5.6 Inverse of a Function

Essential Question

How can you sketch the graph of the inverse of a function?

EXPLORATION 1 Graphing Functions and Their Inverses

Work with a partner. Each pair of functions are *inverses* of each other. Use a graphing calculator to graph f and g in the same viewing window. What do you notice about the graphs?

a. $f(x) = 4x + 3$

$g(x) = \dfrac{x - 3}{4}$

b. $f(x) = x^3 + 1$

$g(x) = \sqrt[3]{x - 1}$

c. $f(x) = \sqrt{x - 3}$

$g(x) = x^2 + 3,\ x \geq 0$

d. $f(x) = \dfrac{4x + 4}{x + 5}$

$g(x) = \dfrac{4 - 5x}{x - 4}$

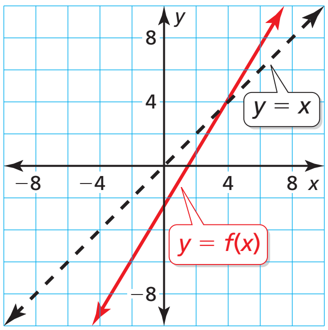
CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to reason inductively and make a plausible argument.

EXPLORATION 2 Sketching Graphs of Inverse Functions

Work with a partner. Use the graph of f to sketch the graph of g, the inverse function of f, on the same set of coordinate axes. Explain your reasoning.

a.



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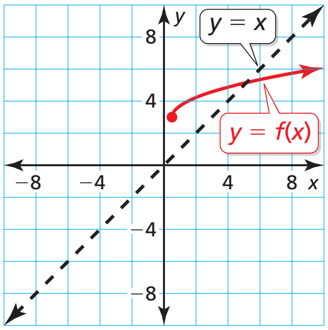

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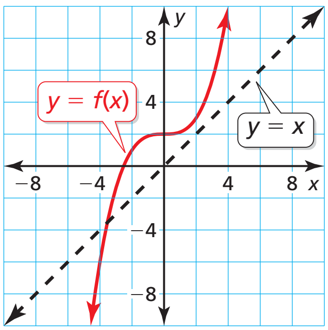

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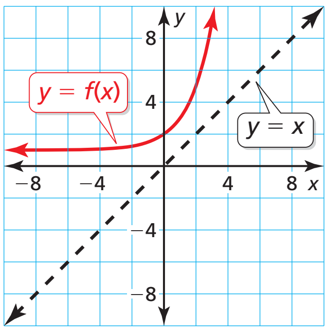

Communicate Your Answer

3. How can you sketch the graph of the inverse of a function?

4. In Exploration 1, what do you notice about the relationship between the equations of f and g? Use your answer to find g, the inverse function of

$f(x) = 2x - 3.$

Use a graph to check your answer.

5.6 Lesson

Core Vocabulary

inverse functions, *p. 277*

Previous
input
output
inverse operations
reflection
line of reflection

What You Will Learn

- Explore inverses of functions.
- Find and verify inverses of nonlinear functions.
- Solve real-life problems using inverse functions.

Exploring Inverses of Functions

You have used given inputs to find corresponding outputs of $y = f(x)$ for various types of functions. You have also used given outputs to find corresponding inputs. Now you will solve equations of the form $y = f(x)$ for x to obtain a general formula for finding the input given a specific output of a function f.

EXAMPLE 1 Writing a Formula for the Input of a Function

Let $f(x) = 2x + 3$.

a. Solve $y = f(x)$ for x.

b. Find the input when the output is -7.

SOLUTION

a.

$y = 2x + 3$ — Set y equal to $f(x)$.

$y - 3 = 2x$ — Subtract 3 from each side.

$\dfrac{y - 3}{2} = x$ — Divide each side by 2.

b. Find the input when $y = -7$.

$x = \dfrac{-7 - 3}{2}$ — Substitute -7 for y.

$= \dfrac{-10}{2}$ — Subtract.

