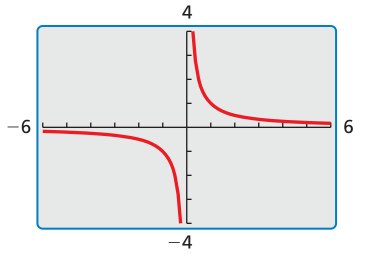
7.2 Graphing Rational Functions

Essential Question What are some of the characteristics of the graph of a rational function?

The parent function for rational functions with a linear numerator and a linear denominator is

$$f(x) = \frac{1}{x}. \qquad \text{Parent function}$$



The graph of this function, shown at the right, is a *hyperbola*.

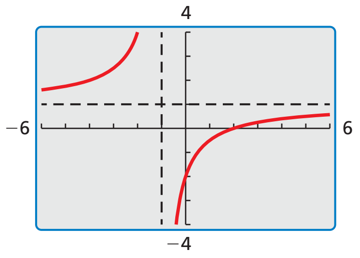
EXPLORATION 1 Identifying Graphs of Rational Functions

Work with a partner. Each function is a transformation of the graph of the parent function $f(x) = \frac{1}{x}$. Match the function with its graph. Explain your reasoning. Then describe the transformation.

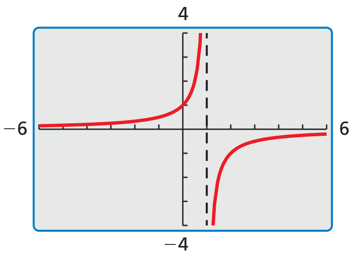
a. $g(x) = \dfrac{1}{x-1}$ **b.** $g(x) = \dfrac{-1}{x-1}$ **c.** $g(x) = \dfrac{x+1}{x-1}$

d. $g(x) = \dfrac{x-2}{x+1}$ **e.** $g(x) = \dfrac{x}{x+2}$ **f.** $g(x) = \dfrac{-x}{x+2}$

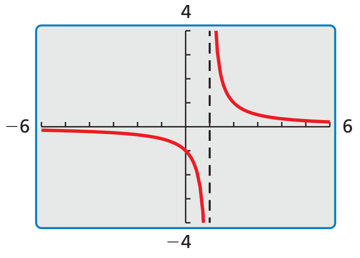
A.



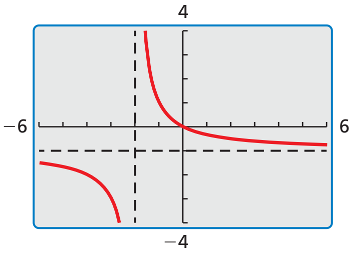
B.



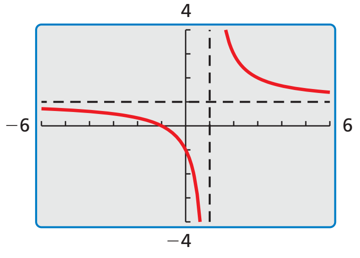
C.



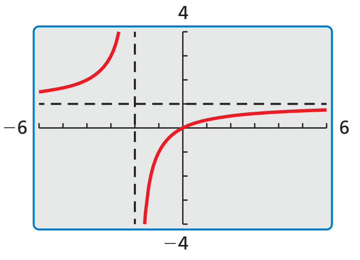
D.



E.



F.



Communicate Your Answer

2. What are some of the characteristics of the graph of a rational function?

3. Determine the intercepts, asymptotes, domain, and range of the rational function $g(x) = \dfrac{x-a}{x-b}$.

LOOKING FOR STRUCTURE

To be proficient in math, you need to look closely to discern a pattern or structure.

7.2 Lesson

Core Vocabulary

rational function, *p. 366*

Previous
domain
range
asymptote
long division

What You Will Learn

- Graph simple rational functions.
- Translate simple rational functions.
- Graph other rational functions.

Graphing Simple Rational Functions

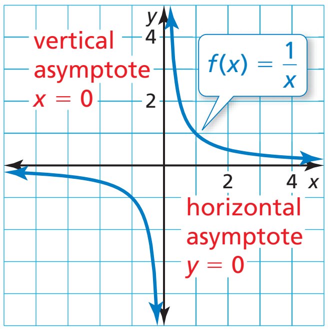
A **rational function** has the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. The inverse variation function $f(x) = \frac{a}{x}$ is a rational function. The graph of this function when $a = 1$ is shown below.

