### Essential Question
What are some of the characteristics of the graph of a rational function?

The parent function for rational functions with a linear numerator and a linear denominator is

\[ f(x) = \frac{1}{x}. \]  

Parent function

The graph of this function, shown at the right, is a hyperbola.

#### Identifying Graphs of Rational Functions

**Work with a partner.** Each function is a transformation of the graph of the parent function \( f(x) = \frac{1}{x} \). Match the function with its graph. Explain your reasoning. Then describe the transformation.

**a.** \( g(x) = \frac{1}{x} - 1 \)  
**b.** \( g(x) = \frac{-1}{x - 1} \)  
**c.** \( g(x) = \frac{x + 1}{x - 1} \)  
**d.** \( g(x) = \frac{x - 2}{x + 1} \)  
**e.** \( g(x) = \frac{x}{x + 2} \)  
**f.** \( g(x) = \frac{-x}{x + 2} \)

**A.**  
**B.**  
**C.**  
**D.**  
**E.**  
**F.**

### Communicate Your Answer

2. What are some of the characteristics of the graph of a rational function?

3. Determine the intercepts, asymptotes, domain, and range of the rational function \( g(x) = \frac{x - a}{x - b} \).
What You Will Learn

- Graph simple rational functions.
- Translate simple rational functions.
- Graph other rational functions.

Graphing Simple Rational Functions

A rational function has the form \( f(x) = \frac{p(x)}{q(x)} \), where \( p(x) \) and \( q(x) \) are polynomials and \( q(x) \neq 0 \). The inverse variation function \( f(x) = \frac{a}{x} \) is a rational function. The graph of this function when \( a = 1 \) is shown below.

Example 1: Graphing a Rational Function of the Form \( y = \frac{a}{x} \)

Graph \( g(x) = \frac{a}{x} \). Compare the graph with the graph of \( f(x) = \frac{1}{x} \).

Solution

Step 1: The function is of the form \( g(x) = \frac{a}{x} \), so the asymptotes are \( x = 0 \) and \( y = 0 \). Draw the asymptotes.

Step 2: Make a table of values and plot the points. Include both positive and negative values of \( x \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( -\frac{4}{3} )</td>
<td>-2</td>
<td>-4</td>
<td>4</td>
<td>2</td>
<td>( \frac{4}{3} )</td>
</tr>
</tbody>
</table>

Step 3: Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.

The graph of \( g \) lies farther from the axes than the graph of \( f \). Both graphs lie in the first and third quadrants and have the same asymptotes, domain, and range.

Monitoring Progress

1. Graph \( g(x) = \frac{-6}{x} \). Compare the graph with the graph of \( f(x) = \frac{1}{x} \).
Translating Simple Rational Functions

**Core Concept**

Graphing Translations of Simple Rational Functions

To graph a rational function of the form \( y = \frac{ax + b}{cx + d} \), follow these steps:

**Step 1** Draw the asymptotes \( x = h \) and \( y = k \).

**Step 2** Plot points to the left and to the right of the vertical asymptote.

**Step 3** Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.

**EXAMPLE 2** Graphing a Translation of a Rational Function

Graph \( g(x) = \frac{-4}{x + 2} - 1 \). State the domain and range.

**SOLUTION**

**Step 1** Draw the asymptotes \( x = -2 \) and \( y = -1 \).

**Step 2** Plot points to the left of the vertical asymptote, such as \((-3, 3), (-4, 1), (-6, 0)\). Plot points to the right of the vertical asymptote, such as \((-1, -5), (0, -3), (2, -2)\).

**Step 3** Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.

The domain is all real numbers except \(-2\) and the range is all real numbers except \(-1\).

**Monitoring Progress**

Graph the function. State the domain and range.

2. \( y = \frac{3}{x} - 2 \)  
3. \( y = \frac{-1}{x + 4} \)  
4. \( y = \frac{1}{x - 1} + 5 \)

**Graphing Other Rational Functions**

All rational functions of the form \( y = \frac{ax + b}{cx + d} \) also have graphs that are hyperbolas.

- The vertical asymptote of the graph is the line \( x = -\frac{d}{c} \) because the function is undefined when the denominator \( cx + d \) is zero.

- The horizontal asymptote is the line \( y = \frac{a}{c} \).
Graphing a Rational Function of the Form \( y = \frac{ax + b}{cx + d} \)

Graph \( f(x) = \frac{2x + 1}{x - 3} \). State the domain and range.

**SOLUTION**

**Step 1** Draw the asymptotes. Solve \( x - 3 = 0 \) for \( x \) to find the vertical asymptote \( x = 3 \). The horizontal asymptote is the line \( y = \frac{a}{c} = \frac{2}{1} = 2 \).