$= -5$ — Divide.

So, the input is -5 when the output is -7.

Check

$f(-5) = 2(-5) + 3$
$= -10 + 3$
$= -7$ ✓

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Solve $y = f(x)$ for x. Then find the input(s) when the output is 2.

1. $f(x) = x - 2$ **2.** $f(x) = 2x^2$ **3.** $f(x) = -x^3 + 3$

In Example 1, notice the steps involved after substituting for x in $y = 2x + 3$ and after substituting for y in $x = \dfrac{y - 3}{2}$.

$y = 2x + 3$	$x = \dfrac{y-3}{2}$
Step 1 Multiply by 2.	**Step 1** Subtract 3.
Step 2 Add 3.	**Step 2** Divide by 2.

inverse operations in the reverse order

UNDERSTANDING MATHEMATICAL TERMS

The term *inverse functions* does not refer to a new type of function. Rather, it describes any pair of functions that are inverses.

Notice that these steps *undo* each other. Functions that undo each other are called **inverse functions**. In Example 1, you can use the equation solved for x to write the inverse of f by switching the roles of x and y.

$f(x) = 2x + 3$ original function $\qquad g(x) = \frac{x-3}{2}$ inverse function

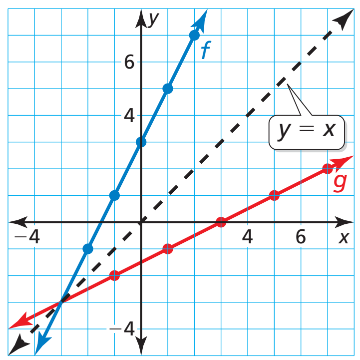
Because inverse functions interchange the input and output values of the original function, the domain and range are also interchanged.

Original function: $f(x) = 2x + 3$

x	−2	−1	0	1	2
y	−1	1	3	5	7

Inverse function: $g(x) = \frac{x-3}{2}$

x	−1	1	3	5	7
y	−2	−1	0	1	2



The graph of an inverse function is a *reflection* of the graph of the original function. The *line of reflection* is $y = x$. To find the inverse of a function algebraically, switch the roles of x and y, and then solve for y.

EXAMPLE 2 Finding the Inverse of a Linear Function

Find the inverse of $f(x) = 3x - 1$.

SOLUTION

Method 1 Use inverse operations in the reverse order.

$f(x) = 3x - 1$ Multiply the input x by 3 and then subtract 1.

To find the inverse, apply inverse operations in the reverse order.

$g(x) = \frac{x+1}{3}$ Add 1 to the input x and then divide by 3.

▶ The inverse of f is $g(x) = \frac{x+1}{3}$, or $g(x) = \frac{1}{3}x + \frac{1}{3}$.

Method 2 Set y equal to $f(x)$. Switch the roles of x and y and solve for y.

$y = 3x - 1$ Set y equal to $f(x)$.

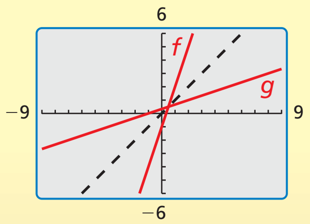
$x = 3y - 1$ Switch x and y.

$x + 1 = 3y$ Add 1 to each side.

$\frac{x+1}{3} = y$ Divide each side by 3.

▶ The inverse of f is $g(x) = \frac{x+1}{3}$, or $g(x) = \frac{1}{3}x + \frac{1}{3}$.

Check



The graph of g appears to be a reflection of the graph of f in the line $y = x$. ✓

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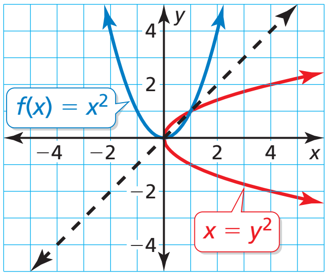
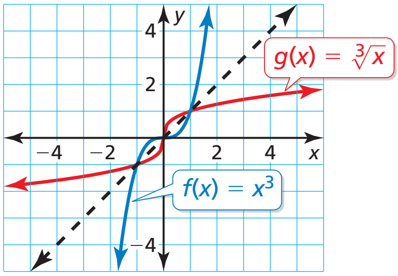
Find the inverse of the function. Then graph the function and its inverse.

