# 6.3 Logarithms and Logarithmic Functions

## Essential Question

What are some of the characteristics of the graph of a logarithmic function?

Every exponential function of the form $f(x) = b^x$, where $b$ is a positive real number other than 1, has an inverse function that you can denote by $g(x) = \log_b x$. This inverse function is called a *logarithmic function with base b*.

### EXPLORATION 1 Rewriting Exponential Equations

**Work with a partner.** Find the value of $x$ in each exponential equation. Explain your reasoning. Then use the value of $x$ to rewrite the exponential equation in its equivalent logarithmic form, $x = \log_b y$.

**a.** $2^x = 8$ **b.** $3^x = 9$ **c.** $4^x = 2$

**d.** $5^x = 1$ **e.** $5^x = \frac{1}{5}$ **f.** $8^x = 4$

### EXPLORATION 2 Graphing Exponential and Logarithmic Functions

**Work with a partner.** Complete each table for the given exponential function. Use the results to complete the table for the given logarithmic function. Explain your reasoning. Then sketch the graphs of $f$ and $g$ in the same coordinate plane.

**a.**

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x) = 2^x$ | | | | | |

| $x$ | | | | | |
|---|---|---|---|---|---|
| $g(x) = \log_2 x$ | $-2$ | $-1$ | 0 | 1 | 2 |

**b.**

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x) = 10^x$ | | | | | |

| $x$ | | | | | |
|---|---|---|---|---|---|
| $g(x) = \log_{10} x$ | $-2$ | $-1$ | 0 | 1 | 2 |

**CONSTRUCTING VIABLE ARGUMENTS**

To be proficient in math, you need to justify your conclusions and communicate them to others.

### EXPLORATION 3 Characteristics of Graphs of Logarithmic Functions

**Work with a partner.** Use the graphs you sketched in Exploration 2 to determine the domain, range, $x$-intercept, and asymptote of the graph of $g(x) = \log_b x$, where $b$ is a positive real number other than 1. Explain your reasoning.

## Communicate Your Answer

4. What are some of the characteristics of the graph of a logarithmic function?
5. How can you use the graph of an exponential function to obtain the graph of a logarithmic function?

# 6.3 Lesson

## Core Vocabulary

logarithm of $y$ with base $b$, *p. 310*
common logarithm, *p. 311*
natural logarithm, *p. 311*

***Previous***
inverse functions

## What You Will Learn

- Define and evaluate logarithms.
- Use inverse properties of logarithmic and exponential functions.
- Graph logarithmic functions.

## Logarithms

You know that $2^2 = 4$ and $2^3 = 8$. However, for what value of $x$ does $2^x = 6$? Mathematicians define this $x$-value using a *logarithm* and write $x = \log_2 6$. The definition of a logarithm can be generalized as follows.

## Core Concept

### Definition of Logarithm with Base *b*

Let $b$ and $y$ be positive real numbers with $b \neq 1$. The **logarithm of *y* with base *b*** is denoted by $\log_b y$ and is defined as

$$\log_b y = x \qquad \text{if and only if} \qquad b^x = y.$$

The expression $\log_b y$ is read as "log base $b$ of $y$."

This definition tells you that the equations $\log_b y = x$ and $b^x = y$ are equivalent. The first is in *logarithmic form*, and the second is in *exponential form*.

### EXAMPLE 1 Rewriting Logarithmic Equations

Rewrite each equation in exponential form.

**a.** $\log_2 16 = 4$ **b.** $\log_4 1 = 0$ **c.** $\log_{12} 12 = 1$ **d.** $\log_{1/4} 4 = -1$

**SOLUTION**

| Logarithmic Form | Exponential Form |
|---|---|
| **a.** $\log_2 16 = 4$ | $2^4 = 16$ |
| **b.** $\log_4 1 = 0$ | $4^0 = 1$ |
| **c.** $\log_{12} 12 = 1$ | $12^1 = 12$ |
| **d.** $\log_{1/4} 4 = -1$ | $\left(\frac{1}{4}\right)^{-1} = 4$ |

### EXAMPLE 2 Rewriting Exponential Equations

Rewrite each equation in logarithmic form.

