.

Layer, n	1	2	3
Number of apples, a_n	 $1 = 1^2$	 $4 = 2^2$	 $9 = 3^2$

Step 2 Write a rule for the number of apples in each layer. From the table, you can see that $a_n = n^2$.

Step 3 Plot the points (1, 1), (2, 4), (3, 9), (4, 16), (5, 25), (6, 36), and (7, 49). The graph is shown at the right.



COMMON ERROR

Although the plotted points in Example 3 follow a curve, do not draw the curve because the sequence is defined only for integer values of n, specifically $n = 1, 2, 3, 4, 5, 6$, and 7.

Monitoring Progress

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Describe the pattern, write the next term, graph the first five terms, and write a rule for the nth term of the sequence.

4. $3, 5, 7, 9, \ldots$

5. $3, 8, 15, 24, \ldots$

6. $1, -2, 4, -8, \ldots$

7. $2, 5, 10, 17, \ldots$

8. **WHAT IF?** In Example 3, suppose there are nine layers of apples. How many apples are in the ninth layer?

Writing Rules for Series

Core Concept

Series and Summation Notation

When the terms of a sequence are added together, the resulting expression is a **series**. A series can be finite or infinite.

Finite series: $2 + 4 + 6 + 8$

Infinite series: $2 + 4 + 6 + 8 + \cdots$

You can use **summation notation** to write a series. For example, the two series above can be written in summation notation as follows:

Finite series: $2 + 4 + 6 + 8 = \sum_{i=1}^{4} 2i$

Infinite series: $2 + 4 + 6 + 8 + \cdots = \sum_{i=1}^{\infty} 2i$

For both series, the *index of summation* is i and the *lower limit of summation* is 1. The *upper limit of summation* is 4 for the finite series and ∞ (infinity) for the infinite series. Summation notation is also called **sigma notation** because it uses the uppercase Greek letter *sigma*, written Σ.

READING

When written in summation notation, this series is read as "the sum of $2i$ for values of i from 1 to 4."

EXAMPLE 4 Writing Series Using Summation Notation

Write each series using summation notation.

a. $25 + 50 + 75 + \cdots + 250$

b. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$

SOLUTION

a. Notice that the first term is 25(1), the second is 25(2), the third is 25(3), and the last is 25(10). So, the terms of the series can be written as:

$$a_i = 25i, \text{ where } i = 1, 2, 3, \ldots, 10$$

The lower limit of summation is 1 and the upper limit of summation is 10.

▶ The summation notation for the series is $\sum_{i=1}^{10} 25i$.

b. Notice that for each term, the denominator of the fraction is 1 more than the numerator. So, the terms of the series can be written as:

$$a_i = \frac{i}{i+1}, \text{ where } i = 1, 2, 3, 4, \ldots$$

The lower limit of summation is 1 and the upper limit of summation is infinity.

▶ The summation notation for the series is $\sum_{i=1}^{\infty} \frac{i}{i+1}$.

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Write the series using summation notation.

9. $5 + 10 + 15 + \cdots + 100$

10. $\frac{1}{2} + \frac{4}{5} + \frac{9}{10} + \frac{16}{17} + \cdots$

11. $6 + 36 + 216 + 1296 + \cdots$

12. $5 + 6 + 7 + \cdots + 12$

COMMON ERROR

Be sure to use the correct lower and upper limits of summation when finding the sum of a series.

The index of summation for a series does not have to be i—any letter can be used. Also, the index does not have to begin at 1. For instance, the index begins at 4 in the next example.

EXAMPLE 5 Finding the Sum of a Series

Find the sum $\sum_{k=4}^{8} (3 + k^2)$.

SOLUTION

$$\begin{aligned}\sum_{k=4}^{8} (3 + k^2) &= (3 + 4^2) + (3 + 5^2) + (3 + 6^2) + (3 + 7^2) + (3 + 8^2)\\ &= 19 + 28 + 39 + 52 + 67\\ &= 205\end{aligned}$$

For series with many terms, finding the sum by adding the terms can be tedious. Below are formulas you can use to find the sums of three special types of series.

Core Concept

Formulas for Special Series

Sum of n terms of 1: $\sum_{i=1}^{n} 1 = n$

Sum of first n positive integers: $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$

Sum of squares of first n positive integers: $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$

EXAMPLE 6 Using a Formula for a Sum

How many apples are in the stack in Example 3?

SOLUTION

From Example 3, you know that the ith term of the series is given by $a_i = i^2$, where $i = 1, 2, 3, \ldots, 7$. Using summation notation and the third formula listed above, you can find the total number of apples as follows:

$$1^2 + 2^2 + \cdots + 7^2 = \sum_{i=1}^{7} i^2 = \frac{7(7+1)(2 \bullet 7 + 1)}{6} = \frac{7(8)(15)}{6} = 140$$

▶ There are 140 apples in the stack. Check this by adding the number of apples in each of the seven layers.

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Find the sum.