4. $f(x) = 2x$ **5.** $f(x) = -x + 1$ **6.** $f(x) = \frac{1}{3}x - 2$

Inverses of Nonlinear Functions

In the previous examples, the inverses of the linear functions were also functions. However, inverses are not always functions. The graphs of $f(x) = x^2$ and $f(x) = x^3$ are shown along with their reflections in the line $y = x$. Notice that the inverse of $f(x) = x^3$ is a function, but the inverse of $f(x) = x^2$ is *not* a function.





When the domain of $f(x) = x^2$ is *restricted* to only nonnegative real numbers, the inverse of f *is* a function.

EXAMPLE 3 Finding the Inverse of a Quadratic Function

Find the inverse of $f(x) = x^2$, $x \geq 0$. Then graph the function and its inverse.

SOLUTION

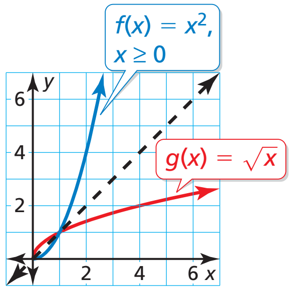
$f(x) = x^2$	Write the original function.
$y = x^2$	Set y equal to $f(x)$.
$x = y^2$	Switch x and y.
$\pm\sqrt{x} = y$	Take square root of each side.

STUDY TIP

If the domain of f were restricted to $x \leq 0$, then the inverse would be $g(x) = -\sqrt{x}$.

The domain of f is restricted to nonnegative values of x. So, the range of the inverse must also be restricted to nonnegative values.

▶ So, the inverse of f is $g(x) = \sqrt{x}$.



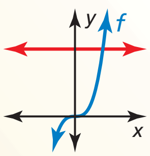
You can use the graph of a function f to determine whether the inverse of f is a function by applying the *horizontal line test*.

Core Concept

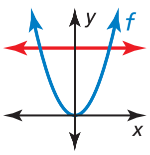
Horizontal Line Test

The inverse of a function f is also a function if and only if no horizontal line intersects the graph of f more than once.

Inverse is a function



Inverse is not a function

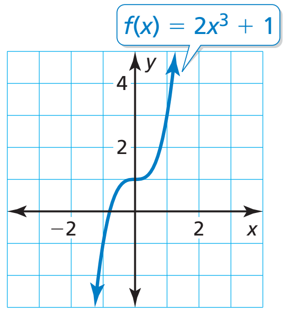


EXAMPLE 4 Finding the Inverse of a Cubic Function

Consider the function $f(x) = 2x^3 + 1$. Determine whether the inverse of f is a function. Then find the inverse.

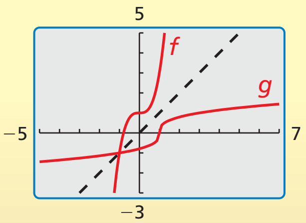
SOLUTION

Graph the function f. Notice that no horizontal line intersects the graph more than once. So, the inverse of f is a function. Find the inverse.



$y = 2x^3 + 1$	Set y equal to $f(x)$.
$x = 2y^3 + 1$	Switch x and y.
$x - 1 = 2y^3$	Subtract 1 from each side.
$\dfrac{x-1}{2} = y^3$	Divide each side by 2.
$\sqrt[3]{\dfrac{x-1}{2}} = y$	Take cube root of each side.