Core Concept

Parent Function for Simple Rational Functions

The graph of the parent function $f(x) = \frac{1}{x}$ is a *hyperbola*, which consists of two symmetrical parts called branches. The domain and range are all nonzero real numbers.

Any function of the form $g(x) = \frac{a}{x}$ $(a \neq 0)$ has the same asymptotes, domain, and range as the function $f(x) = \frac{1}{x}$.



STUDY TIP

Notice that $\frac{1}{x} \to 0$ as $x \to \infty$ and as $x \to -\infty$. This explains why $y = 0$ is a horizontal asymptote of the graph of $f(x) = \frac{1}{x}$. You can also analyze y-values as x approaches 0 to see why $x = 0$ is a vertical asymptote.

EXAMPLE 1 Graphing a Rational Function of the Form $y = \frac{a}{x}$

Graph $g(x) = \frac{4}{x}$. Compare the graph with the graph of $f(x) = \frac{1}{x}$.

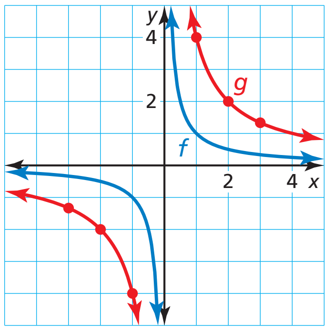
SOLUTION

Step 1 The function is of the form $g(x) = \frac{a}{x}$, so the asymptotes are $x = 0$ and $y = 0$. Draw the asymptotes.

Step 2 Make a table of values and plot the points. Include both positive and negative values of x.

x	-3	-2	-1	1	2	3
y	$-\frac{4}{3}$	-2	-4	4	2	$\frac{4}{3}$

Step 3 Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.



▶ The graph of g lies farther from the axes than the graph of f. Both graphs lie in the first and third quadrants and have the same asymptotes, domain, and range.

LOOKING FOR STRUCTURE

Because the function is of the form $g(x) = a \cdot f(x)$, where $a = 4$, the graph of g is a vertical stretch by a factor of 4 of the graph of f.

Monitoring Progress

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1. Graph $g(x) = \frac{-6}{x}$. Compare the graph with the graph of $f(x) = \frac{1}{x}$.

Translating Simple Rational Functions

Core Concept

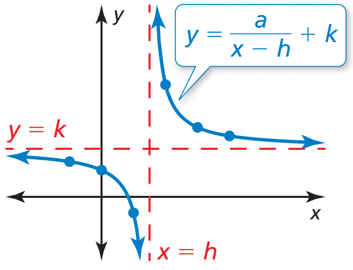
Graphing Translations of Simple Rational Functions

To graph a rational function of the form $y = \frac{a}{x-h} + k$, follow these steps:

Step 1 Draw the asymptotes $x = h$ and $y = k$.

Step 2 Plot points to the left and to the right of the vertical asymptote.

Step 3 Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.



EXAMPLE 2 Graphing a Translation of a Rational Function

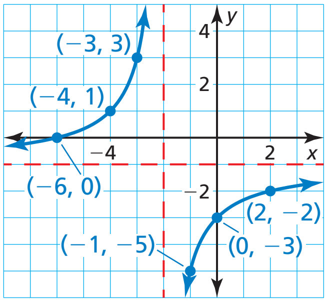
Graph $g(x) = \frac{-4}{x+2} - 1$. State the domain and range.

SOLUTION

Step 1 Draw the asymptotes $x = -2$ and $y = -1$.

Step 2 Plot points to the left of the vertical asymptote, such as $(-3, 3)$, $(-4, 1)$, and $(-6, 0)$. Plot points to the right of the vertical asymptote, such as $(-1, -5)$, $(0, -3)$, and $(2, -2)$.

Step 3 Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.



▶ The domain is all real numbers except −2 and the range is all real numbers except −1.

LOOKING FOR STRUCTURE

Let $f(x) = \frac{-4}{x}$. Notice that g is of the form $g(x) = f(x-h) + k$, where $h = -2$ and $k = -1$. So, the graph of g is a translation 2 units left and 1 unit down of the graph of f.

Monitoring Progress

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Graph the function. State the domain and range.

