6.5 Properties of Logarithms

Essential Question

How can you use properties of exponents to derive properties of logarithms?

Let

$x = \log_b m$ and $y = \log_b n$.

The corresponding exponential forms of these two equations are

$b^x = m$ and $b^y = n$.

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results.

EXPLORATION 1 Product Property of Logarithms

Work with a partner. To derive the Product Property, multiply m and n to obtain

$$mn = b^x b^y = b^{x+y}.$$

The corresponding logarithmic form of $mn = b^{x+y}$ is $\log_b mn = x + y$. So,

$\log_b mn =$ ______. Product Property of Logarithms

EXPLORATION 2 Quotient Property of Logarithms

Work with a partner. To derive the Quotient Property, divide m by n to obtain

$$\frac{m}{n} = \frac{b^x}{b^y} = b^{x-y}.$$

The corresponding logarithmic form of $\frac{m}{n} = b^{x-y}$ is $\log_b \frac{m}{n} = x - y$. So,

$\log_b \frac{m}{n} =$ ______. Quotient Property of Logarithms

EXPLORATION 3 Power Property of Logarithms

Work with a partner. To derive the Power Property, substitute b^x for m in the expression $\log_b m^n$, as follows.

$$\log_b m^n = \log_b (b^x)^n \quad \text{Substitute } b^x \text{ for } m.$$
$$= \log_b b^{nx} \quad \text{Power of a Power Property of Exponents}$$
$$= nx \quad \text{Inverse Property of Logarithms}$$

So, substituting $\log_b m$ for x, you have

$\log_b m^n =$ ______. Power Property of Logarithms

Communicate Your Answer

4. How can you use properties of exponents to derive properties of logarithms?

5. Use the properties of logarithms that you derived in Explorations 1–3 to evaluate each logarithmic expression.

a. $\log_4 16^3$ **b.** $\log_3 81^{-3}$

c. $\ln e^2 + \ln e^5$ **d.** $2 \ln e^6 - \ln e^5$

e. $\log_5 75 - \log_5 3$ **f.** $\log_4 2 + \log_4 32$

6.5 Lesson

Core Vocabulary

Previous
base
properties of exponents

What You Will Learn

- Use the properties of logarithms to evaluate logarithms.
- Use the properties of logarithms to expand or condense logarithmic expressions.
- Use the change-of-base formula to evaluate logarithms.

Properties of Logarithms

You know that the logarithmic function with base b is the inverse function of the exponential function with base b. Because of this relationship, it makes sense that logarithms have properties similar to properties of exponents.

STUDY TIP

These three properties of logarithms correspond to these three properties of exponents.

$a^m a^n = a^{m+n}$

$\dfrac{a^m}{a^n} = a^{m-n}$

$(a^m)^n = a^{mn}$

Core Concept

Properties of Logarithms

Let b, m, and n be positive real numbers with $b \neq 1$.

Product Property $\log_b mn = \log_b m + \log_b n$

Quotient Property $\log_b \dfrac{m}{n} = \log_b m - \log_b n$

Power Property $\log_b m^n = n \log_b m$

EXAMPLE 1 Using Properties of Logarithms

Use $\log_2 3 \approx 1.585$ and $\log_2 7 \approx 2.807$ to evaluate each logarithm.

a. $\log_2 \frac{3}{7}$ **b.** $\log_2 21$ **c.** $\log_2 49$

COMMON ERROR

Note that in general $\log_b \dfrac{m}{n} \neq \dfrac{\log_b m}{\log_b n}$ and $\log_b mn \neq (\log_b m)(\log_b n)$.

SOLUTION

a. $\log_2 \frac{3}{7} = \log_2 3 - \log_2 7$ Quotient Property

$\approx 1.585 - 2.807$ Use the given values of $\log_2 3$ and $\log_2 7$.

$= -1.222$ Subtract.

b. $\log_2 21 = \log_2(3 \cdot 7)$ Write 21 as $3 \cdot 7$.