**a.** $5^2 = 25$ **b.** $10^{-1} = 0.1$ **c.** $8^{2/3} = 4$ **d.** $6^{-3} = \frac{1}{216}$

**SOLUTION**

| Exponential Form | Logarithmic Form |
|---|---|
| **a.** $5^2 = 25$ | $\log_5 25 = 2$ |
| **b.** $10^{-1} = 0.1$ | $\log_{10} 0.1 = -1$ |
| **c.** $8^{2/3} = 4$ | $\log_8 4 = \frac{2}{3}$ |
| **d.** $6^{-3} = \frac{1}{216}$ | $\log_6 \frac{1}{216} = -3$ |

Parts (b) and (c) of Example 1 illustrate two special logarithm values that you should learn to recognize. Let $b$ be a positive real number such that $b \neq 1$.

**Logarithm of 1**

$\log_b 1 = 0$ because $b^0 = 1$.

**Logarithm of *b* with Base *b***

$\log_b b = 1$ because $b^1 = b$.

## EXAMPLE 3 Evaluating Logarithmic Expressions

Evaluate each logarithm.

**a.** $\log_4 64$ **b.** $\log_5 0.2$ **c.** $\log_{1/5} 125$ **d.** $\log_{36} 6$

### SOLUTION

To help you find the value of $\log_b y$, ask yourself what power of $b$ gives you $y$.

**a.** What power of 4 gives you 64? $4^3 = 64$, so $\log_4 64 = 3$.

**b.** What power of 5 gives you 0.2? $5^{-1} = 0.2$, so $\log_5 0.2 = -1$.

**c.** What power of $\frac{1}{5}$ gives you 125? $\left(\frac{1}{5}\right)^{-3} = 125$, so $\log_{1/5} 125 = -3$.

**d.** What power of 36 gives you 6? $36^{1/2} = 6$, so $\log_{36} 6 = \frac{1}{2}$.

A **common logarithm** is a logarithm with base 10. It is denoted by $\log_{10}$ or simply by log. A **natural logarithm** is a logarithm with base $e$. It can be denoted by $\log_e$ but is usually denoted by ln.

**Common Logarithm**

$\log_{10} x = \log x$

**Natural Logarithm**

$\log_e x = \ln x$

## EXAMPLE 4 Evaluating Common and Natural Logarithms

Evaluate (a) log 8 and (b) ln 0.3 using a calculator. Round your answer to three decimal places.

### SOLUTION

Most calculators have keys for evaluating common and natural logarithms.

```
log(8)
     .903089987
ln(0.3)
    -1.203972804
```

**a.** $\log 8 \approx 0.903$

**b.** $\ln 0.3 \approx -1.204$

Check your answers by rewriting each logarithm in exponential form and evaluating.

**Check**

```
10^(0.903)
      7.99834255
e^(-1.204)
     .2999918414
```

## Monitoring Progress

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**Rewrite the equation in exponential form.**

**1.** $\log_3 81 = 4$ **2.** $\log_7 7 = 1$ **3.** $\log_{14} 1 = 0$ **4.** $\log_{1/2} 32 = -5$

**Rewrite the equation in logarithmic form.**

**5.** $7^2 = 49$ **6.** $50^0 = 1$ **7.** $4^{-1} = \frac{1}{4}$ **8.** $256^{1/8} = 2$

**Evaluate the logarithm. If necessary, use a calculator and round your answer to three decimal places.**

**9.** $\log_2 32$ **10.** $\log_{27} 3$ **11.** $\log 12$ **12.** $\ln 0.75$

## Using Inverse Properties

By the definition of a logarithm, it follows that the logarithmic function $g(x) = \log_b x$ is the inverse of the exponential function $f(x) = b^x$. This means that

$$g(f(x)) = \log_b b^x = x \qquad \text{and} \qquad f(g(x)) = b^{\log_b x} = x.$$

In other words, exponential functions and logarithmic functions "undo" each other.