2. $y = \frac{3}{x} - 2$ **3.** $y = \frac{-1}{x+4}$ **4.** $y = \frac{1}{x-1} + 5$

Graphing Other Rational Functions

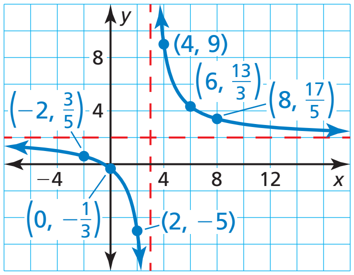
All rational functions of the form $y = \frac{ax+b}{cx+d}$ also have graphs that are hyperbolas.

- The vertical asymptote of the graph is the line $x = -\frac{d}{c}$ because the function is undefined when the denominator $cx + d$ is zero.
- The horizontal asymptote is the line $y = \frac{a}{c}$.

EXAMPLE 3 Graphing a Rational Function of the Form $y = \frac{ax + b}{cx + d}$

Graph $f(x) = \frac{2x + 1}{x - 3}$. State the domain and range.

SOLUTION



Step 1 Draw the asymptotes. Solve $x - 3 = 0$ for x to find the vertical asymptote $x = 3$. The horizontal asymptote is the line $y = \frac{a}{c} = \frac{2}{1} = 2$.

Step 2 Plot points to the left of the vertical asymptote, such as $(2, -5)$, $\left(0, -\frac{1}{3}\right)$, and $\left(-2, \frac{3}{5}\right)$. Plot points to the right of the vertical asymptote, such as $(4, 9)$, $\left(6, \frac{13}{3}\right)$, and $\left(8, \frac{17}{5}\right)$.

Step 3 Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.

▶ The domain is all real numbers except 3 and the range is all real numbers except 2.

Rewriting a rational function may reveal properties of the function and its graph. For example, rewriting a rational function in the form $y = \frac{a}{x - h} + k$ reveals that it is a translation of $y = \frac{a}{x}$ with vertical asymptote $x = h$ and horizontal asymptote $y = k$.

EXAMPLE 4 Rewriting and Graphing a Rational Function

Rewrite $g(x) = \frac{3x + 5}{x + 1}$ in the form $g(x) = \frac{a}{x - h} + k$. Graph the function. Describe the graph of g as a transformation of the graph of $f(x) = \frac{a}{x}$.

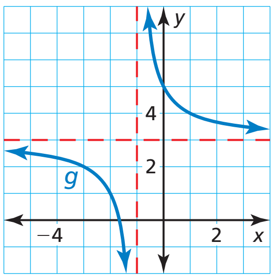
ANOTHER WAY

You will use a different method to rewrite g in Example 5 of Lesson 7.4.

SOLUTION

Rewrite the function by using long division:

$$\begin{array}{r} 3 \\ x + 1 \overline{)\, 3x + 5} \\ \underline{3x + 3} \\ 2 \end{array}$$



▶ The rewritten function is $g(x) = \frac{2}{x + 1} + 3$. The graph of g is a translation 1 unit left and 3 units up of the graph of $f(x) = \frac{2}{x}$.

Monitoring Progress

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Graph the function. State the domain and range.

5. $f(x) = \frac{x - 1}{x + 3}$

6. $f(x) = \frac{2x + 1}{4x - 2}$

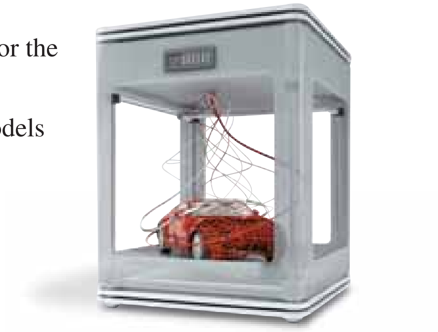
7. $f(x) = \frac{-3x + 2}{-x - 1}$

8. Rewrite $g(x) = \frac{2x + 3}{x + 1}$ in the form $g(x) = \frac{a}{x - h} + k$. Graph the function. Describe the graph of g as a transformation of the graph of $f(x) = \frac{a}{x}$.

EXAMPLE 5 Modeling with Mathematics

A 3-D printer builds up layers of materials to make three-dimensional models. Each deposited layer bonds to the layer below it. A company decides to make small display models of engine components using a 3-D printer. The printer costs \$1000. The material for each model costs \$50.

- Estimate how many models must be printed for the average cost per model to fall to \$90.
- What happens to the average cost as more models are printed?