Check



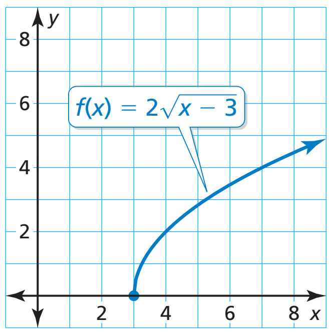
▶ So, the inverse of f is $g(x) = \sqrt[3]{\dfrac{x-1}{2}}$.

EXAMPLE 5 Finding the Inverse of a Radical Function

Consider the function $f(x) = 2\sqrt{x-3}$. Determine whether the inverse of f is a function. Then find the inverse.

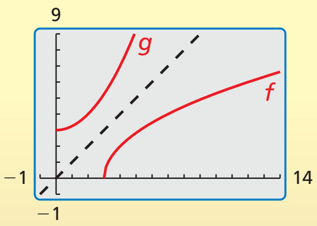
SOLUTION

Graph the function f. Notice that no horizontal line intersects the graph more than once. So, the inverse of f is a function. Find the inverse.



$y = 2\sqrt{x-3}$	Set y equal to $f(x)$.
$x = 2\sqrt{y-3}$	Switch x and y.
$x^2 = \left(2\sqrt{y-3}\right)^2$	Square each side.
$x^2 = 4(y-3)$	Simplify.
$x^2 = 4y - 12$	Distributive Property
$x^2 + 12 = 4y$	Add 12 to each side.
$\frac{1}{4}x^2 + 3 = y$	Divide each side by 4.

Check



Because the range of f is $y \geq 0$, the domain of the inverse must be restricted to $x \geq 0$.

▶ So, the inverse of f is $g(x) = \frac{1}{4}x^2 + 3$, where $x \geq 0$.

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Find the inverse of the function. Then graph the function and its inverse.

7. $f(x) = -x^2, x \leq 0$ **8.** $f(x) = -x^3 + 4$ **9.** $f(x) = \sqrt{x+2}$

REASONING ABSTRACTLY

Inverse functions *undo* each other. So, when you evaluate a function for a specific input, and then evaluate its inverse using the output, you obtain the original input.

Let f and g be inverse functions. If $f(a) = b$, then $g(b) = a$. So, in general,

$$f(g(x)) = x \qquad \text{and} \qquad g(f(x)) = x.$$

EXAMPLE 6 Verifying Functions Are Inverses

Verify that $f(x) = 3x - 1$ and $g(x) = \dfrac{x+1}{3}$ are inverse functions.

SOLUTION

Step 1 Show that $f(g(x)) = x$.

$$f(g(x)) = f\left(\frac{x+1}{3}\right)$$
$$= 3\left(\frac{x+1}{3}\right) - 1$$
$$= x + 1 - 1$$
$$= x \ ✓$$

Step 2 Show that $g(f(x)) = x$.

$$g(f(x)) = g(3x - 1)$$
$$= \frac{3x - 1 + 1}{3}$$
$$= \frac{3x}{3}$$
$$= x \ ✓$$

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Determine whether the functions are inverse functions.

10. $f(x) = x + 5,\ g(x) = x - 5$

11. $f(x) = 8x^3,\ g(x) = \sqrt[3]{2x}$

Solving Real-Life Problems

In many real-life problems, formulas contain meaningful variables, such as the radius r in the formula for the surface area S of a sphere, $S = 4\pi r^2$. In this situation, switching the variables to find the inverse would create confusion by switching the meanings of S and r. So, when finding the inverse, solve for r without switching the variables.

EXAMPLE 7 Solving a Multi-Step Problem

Find the inverse of the function that represents the surface area of a sphere, $S = 4\pi r^2$. Then find the radius of a sphere that has a surface area of 100π square feet.

SOLUTION

Step 1 Find the inverse of the function.

$$S = 4\pi r^2$$
$$\frac{S}{4\pi} = r^2$$
$$\sqrt{\frac{S}{4\pi}} = r$$

The radius r must be positive, so disregard the negative square root.