**Step 2** Plot points to the left of the vertical asymptote, such as \((2, -5)\), \((0, -\frac{1}{3})\), and \((-2, \frac{3}{2})\). Plot points to the right of the vertical asymptote, such as \((4, 9)\), \((6, \frac{13}{3})\), and \((8, \frac{17}{3})\).

**Step 3** Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.

\[ g(x) = \frac{3x + 5}{x + 1} \]

Rewriting a rational function may reveal properties of the function and its graph. For example, rewriting a rational function in the form \( y = \frac{a}{x - h} + k \) reveals that it is a translation of \( y = \frac{a}{x} \) with vertical asymptote \( x = h \) and horizontal asymptote \( y = k \).

**EXAMPLE 4** Rewriting and Graphing a Rational Function

Rewrite \( g(x) = \frac{3x + 5}{x + 1} \) in the form \( g(x) = \frac{a}{x - h} + k \). Graph the function. Describe the graph of \( g \) as a transformation of the graph of \( f(x) = \frac{a}{x} \).

**SOLUTION**

Rewrite the function by using long division:

\[
\frac{3x + 5}{x + 1} = \frac{3x + 3}{x + 1} + \frac{2}{x + 1}
\]

\[ x + 1 | 3x + 5 \]

\[ 3x + 3 \]

\[ 2 \]

\[ g(x) = \frac{2}{x + 1} + 3. \]

The rewritten function is \( g(x) = \frac{2}{x + 1} + 3. \)

The graph of \( g \) is a translation 1 unit left and 3 units up of the graph of \( f(x) = \frac{2}{x} \).

**Monitoring Progress**

Graph the function. State the domain and range.

5. \( f(x) = \frac{x - 1}{x + 3} \)
6. \( f(x) = \frac{2x + 1}{4x - 2} \)
7. \( f(x) = \frac{-3x + 2}{-x - 1} \)

8. Rewrite \( g(x) = \frac{2x + 3}{x + 1} \) in the form \( g(x) = \frac{a}{x - h} + k \). Graph the function.

Describe the graph of \( g \) as a transformation of the graph of \( f(x) = \frac{a}{x} \).
Modeling with Mathematics

A 3-D printer builds up layers of materials to make three-dimensional models. Each deposited layer bonds to the layer below it. A company decides to make small display models of engine components using a 3-D printer. The printer costs $1000. The material for each model costs $50.

- Estimate how many models must be printed for the average cost per model to fall to $90.
- What happens to the average cost as more models are printed?

SOLUTION

1. Understand the Problem You are given the cost of a printer and the cost to create a model using the printer. You are asked to find the number of models for which the average cost falls to $90.

2. Make a Plan Write an equation that represents the average cost. Use a graphing calculator to estimate the number of models for which the average cost is about $90. Then analyze the horizontal asymptote of the graph to determine what happens to the average cost as more models are printed.

3. Solve the Problem Let \( c \) be the average cost (in dollars) and \( m \) be the number of models printed.

\[
c = \frac{(\text{Unit cost})(\text{Number printed}) + (\text{Cost of printer})}{\text{Number printed}} = \frac{50m + 1000}{m}
\]

Use a graphing calculator to graph the function.

Using the trace feature, the average cost falls to $90 per model after about 25 models are printed. Because the horizontal asymptote is \( c = 50 \), the average cost approaches $50 as more models are printed.

4. Look Back Use a graphing calculator to create tables of values for large values of \( m \). The tables show that the average cost approaches $50 as more models are printed.

<table>
<thead>
<tr>
<th>( m )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>ERROR</td>
</tr>
<tr>
<td>50</td>
<td>70</td>
</tr>
<tr>
<td>100</td>
<td>60</td>
</tr>
<tr>
<td>150</td>
<td>56.667</td>
</tr>
<tr>
<td>200</td>
<td>55</td>
</tr>
<tr>
<td>250</td>
<td>54</td>
</tr>
<tr>
<td>300</td>
<td>53.333</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( m )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>ERROR</td>
</tr>
<tr>
<td>10000</td>
<td>50.1</td>
</tr>
<tr>
<td>20000</td>
<td>50.05</td>
</tr>
<tr>
<td>30000</td>
<td>50.033</td>
</tr>
<tr>
<td>40000</td>
<td>50.025</td>
</tr>
<tr>
<td>50000</td>
<td>50.02</td>
</tr>
<tr>
<td>60000</td>
<td>50.017</td>
</tr>
</tbody>
</table>

Monitoring Progress

9. **WHAT IF?** How do the answers in Example 5 change when the cost of the 3-D printer is $800?
Chapter 7  Rational Functions

7.2 Exercises

Vocabulary and Core Concept Check

1. COMPLETE THE SENTENCE  The function $y = \frac{7}{x+4} + 3$ has an(a) __________ of all real numbers except 3 and an(a) __________ of all real numbers except $-4$.