### EXAMPLE 5 Using Inverse Properties

Simplify (a) $10^{\log 4}$ and (b) $\log_5 25^x$.

**SOLUTION**

**a.** $10^{\log 4} = 4$     $b^{\log_b x} = x$

**b.** $\log_5 25^x = \log_5 (5^2)^x$     Express 25 as a power with base 5.

$= \log_5 5^{2x}$     Power of a Power Property

$= 2x$     $\log_b b^x = x$

### EXAMPLE 6 Finding Inverse Functions

Find the inverse of each function.

**a.** $f(x) = 6^x$      **b.** $y = \ln(x + 3)$

**SOLUTION**

**a.** From the definition of logarithm, the inverse of $f(x) = 6^x$ is $g(x) = \log_6 x$.

**b.**

$y = \ln(x + 3)$     Write original function.

$x = \ln(y + 3)$     Switch $x$ and $y$.

$e^x = y + 3$     Write in exponential form.

$e^x - 3 = y$     Subtract 3 from each side.

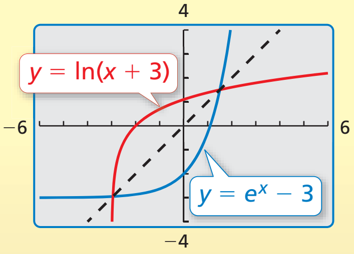
▶ The inverse of $y = \ln(x + 3)$ is $y = e^x - 3$.

**Check**

**a.** $f(g(x)) = 6^{\log_6 x} = x$ ✓

$g(f(x)) = \log_6 6^x = x$ ✓

**b.**




The graphs appear to be reflections of each other in the line $y = x$. ✓

## Monitoring Progress

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**Simplify the expression.**

**13.** $8^{\log_8 x}$     **14.** $\log_7 7^{-3x}$     **15.** $\log_2 64^x$     **16.** $e^{\ln 20}$

**17.** Find the inverse of $y = 4^x$.     **18.** Find the inverse of $y = \ln(x - 5)$.

## Graphing Logarithmic Functions

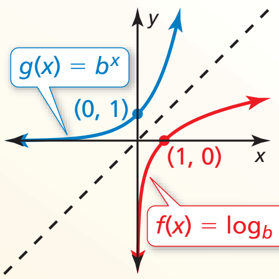
You can use the inverse relationship between exponential and logarithmic functions to graph logarithmic functions.

## Core Concept

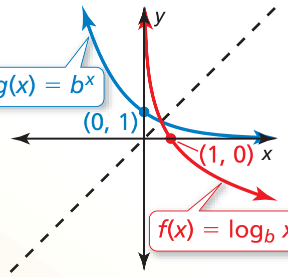
### Parent Graphs for Logarithmic Functions

The graph of $f(x) = \log_b x$ is shown below for $b > 1$ and for $0 < b < 1$. Because $f(x) = \log_b x$ and $g(x) = b^x$ are inverse functions, the graph of $f(x) = \log_b x$ is the reflection of the graph of $g(x) = b^x$ in the line $y = x$.

**Graph of $f(x) = \log_b x$ for $b > 1$**




**Graph of $f(x) = \log_b x$ for $0 < b < 1$**




Note that the $y$-axis is a vertical asymptote of the graph of $f(x) = \log_b x$. The domain of $f(x) = \log_b x$ is $x > 0$, and the range is all real numbers.

### EXAMPLE 7 Graphing a Logarithmic Function

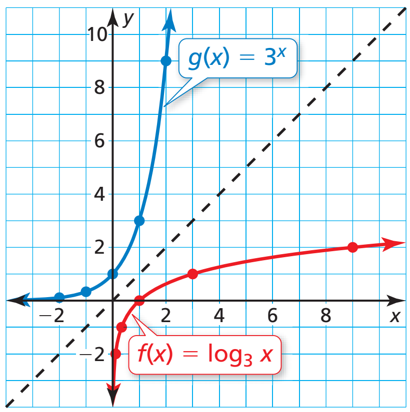
Graph $f(x) = \log_3 x$.