Step 2 Evaluate the inverse when $S = 100\pi$.

$$r = \sqrt{\frac{100\pi}{4\pi}}$$
$$= \sqrt{25} = 5$$

▶ The radius of the sphere is 5 feet.

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12. The distance d (in meters) that a dropped object falls in t seconds on Earth is represented by $d = 4.9t^2$. Find the inverse of the function. How long does it take an object to fall 50 meters?

5.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** In your own words, state the definition of inverse functions.

2. **WRITING** Explain how to determine whether the inverse of a function is also a function.

3. **COMPLETE THE SENTENCE** Functions f and g are inverses of each other provided that $f(g(x)) =$ ____ and $g(f(x)) =$ ____.

4. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

Let $f(x) = 5x - 2$. Solve $y = f(x)$ for x and then switch the roles of x and y.	Write an equation that represents a reflection of the graph of $f(x) = 5x - 2$ in the x-axis.
Write an equation that represents a reflection of the graph of $f(x) = 5x - 2$ in the line $y = x$.	Find the inverse of $f(x) = 5x - 2$.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, solve $y = f(x)$ for x. Then find the input(s) when the output is -3. *(See Example 1.)*

5. $f(x) = 3x + 5$

6. $f(x) = -7x - 2$

7. $f(x) = \frac{1}{2}x - 3$

8. $f(x) = -\frac{2}{3}x + 1$

9. $f(x) = 3x^3$

10. $f(x) = 2x^4 - 5$

11. $f(x) = (x - 2)^2 - 7$

12. $f(x) = (x - 5)^3 - 1$

In Exercises 13–20, find the inverse of the function. Then graph the function and its inverse. *(See Example 2.)*

13. $f(x) = 6x$

14. $f(x) = -3x$

15. $f(x) = -2x + 5$

16. $f(x) = 6x - 3$

17. $f(x) = -\frac{1}{2}x + 4$

18. $f(x) = \frac{1}{3}x - 1$

19. $f(x) = \frac{2}{3}x - \frac{1}{3}$

20. $f(x) = -\frac{4}{5}x + \frac{1}{5}$

21. **COMPARING METHODS** Find the inverse of the function $f(x) = -3x + 4$ by switching the roles of x and y and solving for y. Then find the inverse of the function f by using inverse operations in the reverse order. Which method do you prefer? Explain.

22. **REASONING** Determine whether each pair of functions f and g are inverses. Explain your reasoning.

a.

x	-2	-1	0	1	2
$f(x)$	-2	1	4	7	10

x	-2	1	4	7	10
$g(x)$	-2	-1	0	1	2

b.

x	2	3	4	5	6
$f(x)$	8	6	4	2	0

x	2	3	4	5	6
$g(x)$	-8	-6	-4	-2	0

c.

x	-4	-2	0	2	4
$f(x)$	2	10	18	26	34

x	-4	-2	0	2	4
$g(x)$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{18}$	$\frac{1}{26}$	$\frac{1}{34}$

In Exercises 23–28, find the inverse of the function. Then graph the function and its inverse. *(See Example 3.)*

23. $f(x) = 4x^2, x \le 0$

24. $f(x) = 9x^2, x \le 0$

25. $f(x) = (x - 3)^3$

26. $f(x) = (x + 4)^3$

27. $f(x) = 2x^4, x \ge 0$

28. $f(x) = -x^6, x \ge 0$

ERROR ANALYSIS **In Exercises 29 and 30, describe and correct the error in finding the inverse of the function.**

29. ✗

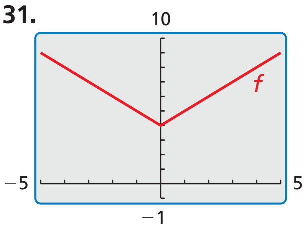
$$f(x) = -x + 3$$
$$y = -x + 3$$
$$-x = y + 3$$
$$-x - 3 = y$$

30. ✗

$$f(x) = \tfrac{1}{7}x^2, x \ge 0$$
$$y = \tfrac{1}{7}x^2$$
$$x = \tfrac{1}{7}y^2$$
$$7x = y^2$$
$$\pm\sqrt{7x} = y$$

USING TOOLS **In Exercises 31–34, use the graph to determine whether the inverse of *f* is a function. Explain your reasoning.**

31.