2. WRITING  Is $f(x) = -\frac{3x + 5}{2x + 1}$ a rational function? Explain your reasoning.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, graph the function. Compare the graph with the graph of $f(x) = \frac{1}{x}$ (See Example 1.)

3. $g(x) = \frac{3}{x}$
4. $g(x) = \frac{10}{x}$
5. $g(x) = -\frac{5}{x}$
6. $g(x) = -\frac{9}{x}$
7. $g(x) = \frac{15}{x}$
8. $g(x) = -\frac{12}{x}$
9. $g(x) = -\frac{0.5}{x}$
10. $g(x) = \frac{0.1}{x}$

In Exercises 11–18, graph the function. State the domain and range. (See Example 2.)

11. $g(x) = \frac{4}{x} + 3$
12. $y = \frac{2}{x} - 3$
13. $h(x) = \frac{6}{x - 1}$
14. $y = \frac{1}{x + 2}$
15. $h(x) = \frac{-3}{x + 2}$
16. $f(x) = -\frac{2}{x - 7}$
17. $g(x) = \frac{-3}{x - 4} - 1$
18. $y = \frac{10}{x + 7} - 5$

ERROR ANALYSIS  In Exercises 19 and 20, describe and correct the error in graphing the rational function.

19. $y = \frac{-8}{x}$

20. $y = \frac{2}{x - 1} - 2$

ANALYZING RELATIONSHIPS  In Exercises 21–24, match the function with its graph. Explain your reasoning.

21. $g(x) = \frac{2}{x - 3} + 1$  22. $h(x) = \frac{2}{x + 3} + 1$
23. $f(x) = \frac{2}{x - 3} - 1$  24. $y = \frac{2}{x + 3} - 1$

A.  B.  C.  D.
In Exercises 25–32, graph the function. State the domain and range. (See Example 3.)

25. \( f(x) = \frac{x + 4}{x - 3} \) 26. \( y = \frac{x - 1}{x + 5} \)

27. \( y = \frac{x + 6}{4x - 8} \) 28. \( h(x) = \frac{8x + 3}{2x - 6} \)

29. \( f(x) = -\frac{5x + 2}{4x + 5} \) 30. \( g(x) = \frac{6x - 1}{3x - 1} \)

31. \( h(x) = -\frac{5x}{-2x - 3} \) 32. \( y = -\frac{2x + 3}{-x + 10} \)

In Exercises 33–40, rewrite the function in the form \( g(x) = \frac{a}{x - h} + k. \) Graph the function. Describe the graph of \( g \) as a transformation of the graph of \( f(x) = \frac{a}{x}. \) (See Example 4.)

33. \( g(x) = \frac{5x + 6}{x + 1} \) 34. \( g(x) = \frac{7x + 4}{x - 3} \)

35. \( g(x) = \frac{2x - 4}{x - 5} \) 36. \( g(x) = \frac{4x - 11}{x - 2} \)

37. \( g(x) = \frac{x + 18}{x - 6} \) 38. \( g(x) = \frac{x + 2}{x - 8} \)

39. \( g(x) = \frac{7x - 20}{x + 13} \) 40. \( g(x) = \frac{9x - 3}{x + 7} \)

41. **PROBLEM SOLVING** Your school purchases a math software program. The program has an initial cost of $500 plus $20 for each student that uses the program. (See Example 5.)

   a. Estimate how many students must use the program for the average cost per student to fall to $30.

   b. What happens to the average cost as more students use the program?

42. **PROBLEM SOLVING** To join a rock climbing gym, you must pay an initial fee of $100 and a monthly fee of $59.

   a. Estimate how many months you must purchase a membership for the average cost per month to fall to $69.

   b. What happens to the average cost as the number of months that you are a member increases?

43. **USING STRUCTURE** What is the vertical asymptote of the graph of the function \( y = \frac{2}{x + 4} + 7? \)

   A) \( x = -7 \)  B) \( x = -4 \)  C) \( x = 4 \)  D) \( x = 7 \)

44. **REASONING** What are the \( x \)-intercept(s) of the graph of the function \( y = \frac{x - 5}{x^2 - 1}? \)

   A) \( 1, -1 \)  B) \( 5 \)  C) \( 1 \)  D) \( -5 \)

45. **USING TOOLS** The time \( t \) (in seconds) it takes for sound to travel 1 kilometer can be modeled by

\[
t = \frac{1000}{0.6T + 331}
\]

where \( T \) is the air temperature (in degrees Celsius).

   a. You are 1 kilometer from a lightning strike. You hear the thunder 2.9 seconds later. Use a graph to find the approximate air temperature.

   b. Find the average rate of change in the time it takes sound to travel 1 kilometer as the air temperature increases from 0°C to 10°C.