**SOLUTION**

**Step 1** Find the inverse of $f$. From the definition of logarithm, the inverse of $f(x) = \log_3 x$ is $g(x) = 3^x$.

**Step 2** Make a table of values for $g(x) = 3^x$.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $g(x)$ | $\frac{1}{9}$ | $\frac{1}{3}$ | $1$ | $3$ | $9$ |




**Step 3** Plot the points from the table and connect them with a smooth curve.

**Step 4** Because $f(x) = \log_3 x$ and $g(x) = 3^x$ are inverse functions, the graph of $f$ is obtained by reflecting the graph of $g$ in the line $y = x$. To do this, reverse the coordinates of the points on $g$ and plot these new points on the graph of $f$.

## Monitoring Progress

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**Graph the function.**

**19.** $y = \log_2 x$ **20.** $f(x) = \log_5 x$ **21.** $y = \log_{1/2} x$

# 6.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

## Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A logarithm with base 10 is called a(n) ___________ logarithm.

2. **COMPLETE THE SENTENCE** The expression $\log_3 9$ is read as ______________.

3. **WRITING** Describe the relationship between $y = 7^x$ and $y = \log_7 x$.

4. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

| | |
|---|---|
| What power of 4 gives you 16? | What is log base 4 of 16? |
| Evaluate $4^2$. | Evaluate $\log_4 16$. |

## Monitoring Progress and Modeling with Mathematics

**In Exercises 5–10, rewrite the equation in exponential form.** *(See Example 1.)*

5. $\log_3 9 = 2$
6. $\log_4 4 = 1$
7. $\log_6 1 = 0$
8. $\log_7 343 = 3$
9. $\log_{1/2} 16 = -4$
10. $\log_3 \frac{1}{3} = -1$

**In Exercises 11–16, rewrite the equation in logarithmic form.** *(See Example 2.)*

11. $6^2 = 36$
12. $12^0 = 1$
13. $16^{-1} = \frac{1}{16}$
14. $5^{-2} = \frac{1}{25}$
15. $125^{2/3} = 25$
16. $49^{1/2} = 7$

**In Exercises 17–24, evaluate the logarithm.** *(See Example 3.)*

17. $\log_3 81$
18. $\log_7 49$
19. $\log_3 3$
20. $\log_{1/2} 1$
21. $\log_5 \frac{1}{625}$
22. $\log_8 \frac{1}{512}$
23. $\log_4 0.25$
24. $\log_{10} 0.001$

25. **NUMBER SENSE** Order the logarithms from least value to greatest value.

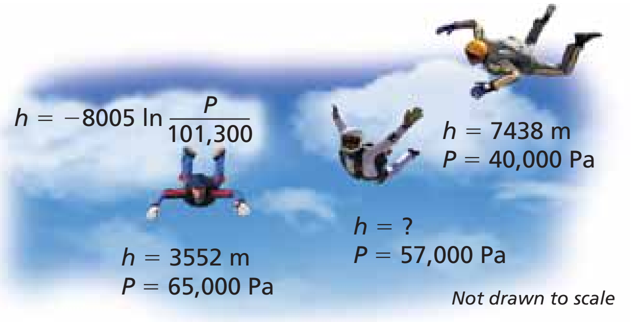
| | | | |
|---|---|---|---|
| $\log_5 23$ | $\log_6 38$ | $\log_7 8$ | $\log_2 10$ |

26. **WRITING** Explain why the expressions $\log_2(-1)$ and $\log_1 1$ are not defined.

**In Exercises 27–32, evaluate the logarithm using a calculator. Round your answer to three decimal places.** *(See Example 4.)*

27. $\log 6$
28. $\ln 12$
29. $\ln \frac{1}{3}$
30. $\log \frac{2}{7}$
31. $3 \ln 0.5$
32. $\log 0.6 + 1$

33. **MODELING WITH MATHEMATICS** Skydivers use an instrument called an *altimeter* to track their altitude as they fall. The altimeter determines altitude by measuring air pressure. The altitude $h$ (in meters) above sea level is related to the air pressure $P$ (in pascals) by the function shown in the diagram. What is the altitude above sea level when the air pressure is 57,000 pascals?