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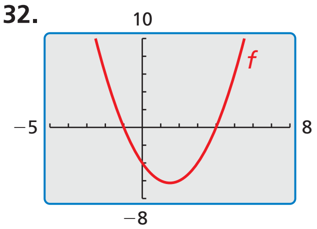

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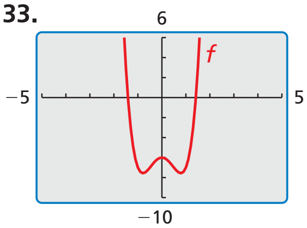

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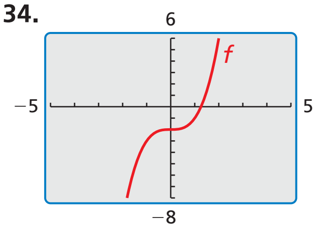


In Exercises 35–46, determine whether the inverse of *f* is a function. Then find the inverse. *(See Examples 4 and 5.)*

35. $f(x) = x^3 - 1$

36. $f(x) = -x^3 + 3$

37. $f(x) = \sqrt{x + 4}$

38. $f(x) = \sqrt{x - 6}$

39. $f(x) = 2\sqrt[3]{x - 5}$

40. $f(x) = 2x^2 - 5$

41. $f(x) = x^4 + 2$

42. $f(x) = 2x^3 - 5$

43. $f(x) = 3\sqrt[3]{x + 1}$

44. $f(x) = -\sqrt[3]{\dfrac{2x + 4}{3}}$

45. $f(x) = \dfrac{1}{2}x^5$

46. $f(x) = -3\sqrt{\dfrac{4x - 7}{3}}$

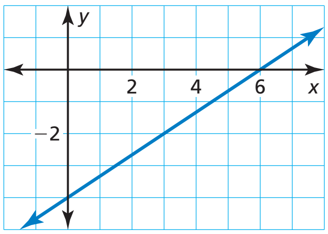
47. **WRITING EQUATIONS** What is the inverse of the function whose graph is shown?

Ⓐ $g(x) = \frac{3}{2}x - 6$

Ⓑ $g(x) = \frac{3}{2}x + 6$

Ⓒ $g(x) = \frac{2}{3}x - 6$

Ⓓ $g(x) = \frac{2}{3}x + 12$



48. **WRITING EQUATIONS** What is the inverse of $f(x) = -\frac{1}{64}x^3$?

Ⓐ $g(x) = -4x^3$

Ⓑ $g(x) = 4\sqrt[3]{x}$

Ⓒ $g(x) = -4\sqrt[3]{x}$

Ⓓ $g(x) = \sqrt[3]{-4x}$

In Exercises 49–52, determine whether the functions are inverses. *(See Example 6.)*

49. $f(x) = 2x - 9, g(x) = \dfrac{x}{2} + 9$

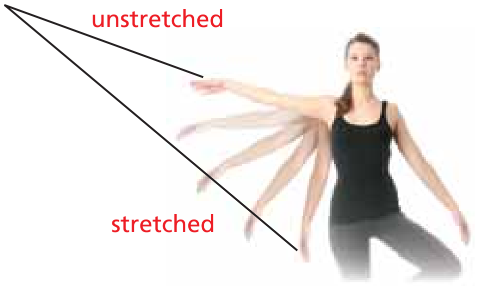
50. $f(x) = \dfrac{x - 3}{4}, g(x) = 4x + 3$