$= \log_2 3 + \log_2 7$ Product Property

$\approx 1.585 + 2.807$ Use the given values of $\log_2 3$ and $\log_2 7$.

$= 4.392$ Add.

c. $\log_2 49 = \log_2 7^2$ Write 49 as 7^2.

$= 2 \log_2 7$ Power Property

$\approx 2(2.807)$ Use the given value $\log_2 7$.

$= 5.614$ Multiply.

Monitoring Progress

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Use $\log_6 5 \approx 0.898$ and $\log_6 8 \approx 1.161$ to evaluate the logarithm.

1. $\log_6 \frac{5}{8}$ **2.** $\log_6 40$ **3.** $\log_6 64$ **4.** $\log_6 125$

Rewriting Logarithmic Expressions

You can use the properties of logarithms to expand and condense logarithmic expressions.

EXAMPLE 2 Expanding a Logarithmic Expression

Expand $\ln \frac{5x^7}{y}$.

STUDY TIP

When you are expanding or condensing an expression involving logarithms, you can assume that any variables are positive.

SOLUTION

$$\ln \frac{5x^7}{y} = \ln 5x^7 - \ln y \qquad \text{Quotient Property}$$

$$= \ln 5 + \ln x^7 - \ln y \qquad \text{Product Property}$$

$$= \ln 5 + 7 \ln x - \ln y \qquad \text{Power Property}$$

EXAMPLE 3 Condensing a Logarithmic Expression

Condense $\log 9 + 3 \log 2 - \log 3$.

SOLUTION

$$\log 9 + 3 \log 2 - \log 3 = \log 9 + \log 2^3 - \log 3 \qquad \text{Power Property}$$

$$= \log(9 \cdot 2^3) - \log 3 \qquad \text{Product Property}$$

$$= \log \frac{9 \cdot 2^3}{3} \qquad \text{Quotient Property}$$

$$= \log 24 \qquad \text{Simplify.}$$

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Expand the logarithmic expression.

5. $\log_6 3x^4$

6. $\ln \frac{5}{12x}$

Condense the logarithmic expression.

7. $\log x - \log 9$

8. $\ln 4 + 3 \ln 3 - \ln 12$

Change-of-Base Formula

Logarithms with any base other than 10 or e can be written in terms of common or natural logarithms using the *change-of-base formula*. This allows you to evaluate any logarithm using a calculator.

Core Concept

Change-of-Base Formula

If a, b, and c are positive real numbers with $b \neq 1$ and $c \neq 1$, then

$$\log_c a = \frac{\log_b a}{\log_b c}.$$

In particular, $\log_c a = \frac{\log a}{\log c}$ and $\log_c a = \frac{\ln a}{\ln c}$.

EXAMPLE 4 Changing a Base Using Common Logarithms

Evaluate $\log_3 8$ using common logarithms.

SOLUTION

$$\log_3 8 = \frac{\log 8}{\log 3} \qquad \log_c a = \frac{\log a}{\log c}$$

$$\approx \frac{0.9031}{0.4771} \approx 1.893 \qquad \text{Use a calculator. Then divide.}$$

ANOTHER WAY

In Example 4, $\log_3 8$ can be evaluated using natural logarithms.

$$\log_3 8 = \frac{\ln 8}{\ln 3} \approx 1.893$$

Notice that you get the same answer whether you use natural logarithms or common logarithms in the change-of-base formula.

EXAMPLE 5 Changing a Base Using Natural Logarithms

Evaluate $\log_6 24$ using natural logarithms.

SOLUTION

$$\log_6 24 = \frac{\ln 24}{\ln 6} \qquad \log_c a = \frac{\ln a}{\ln c}$$

$$\approx \frac{3.1781}{1.7918} \approx 1.774 \qquad \text{Use a calculator. Then divide.}$$

EXAMPLE 6 Solving a Real-Life Problem

For a sound with intensity I (in watts per square meter), the loudness $L(I)$ of the sound (in decibels) is given by the function

$$L(I) = 10 \log \frac{I}{I_0}$$

where I_0 is the intensity of a barely audible sound (about 10^{-12} watts per square meter). An artist in a recording studio turns up the volume of a track so that the intensity of the sound doubles. By how many decibels does the loudness increase?