34. **MODELING WITH MATHEMATICS** The pH value for a substance measures how acidic or alkaline the substance is. It is given by the formula $\text{pH} = -\log[H^+]$, where $H^+$ is the hydrogen ion concentration (in moles per liter). Find the pH of each substance.

   a. baking soda: $[H^+] = 10^{-8}$ moles per liter

   b. vinegar: $[H^+] = 10^{-3}$ moles per liter

**In Exercises 35–40, simplify the expression.** *(See Example 5.)*

**35.** $7^{\log_7 x}$ **36.** $3^{\log_3 5x}$

**37.** $e^{\ln 4}$ **38.** $10^{\log 15}$

**39.** $\log_3 3^{2x}$ **40.** $\ln e^{x+1}$

**41. ERROR ANALYSIS** Describe and correct the error in rewriting $4^{-3} = \frac{1}{64}$ in logarithmic form.

$$\log_4 (-3) = \frac{1}{64}$$

**42. ERROR ANALYSIS** Describe and correct the error in simplifying the expression $\log_4 64^x$.

$$\begin{aligned}\log_4 64^x &= \log_4(16 \cdot 4^x)\\ &= \log_4(4^2 \cdot 4^x)\\ &= \log_4 4^{2+x}\\ &= 2 + x\end{aligned}$$

**In Exercises 43–52, find the inverse of the function.** *(See Example 6.)*

**43.** $y = 0.3^x$ **44.** $y = 11^x$

**45.** $y = \log_2 x$ **46.** $y = \log_{1/5} x$

**47.** $y = \ln(x - 1)$ **48.** $y = \ln 2x$

**49.** $y = e^{3x}$ **50.** $y = e^{x-4}$

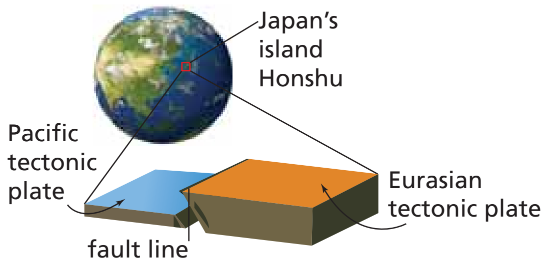
**51.** $y = 5^x - 9$ **52.** $y = 13 + \log x$

**53. PROBLEM SOLVING** The wind speed $s$ (in miles per hour) near the center of a tornado can be modeled by $s = 93 \log d + 65$, where $d$ is the distance (in miles) that the tornado travels.

**a.** In 1925, a tornado traveled 220 miles through three states. Estimate the wind speed near the center of the tornado.

**b.** Find the inverse of the given function. Describe what the inverse represents.

**54. MODELING WITH MATHEMATICS** The energy magnitude $M$ of an earthquake can be modeled by $M = \frac{2}{3} \log E - 9.9$, where $E$ is the amount of energy released (in ergs).




**a.** In 2011, a powerful earthquake in Japan, caused by the slippage of two tectonic plates along a fault, released $2.24 \times 10^{28}$ ergs. What was the energy magnitude of the earthquake?

**b.** Find the inverse of the given function. Describe what the inverse represents.

**In Exercises 55–60, graph the function.** *(See Example 7.)*

**55.** $y = \log_4 x$ **56.** $y = \log_6 x$

**57.** $y = \log_{1/3} x$ **58.** $y = \log_{1/4} x$

**59.** $y = \log_2 x - 1$ **60.** $y = \log_3(x + 2)$

**USING TOOLS In Exercises 61–64, use a graphing calculator to graph the function. Determine the domain, range, and asymptote of the function.**

**61.** $y = \log(x + 2)$ **62.** $y = -\ln x$

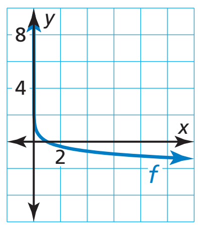
**63.** $y = \ln(-x)$ **64.** $y = 3 - \log x$

**65. MAKING AN ARGUMENT** Your friend states that every logarithmic function will pass through the point (1, 0). Is your friend correct? Explain your reasoning.