13. $\sum_{i=1}^{5} 8i$

14. $\sum_{k=3}^{7} (k^2 - 1)$

15. $\sum_{i=1}^{34} 1$

16. $\sum_{k=1}^{6} k$

17. WHAT IF? Suppose there are nine layers in the apple stack in Example 3. How many apples are in the stack?

8.1 Exercises

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Vocabulary and Core Concept Check

1. **VOCABULARY** What is another name for summation notation?

2. **COMPLETE THE SENTENCE** In a sequence, the numbers are called __________ of the sequence.

3. **WRITING** Compare sequences and series.

4. **WHICH ONE DOESN'T BELONG?** Which does *not* belong with the other three? Explain your reasoning.

$\sum_{i=1}^{6} i^2$	91	$1 + 4 + 9 + 16 + 25 + 36$	$\sum_{i=0}^{5} i^2$

Monitoring Progress and Modeling with Mathematics

In Exercises 5–14, write the first six terms of the sequence. *(See Example 1.)*

5. $a_n = n + 2$

6. $a_n = 6 - n$

7. $a_n = n^2$

8. $f(n) = n^3 + 2$

9. $f(n) = 4^{n-1}$

10. $a_n = -n^2$

11. $a_n = n^2 - 5$

12. $a_n = (n + 3)^2$

13. $f(n) = \dfrac{2n}{n + 2}$

14. $f(n) = \dfrac{n}{2n - 1}$

In Exercises 15–26, describe the pattern, write the next term, and write a rule for the *n*th term of the sequence. *(See Example 2.)*

15. 1, 6, 11, 16, . . .

16. 1, 2, 4, 8, . . .

17. 3.1, 3.8, 4.5, 5.2, . . .

18. 9, 16.8, 24.6, 32.4, . . .

19. 5.8, 4.2, 2.6, 1, −0.6 . . .

20. −4, 8, −12, 16, . . .

21. $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \ldots$

22. $\frac{1}{10}, \frac{3}{20}, \frac{5}{30}, \frac{7}{40}, \ldots$

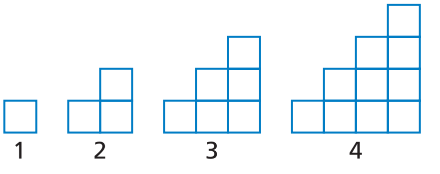
23. $\frac{2}{3}, \frac{2}{6}, \frac{2}{9}, \frac{2}{12}, \ldots$

24. $\frac{2}{3}, \frac{4}{4}, \frac{6}{5}, \frac{8}{6}, \ldots$

25. 2, 9, 28, 65, . . .

26. 1.2, 4.2, 9.2, 16.2, . . .

27. **FINDING A PATTERN** Which rule gives the total number of squares in the *n*th figure of the pattern shown? Justify your answer.



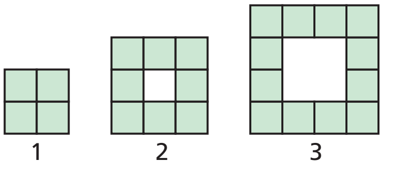
Ⓐ $a_n = 3n - 3$

Ⓑ $a_n = 4n - 5$

Ⓒ $a_n = n$

Ⓓ $a_n = \dfrac{n(n + 1)}{2}$

28. **FINDING A PATTERN** Which rule gives the total number of green squares in the *n*th figure of the pattern shown? Justify your answer.



Ⓐ $a_n = n^2 - 1$

Ⓑ $a_n = \dfrac{n^2}{2}$

Ⓒ $a_n = 4n$

Ⓓ $a_n = 2n + 1$

29. **MODELING WITH MATHEMATICS** Rectangular tables are placed together along their short edges, as shown in the diagram. Write a rule for the number of people that can be seated around n tables arranged in this manner. Then graph the sequence. *(See Example 3.)*

30. **MODELING WITH MATHEMATICS** An employee at a construction company earns \$33,000 for the first year of employment. Employees at the company receive raises of \$2400 each year. Write a rule for the salary of the employee each year. Then graph the sequence.

In Exercises 31–38, write the series using summation notation. *(See Example 4.)*

31. $7 + 10 + 13 + 16 + 19$

32. $5 + 11 + 17 + 23 + 29$

33. $4 + 7 + 12 + 19 + \cdots$

34. $-1 + 2 + 7 + 14 + \cdots$

35. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$

36. $\frac{1}{4} + \frac{2}{5} + \frac{3}{6} + \frac{4}{7} + \cdots$

37. $-3 + 4 - 5 + 6 - 7$

38. $-2 + 4 - 8 + 16 - 32$

In Exercises 39–50, find the sum. *(See Examples 5 and 6.)*

39. $\sum_{i=1}^{6} 2i$

40. $\sum_{i=1}^{5} 7i$

41. $\sum_{n=0}^{4} n^3$

42. $\sum_{k=1}^{4} 3k^2$

43. $\sum_{k=3}^{6} (5k - 2)$

44. $\sum_{n=1}^{5} (n^2 - 1)$

45. $\sum_{i=2}^{8} \frac{2}{i}$

46. $\sum_{k=4}^{6} \frac{k}{k+1}$

47. $\sum_{i=1}^{35} 1$

48. $\sum_{n=1}^{16} n$

49. $\sum_{i=10}^{25} i$

50. $\sum_{n=1}^{18} n^2$

ERROR ANALYSIS **In Exercises 51 and 52, describe and correct the error in finding the sum of the series.**