SOLUTION

Let I be the original intensity, so that $2I$ is the doubled intensity.

$$\begin{aligned}
\text{increase in loudness} &= L(2I) - L(I) && \text{Write an expression.}\\
&= 10 \log \frac{2I}{I_0} - 10 \log \frac{I}{I_0} && \text{Substitute.}\\
&= 10\left(\log \frac{2I}{I_0} - \log \frac{I}{I_0}\right) && \text{Distributive Property}\\
&= 10\left(\log 2 + \log \frac{I}{I_0} - \log \frac{I}{I_0}\right) && \text{Product Property}\\
&= 10 \log 2 && \text{Simplify.}
\end{aligned}$$

▶ The loudness increases by 10 log 2 decibels, or about 3 decibels.

Monitoring Progress

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Use the change-of-base formula to evaluate the logarithm.

9. $\log_5 8$ **10.** $\log_8 14$ **11.** $\log_{26} 9$ **12.** $\log_{12} 30$

13. **WHAT IF?** In Example 6, the artist turns up the volume so that the intensity of the sound triples. By how many decibels does the loudness increase?

6.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** To condense the expression $\log_3 2x + \log_3 y$, you need to use the __________ Property of Logarithms.

2. **WRITING** Describe two ways to evaluate $\log_7 12$ using a calculator.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, use $\log_7 4 \approx 0.712$ and $\log_7 12 \approx 1.277$ to evaluate the logarithm. *(See Example 1.)*

3. $\log_7 3$
4. $\log_7 48$
5. $\log_7 16$
6. $\log_7 64$
7. $\log_7 \frac{1}{4}$
8. $\log_7 \frac{1}{3}$

In Exercises 9–12, match the expression with the logarithm that has the same value. Justify your answer.

9. $\log_3 6 - \log_3 2$ | **A.** $\log_3 64$
10. $2 \log_3 6$ | **B.** $\log_3 3$
11. $6 \log_3 2$ | **C.** $\log_3 12$
12. $\log_3 6 + \log_3 2$ | **D.** $\log_3 36$

In Exercises 13–20, expand the logarithmic expression. *(See Example 2.)*

13. $\log_3 4x$
14. $\log_8 3x$
15. $\log 10x^5$
16. $\ln 3x^4$
17. $\ln \frac{x}{3y}$
18. $\ln \frac{6x^2}{y^4}$
19. $\log_7 5\sqrt{x}$
20. $\log_5 \sqrt[3]{x^2 y}$

ERROR ANALYSIS In Exercises 21 and 22, describe and correct the error in expanding the logarithmic expression.

21. ✗ $\log_2 5x = (\log_2 5)(\log_2 x)$

22. ✗ $\ln 8x^3 = 3 \ln 8 + \ln x$

In Exercises 23–30, condense the logarithmic expression. *(See Example 3.)*

23. $\log_4 7 - \log_4 10$
24. $\ln 12 - \ln 4$
25. $6 \ln x + 4 \ln y$
26. $2 \log x + \log 11$
27. $\log_5 4 + \frac{1}{3} \log_5 x$
28. $6 \ln 2 - 4 \ln y$
29. $5 \ln 2 + 7 \ln x + 4 \ln y$
30. $\log_3 4 + 2 \log_3 \frac{1}{2} + \log_3 x$

31. **REASONING** Which of the following is *not* equivalent to $\log_5 \frac{y^4}{3x}$? Justify your answer.

 Ⓐ $4 \log_5 y - \log_5 3x$

 Ⓑ $4 \log_5 y - \log_5 3 + \log_5 x$

 Ⓒ $4 \log_5 y - \log_5 3 - \log_5 x$

 Ⓓ $\log_5 y^4 - \log_5 3 - \log_5 x$

32. **REASONING** Which of the following equations is correct? Justify your answer.