46. **MODELING WITH MATHEMATICS** A business is studying the cost to remove a pollutant from the ground at its site. The function \( y = \frac{-15x + 1.1}{x} \) models the estimated cost \( y \) (in thousands of dollars) to remove \( x \) percent (expressed as a decimal) of the pollutant.

   a. Graph the function. Describe a reasonable domain and range.

   b. How much does it cost to remove 20% of the pollutant? 40% of the pollutant? 80% of the pollutant? Does doubling the percentage of the pollutant removed double the cost? Explain.

**USING TOOLS** In Exercises 47–50, use a graphing calculator to graph the function. Then determine whether the function is **even**, **odd**, or **neither**.

47. \( h(x) = \frac{6}{x^2 + 1} \) 48. \( f(x) = -\frac{2x^2}{x^2 - 9} \)

49. \( y = \frac{x^3}{3x^2 + x^4} \) 50. \( f(x) = \frac{4x^2}{2x^3 - x} \)

Section 7.2 Graphing Rational Functions 371
51. **MAKING AN ARGUMENT** Your friend claims it is possible for a rational function to have two vertical asymptotes. Is your friend correct? Justify your answer.

52. **HOW DO YOU SEE IT?** Use the graph of $f$ to determine the equations of the asymptotes. Explain.

53. **DRAWING CONCLUSIONS** In what line(s) is the graph of $y = \frac{1}{x}$ symmetric? What does this symmetry tell you about the inverse of the function $f(x) = \frac{1}{x}$?

54. **THOUGHT PROVOKING** There are four basic types of conic sections: parabola, circle, ellipse, and hyperbola. Each of these can be represented by the intersection of a double-napped cone and a plane. The intersections for a parabola, circle, and ellipse are shown below. Sketch the intersection for a hyperbola.

55. **REASONING** The graph of the rational function $f$ is a hyperbola. The asymptotes of the graph of $f$ intersect at $(3, 2)$. The point $(2, 1)$ is on the graph. Find another point on the graph. Explain your reasoning.

56. **ABSTRACT REASONING** Describe the intervals where the graph of $y = \frac{a}{x}$ is increasing or decreasing when (a) $a > 0$ and (b) $a < 0$. Explain your reasoning.

57. **PROBLEM SOLVING** An Internet service provider charges a $50 installation fee and a monthly fee of $43. The table shows the average monthly costs $y$ of a competing provider for $x$ months of service. Under what conditions would a person choose one provider over the other? Explain your reasoning.

<table>
<thead>
<tr>
<th>Months, $x$</th>
<th>Average monthly cost (dollars), $y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$49.83</td>
</tr>
<tr>
<td>12</td>
<td>$46.92</td>
</tr>
<tr>
<td>18</td>
<td>$45.94</td>
</tr>
<tr>
<td>24</td>
<td>$45.45</td>
</tr>
</tbody>
</table>

58. **MODELING WITH MATHEMATICS** The Doppler effect occurs when the source of a sound is moving relative to a listener, so that the frequency $f_s$ (in hertz) heard by the listener is different from the frequency $f_s$ (in hertz) at the source. In both equations below, $r$ is the speed (in miles per hour) of the sound source.

Moving away: $f_l = \frac{740f_s}{740 + r}$
Approaching: $f_l = \frac{740f_s}{740 - r}$

- a. An ambulance siren has a frequency of 2000 hertz. Write two equations modeling the frequencies heard when the ambulance is approaching and when the ambulance is moving away.
- b. Graph the equations in part (a) using the domain $0 \leq r \leq 60$.
- c. For any speed $r$, how does the frequency heard for an approaching sound source compare with the frequency heard when the source moves away?

---

**Maintaining Mathematical Proficiency**

Reviewing what you learned in previous grades and lessons

- **Factor the polynomial.** *(Skills Review Handbook)*
  - 59. $4x^2 - 4x - 80$
  - 60. $3x^2 - 3x - 6$
  - 61. $2x^2 - 2x - 12$
  - 62. $10x^2 + 31x - 14$

- **Simplify the expression.** *(Section 5.2)*
  - 63. $3^2 \cdot 3^4$
  - 64. $2^{1/2} \cdot 2^{3/5}$
  - 65. $\frac{6^{5/6}}{6^{1/6}}$
  - 66. $\frac{6^8}{6^{10}}$