SOLUTION

1. **Understand the Problem** You are given the cost of a printer and the cost to create a model using the printer. You are asked to find the number of models for which the average cost falls to \$90.

2. **Make a Plan** Write an equation that represents the average cost. Use a graphing calculator to estimate the number of models for which the average cost is about \$90. Then analyze the horizontal asymptote of the graph to determine what happens to the average cost as more models are printed.

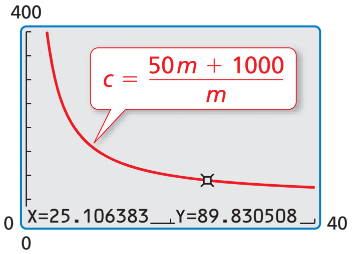
3. **Solve the Problem** Let c be the average cost (in dollars) and m be the number of models printed.

$$c = \frac{\text{(Unit cost)(Number printed)} + \text{(Cost of printer)}}{\text{Number printed}} = \frac{50m + 1000}{m}$$

USING A GRAPHING CALCULATOR

Because the number of models and average cost cannot be negative, choose a viewing window in the first quadrant.

Use a graphing calculator to graph the function.



▶ Using the *trace* feature, the average cost falls to \$90 per model after about 25 models are printed. Because the horizontal asymptote is $c = 50$, the average cost approaches \$50 as more models are printed.

4. **Look Back** Use a graphing calculator to create tables of values for large values of m. The tables show that the average cost approaches \$50 as more models are printed.

X	Y1
0	ERROR
50	70
100	60
150	56.667
200	55
250	54
300	53.333
X=0	

X	Y1
0	ERROR
10000	50.1
20000	50.05
30000	50.033
40000	50.025
50000	50.02
60000	50.017
X=0	

Monitoring Progress

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9. **WHAT IF?** How do the answers in Example 5 change when the cost of the 3-D printer is \$800?

7.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The function $y = \frac{7}{x+4} + 3$ has a(n) __________ of all real numbers except 3 and a(n) __________ of all real numbers except -4.

2. **WRITING** Is $f(x) = \frac{-3x+5}{2^x+1}$ a rational function? Explain your reasoning.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, graph the function. Compare the graph with the graph of $f(x) = \frac{1}{x}$. *(See Example 1.)*

3. $g(x) = \frac{3}{x}$

4. $g(x) = \frac{10}{x}$

5. $g(x) = \frac{-5}{x}$

6. $g(x) = \frac{-9}{x}$

7. $g(x) = \frac{15}{x}$

8. $g(x) = \frac{-12}{x}$

9. $g(x) = \frac{-0.5}{x}$

10. $g(x) = \frac{0.1}{x}$

In Exercises 11–18, graph the function. State the domain and range. *(See Example 2.)*

11. $g(x) = \frac{4}{x} + 3$

12. $y = \frac{2}{x} - 3$

13. $h(x) = \frac{6}{x-1}$

14. $y = \frac{1}{x+2}$

15. $h(x) = \frac{-3}{x+2}$

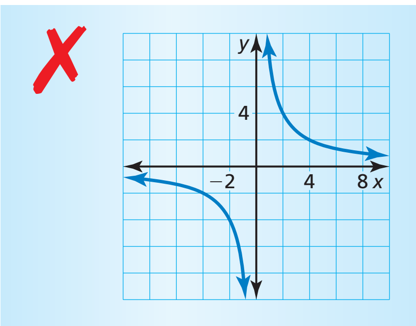
16. $f(x) = \frac{-2}{x-7}$

17. $g(x) = \frac{-3}{x-4} - 1$

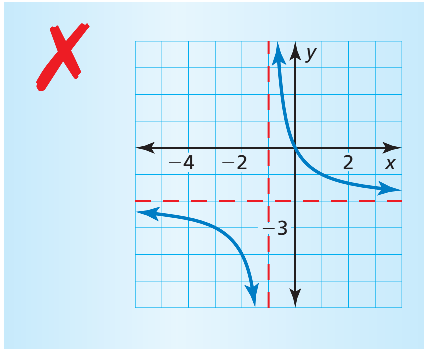
18. $y = \frac{10}{x+7} - 5$

ERROR ANALYSIS In Exercises 19 and 20, describe and correct the error in graphing the rational function.

19. $y = \frac{-8}{x}$



20. $y = \frac{2}{x-1} - 2$



ANALYZING RELATIONSHIPS In Exercises 21–24, match the function with its graph. Explain your reasoning.