51.

$$\sum_{n=1}^{10} (3n - 5) = -2 + 1 + 4 + 7 + 10$$
$$= 20$$

52.

$$\sum_{i=2}^{4} i^2 = \frac{4(4+1)(2 \cdot 4 + 1)}{6}$$
$$= \frac{180}{6}$$
$$= 30$$

53. **PROBLEM SOLVING** You want to save \$500 for a school trip. You begin by saving a penny on the first day. You save an additional penny each day after that. For example, you will save two pennies on the second day, three pennies on the third day, and so on.

a. How much money will you have saved after 100 days?

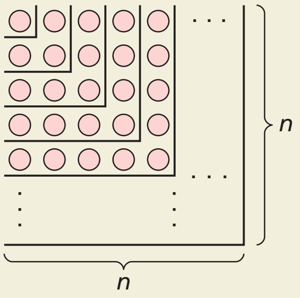
b. Use a series to determine how many days it takes you to save \$500.

54. **MODELING WITH MATHEMATICS** You begin an exercise program. The first week you do 25 push-ups. Each week you do 10 more push-ups than the previous week. How many push-ups will you do in the ninth week? Justify your answer.

55. **MODELING WITH MATHEMATICS** For a display at a sports store, you are stacking soccer balls in a pyramid whose base is an equilateral triangle with five layers. Write a rule for the number of soccer balls in each layer. Then graph the sequence.

56. HOW DO YOU SEE IT? Use the diagram to determine the sum of the series. Explain your reasoning.

$$1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) = ?$$



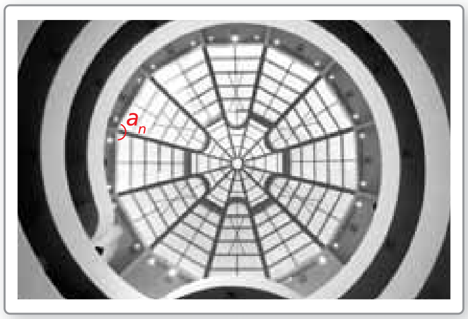
57. MAKING AN ARGUMENT You use a calculator to evaluate $\sum_{i=3}^{1659} i$ because the lower limit of summation is 3, not 1. Your friend claims there is a way to use the formula for the sum of the first n positive integers. Is your friend correct? Explain.

58. MATHEMATICAL CONNECTIONS A *regular* polygon has equal angle measures and equal side lengths. For a regular n-sided polygon ($n \geq 3$), the measure a_n of an interior angle is given by $a_n = \frac{180(n - 2)}{n}$.

a. Write the first five terms of the sequence.

b. Write a rule for the sequence giving the sum T_n of the measures of the interior angles in each regular n-sided polygon.

c. Use your rule in part (b) to find the sum of the interior angle measures in the Guggenheim Museum skylight, which is a regular dodecagon.



Guggenheim Museum Skylight

59. USING STRUCTURE Determine whether each statement is true. If so, provide a proof. If not, provide a counterexample.

a. $\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$

b. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

c. $\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i$

d. $\sum_{i=1}^{n} (a_i)^c = \left(\sum_{i=1}^{n} a_i \right)^c$

60. THOUGHT PROVOKING In this section, you learned the following formulas.

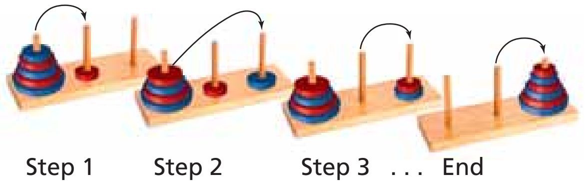
$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}$$

Write a formula for the sum of the cubes of the first n positive integers.

61. MODELING WITH MATHEMATICS In the puzzle called the Tower of Hanoi, the object is to use a series of moves to take the rings from one peg and stack them in order on another peg. A move consists of moving exactly one ring, and no ring may be placed on top of a smaller ring. The minimum number a_n of moves required to move n rings is 1 for 1 ring, 3 for 2 rings, 7 for 3 rings, 15 for 4 rings, and 31 for 5 rings.



a. Write a rule for the sequence.

b. What is the minimum number of moves required to move 6 rings? 7 rings? 8 rings?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the system. Check your solution. *(Section 1.4)*

62. $2x - y - 3z = 6$
$x + y + 4z = -1$
$3x - 2z = 8$

63. $2x - 2y + z = 5$
$-2x + 3y + 2z = -1$
$x - 4y + 5z = 4$

64. $2x - 3y + z = 4$
$x - 2z = 1$
$y + z = 2$