51. $f(x) = \sqrt[5]{\dfrac{x + 9}{5}}, g(x) = 5x^5 - 9$

52. $f(x) = 7x^{3/2} - 4, g(x) = \left(\dfrac{x + 4}{7}\right)^{3/2}$

53. **MODELING WITH MATHEMATICS** The maximum hull speed v (in knots) of a boat with a displacement hull can be approximated by $v = 1.34\sqrt{\ell}$, where ℓ is the waterline length (in feet) of the boat. Find the inverse function. What waterline length is needed to achieve a maximum speed of 7.5 knots? *(See Example 7.)*

54. MODELING WITH MATHEMATICS Elastic bands can be used for exercising to provide a range of resistance. The resistance R (in pounds) of a band can be modeled by $R = \frac{3}{8}L - 5$, where L is the total length (in inches) of the stretched band. Find the inverse function. What length of the stretched band provides 19 pounds of resistance?



ANALYZING RELATIONSHIPS In Exercises 55–58, match the graph of the function with the graph of its inverse.

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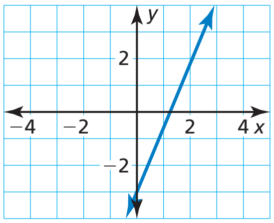

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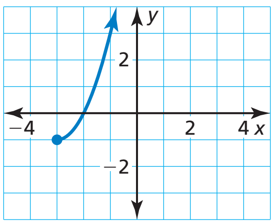
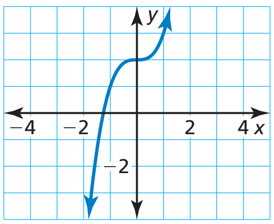


57.



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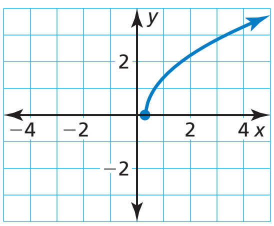
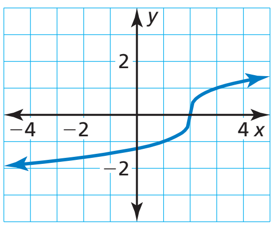
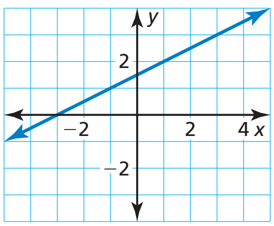


A.



B.



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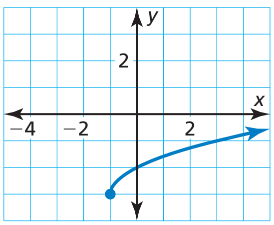
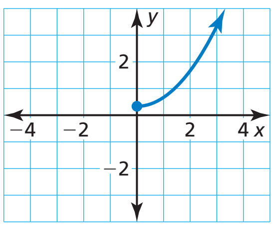


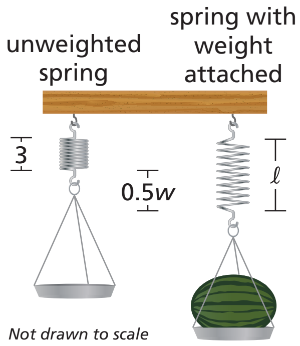
D.



59. REASONING You and a friend are playing a number-guessing game. You ask your friend to think of a positive number, square the number, multiply the result by 2, and then add 3. Your friend's final answer is 53. What was the original number chosen? Justify your answer.

60. MAKING AN ARGUMENT Your friend claims that every quadratic function whose domain is restricted to nonnegative values has an inverse function. Is your friend correct? Explain your reasoning.

61. PROBLEM SOLVING When calibrating a spring scale, you need to know how far the spring stretches for various weights. Hooke's Law states that the length a spring stretches is proportional to the weight attached to it. A model for one scale is $\ell = 0.5w + 3$, where ℓ is the total length (in inches) of the stretched spring and w is the weight (in pounds) of the object.



a. Find the inverse function. Describe what it represents.

b. You place a melon on the scale, and the spring stretches to a total length of 5.5 inches. Determine the weight of the melon.

c. Verify that the function $\ell = 0.5w + 3$ and the inverse model in part (a) are inverse functions.