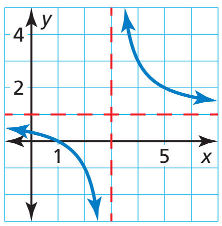
21. $g(x) = \frac{2}{x-3} + 1$

22. $h(x) = \frac{2}{x+3} + 1$

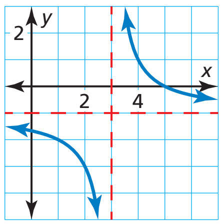
23. $f(x) = \frac{2}{x-3} - 1$

24. $y = \frac{2}{x+3} - 1$

A.



B.



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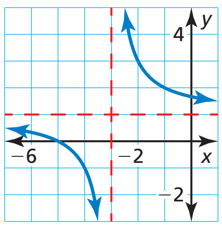
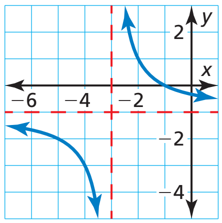


D.



In Exercises 25–32, graph the function. State the domain and range. *(See Example 3.)*

25. $f(x) = \frac{x + 4}{x - 3}$

26. $y = \frac{x - 1}{x + 5}$

27. $y = \frac{x + 6}{4x - 8}$

28. $h(x) = \frac{8x + 3}{2x - 6}$

29. $f(x) = \frac{-5x + 2}{4x + 5}$

30. $g(x) = \frac{6x - 1}{3x - 1}$

31. $h(x) = \frac{-5x}{-2x - 3}$

32. $y = \frac{-2x + 3}{-x + 10}$

In Exercises 33–40, rewrite the function in the form $g(x) = \frac{a}{x - h} + k$. Graph the function. Describe the graph of g as a transformation of the graph of $f(x) = \frac{a}{x}$. *(See Example 4.)*

33. $g(x) = \frac{5x + 6}{x + 1}$

34. $g(x) = \frac{7x + 4}{x - 3}$

35. $g(x) = \frac{2x - 4}{x - 5}$

36. $g(x) = \frac{4x - 11}{x - 2}$

37. $g(x) = \frac{x + 18}{x - 6}$

38. $g(x) = \frac{x + 2}{x - 8}$

39. $g(x) = \frac{7x - 20}{x + 13}$

40. $g(x) = \frac{9x - 3}{x + 7}$

41. PROBLEM SOLVING Your school purchases a math software program. The program has an initial cost of \$500 plus \$20 for each student that uses the program. *(See Example 5.)*

a. Estimate how many students must use the program for the average cost per student to fall to \$30.

b. What happens to the average cost as more students use the program?

42. PROBLEM SOLVING To join a rock climbing gym, you must pay an initial fee of \$100 and a monthly fee of \$59.

a. Estimate how many months you must purchase a membership for the average cost per month to fall to \$69.

b. What happens to the average cost as the number of months that you are a member increases?

43. USING STRUCTURE What is the vertical asymptote of the graph of the function $y = \frac{2}{x + 4} + 7$?

Ⓐ $x = -7$ Ⓑ $x = -4$

Ⓒ $x = 4$ Ⓓ $x = 7$

44. REASONING What are the x-intercept(s) of the graph of the function $y = \frac{x - 5}{x^2 - 1}$?

Ⓐ 1, −1 Ⓑ 5

Ⓒ 1 Ⓓ −5

45. USING TOOLS The time t (in seconds) it takes for sound to travel 1 kilometer can be modeled by

$$t = \frac{1000}{0.6T + 331}$$

where T is the air temperature (in degrees Celsius).

a. You are 1 kilometer from a lightning strike. You hear the thunder 2.9 seconds later. Use a graph to find the approximate air temperature.

b. Find the average rate of change in the time it takes sound to travel 1 kilometer as the air temperature increases from 0°C to 10°C.

46. MODELING WITH MATHEMATICS A business is studying the cost to remove a pollutant from the ground at its site. The function $y = \frac{15x}{1.1 - x}$ models the estimated cost y (in thousands of dollars) to remove x percent (expressed as a decimal) of the pollutant.

a. Graph the function. Describe a reasonable domain and range.

b. How much does it cost to remove 20% of the pollutant? 40% of the pollutant? 80% of the pollutant? Does doubling the percentage of the pollutant removed double the cost? Explain.