 Ⓐ $\log_7 x + 2 \log_7 y = \log_7(x + y^2)$

 Ⓑ $9 \log x - 2 \log y = \log \frac{x^9}{y^2}$

 Ⓒ $5 \log_4 x + 7 \log_2 y = \log_6 x^5 y^7$

 Ⓓ $\log_9 x - 5 \log_9 y = \log_9 \frac{x}{5y}$

In Exercises 33–40, use the change-of-base formula to evaluate the logarithm. *(See Examples 4 and 5.)*

33. $\log_4 7$

34. $\log_5 13$

35. $\log_9 15$

36. $\log_8 22$

37. $\log_6 17$

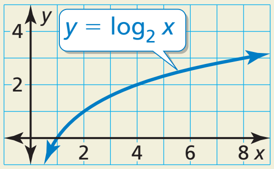
38. $\log_2 28$

39. $\log_7 \frac{3}{16}$

40. $\log_3 \frac{9}{40}$

41. **MAKING AN ARGUMENT** Your friend claims you can use the change-of-base formula to graph $y = \log_3 x$ using a graphing calculator. Is your friend correct? Explain your reasoning.

42. **HOW DO YOU SEE IT?** Use the graph to determine the value of $\frac{\log 8}{\log 2}$.

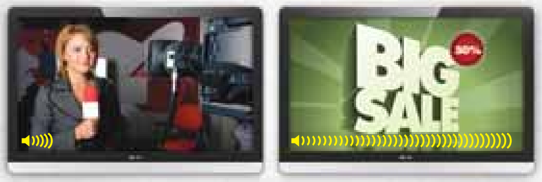


MODELING WITH MATHEMATICS **In Exercises 43 and 44, use the function $L(I)$ given in Example 6.**

43. The blue whale can produce sound with an intensity that is 1 million times greater than the intensity of the loudest sound a human can make. Find the difference in the decibel levels of the sounds made by a blue whale and a human. *(See Example 6.)*

44. The intensity of the sound of a certain television advertisement is 10 times greater than the intensity of the television program. By how many decibels does the loudness increase?

Intensity of Television Sound



During show:
Intensity $= I$

During ad:
Intensity $= 10I$

45. **REWRITING A FORMULA** Under certain conditions, the wind speed s (in knots) at an altitude of h meters above a grassy plain can be modeled by the function

$$s(h) = 2 \ln 100h.$$

a. By what amount does the wind speed increase when the altitude doubles?

b. Show that the given function can be written in terms of common logarithms as

$$s(h) = \frac{2}{\log e}(\log h + 2).$$

46. **THOUGHT PROVOKING** Determine whether the formula

$$\log_b(M + N) = \log_b M + \log_b N$$

is true for all positive, real values of M, N, and b (with $b \neq 1$). Justify your answer.

47. **USING STRUCTURE** Use the properties of exponents to prove the change-of-base formula. (*Hint:* Let $x = \log_b a$, $y = \log_b c$, and $z = \log_c a$.)

48. **CRITICAL THINKING** Describe *three* ways to transform the graph of $f(x) = \log x$ to obtain the graph of $g(x) = \log 100x - 1$. Justify your answers.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the inequality by graphing. *(Section 3.6)*

49. $x^2 - 4 > 0$

50. $2(x - 6)^2 - 5 \geq 37$

51. $x^2 + 13x + 42 < 0$

52. $-x^2 - 4x + 6 \leq -6$

Solve the equation by graphing the related system of equations. *(Section 3.5)*

53. $4x^2 - 3x - 6 = -x^2 + 5x + 3$

54. $-(x + 3)(x - 2) = x^2 - 6x$

55. $2x^2 - 4x - 5 = -(x + 3)^2 + 10$

56. $-(x + 7)^2 + 5 = (x + 10)^2 - 3$