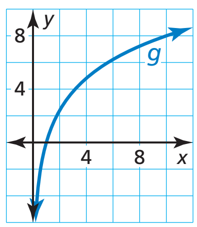
**66. ANALYZING RELATIONSHIPS** Rank the functions in order from the least average rate of change to the greatest average rate of change over the interval $1 \le x \le 10$.

**a.** $y = \log_6 x$ **b.** $y = \log_{3/5} x$

**c.**




**d.**



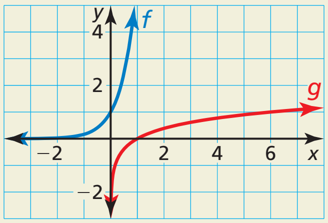
67. **PROBLEM SOLVING** Biologists have found that the length $\ell$ (in inches) of an alligator and its weight $w$ (in pounds) are related by the function $\ell = 27.1 \ln w - 32.8$.

a. Use a graphing calculator to graph the function.

b. Use your graph to estimate the weight of an alligator that is 10 feet long.

c. Use the *zero* feature to find the $x$-intercept of the graph of the function. Does this $x$-value make sense in the context of the situation? Explain.

68. **HOW DO YOU SEE IT?** The figure shows the graphs of the two functions $f$ and $g$.




a. Compare the end behavior of the logarithmic function $g$ to that of the exponential function $f$.

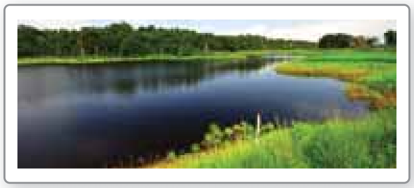
b. Determine whether the functions are inverse functions. Explain.

c. What is the base of each function? Explain.

69. **PROBLEM SOLVING** A study in Florida found that the number $s$ of fish species in a pool or lake can be modeled by the function

$$s = 30.6 - 20.5 \log A + 3.8(\log A)^2$$

where $A$ is the area (in square meters) of the pool or lake.



a. Use a graphing calculator to graph the function on the domain $200 \leq A \leq 35{,}000$.

b. Use your graph to estimate the number of species in a lake with an area of 30,000 square meters.

c. Use your graph to estimate the area of a lake that contains six species of fish.

d. Describe what happens to the number of fish species as the area of a pool or lake increases. Explain why your answer makes sense.

70. **THOUGHT PROVOKING** Write a logarithmic function that has an output of $-4$. Then sketch the graph of your function.

71. **CRITICAL THINKING** Evaluate each logarithm. (*Hint*: For each logarithm $\log_b x$, rewrite $b$ and $x$ as powers of the same base.)

a. $\log_{125} 25$ b. $\log_8 32$

c. $\log_{27} 81$ d. $\log_4 128$

## Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

**Let $f(x) = \sqrt[3]{x}$. Write a rule for $g$ that represents the indicated transformation of the graph of $f$.** *(Section 5.3)*

72. $g(x) = -f(x)$

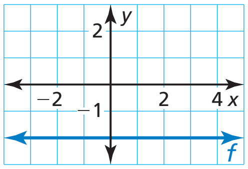
73. $g(x) = f\left(\frac{1}{2}x\right)$

74. $g(x) = f(-x) + 3$

75. $g(x) = f(x + 2)$

**Identify the function family to which $f$ belongs. Compare the graph of $f$ to the graph of its parent function.** *(Section 1.1)*

76.

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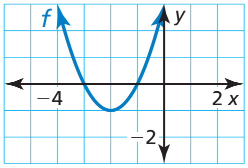


78.