USING TOOLS In Exercises 47–50, use a graphing calculator to graph the function. Then determine whether the function is *even*, *odd*, or *neither*.

47. $h(x) = \frac{6}{x^2 + 1}$

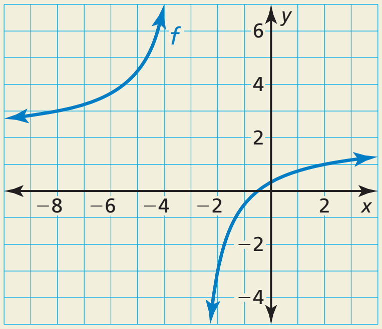
48. $f(x) = \frac{2x^2}{x^2 - 9}$

49. $y = \frac{x^3}{3x^2 + x^4}$

50. $f(x) = \frac{4x^2}{2x^3 - x}$

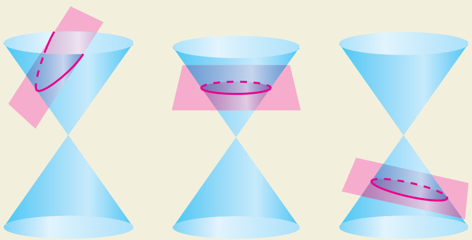
51. MAKING AN ARGUMENT Your friend claims it is possible for a rational function to have two vertical asymptotes. Is your friend correct? Justify your answer.

52. HOW DO YOU SEE IT? Use the graph of f to determine the equations of the asymptotes. Explain.



53. DRAWING CONCLUSIONS In what line(s) is the graph of $y = \frac{1}{x}$ symmetric? What does this symmetry tell you about the inverse of the function $f(x) = \frac{1}{x}$?

54. THOUGHT PROVOKING There are four basic types of conic sections: parabola, circle, ellipse, and hyperbola. Each of these can be represented by the intersection of a double-napped cone and a plane. The intersections for a parabola, circle, and ellipse are shown below. Sketch the intersection for a hyperbola.



Parabola Circle Ellipse

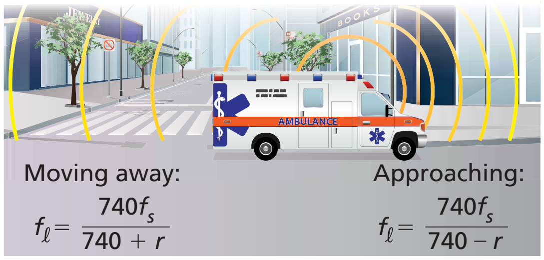
55. REASONING The graph of the rational function f is a hyperbola. The asymptotes of the graph of f intersect at (3, 2). The point (2, 1) is on the graph. Find another point on the graph. Explain your reasoning.

56. ABSTRACT REASONING Describe the intervals where the graph of $y = \frac{a}{x}$ is increasing or decreasing when (a) $a > 0$ and (b) $a < 0$. Explain your reasoning.

57. PROBLEM SOLVING An Internet service provider charges a \$50 installation fee and a monthly fee of \$43. The table shows the average monthly costs y of a competing provider for x months of service. Under what conditions would a person choose one provider over the other? Explain your reasoning.

Months, x	Average monthly cost (dollars), y
6	\$49.83
12	\$46.92
18	\$45.94
24	\$45.45

58. MODELING WITH MATHEMATICS The Doppler effect occurs when the source of a sound is moving relative to a listener, so that the frequency f_ℓ (in hertz) heard by the listener is different from the frequency f_s (in hertz) at the source. In both equations below, r is the speed (in miles per hour) of the sound source.



a. An ambulance siren has a frequency of 2000 hertz. Write two equations modeling the frequencies heard when the ambulance is approaching and when the ambulance is moving away.

b. Graph the equations in part (a) using the domain $0 \le r \le 60$.

c. For any speed r, how does the frequency heard for an approaching sound source compare with the frequency heard when the source moves away?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Factor the polynomial. *(Skills Review Handbook)*

59. $4x^2 - 4x - 80$ **60.** $3x^2 - 3x - 6$ **61.** $2x^2 - 2x - 12$ **62.** $10x^2 + 31x - 14$

Simplify the expression. *(Section 5.2)*

63. $3^2 \cdot 3^4$ **64.** $2^{1/2} \cdot 2^{3/5}$ **65.** $\frac{6^{5/6}}{6^{1/6}}$ **66.** $\frac{6^8}{6^{10}}$