62. THOUGHT PROVOKING Do functions of the form $y = x^{m/n}$, where m and n are positive integers, have inverse functions? Justify your answer with examples.

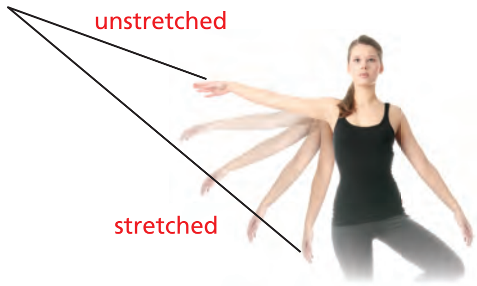
63. PROBLEM SOLVING At the start of a dog sled race in Anchorage, Alaska, the temperature was 5°C. By the end of the race, the temperature was −10°C. The formula for converting temperatures from degrees Fahrenheit F to degrees Celsius C is $C = \frac{5}{9}(F - 32)$.

a. Find the inverse function. Describe what it represents.

b. Find the Fahrenheit temperatures at the start and end of the race.

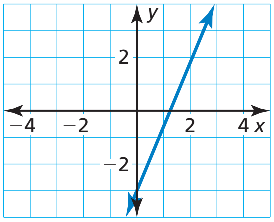
c. Use a graphing calculator to graph the original function and its inverse. Find the temperature that is the same on both temperature scales.

54. MODELING WITH MATHEMATICS Elastic bands can be used for exercising to provide a range of resistance. The resistance R (in pounds) of a band can be modeled by $R = \frac{3}{8}L - 5$, where L is the total length (in inches) of the stretched band. Find the inverse function. What length of the stretched band provides 19 pounds of resistance?



ANALYZING RELATIONSHIPS In Exercises 55–58, match the graph of the function with the graph of its inverse.

55.



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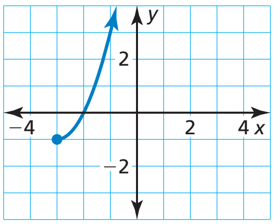

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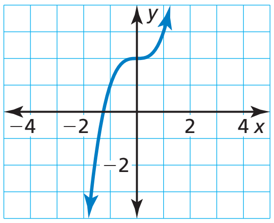

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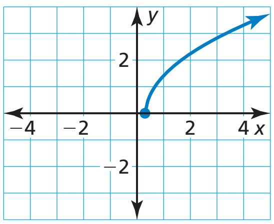
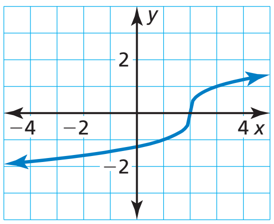
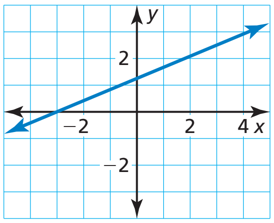


A.



B.



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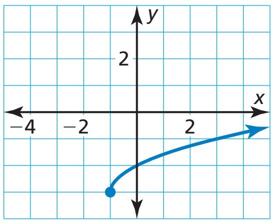

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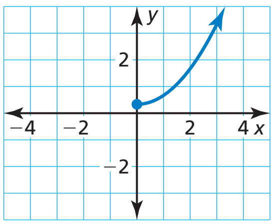

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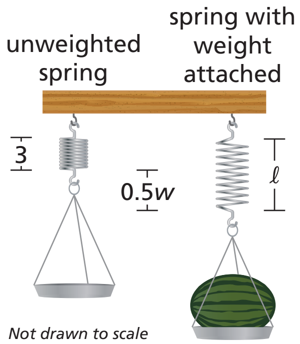

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