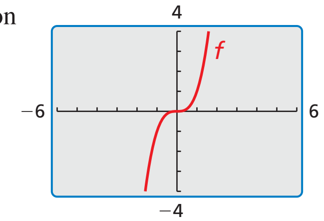
4.7 Transformations of Polynomial Functions

Essential Question How can you transform the graph of a polynomial function?

EXPLORATION 1 Transforming the Graph of a Cubic Function

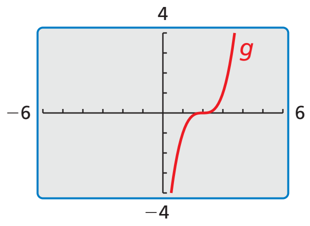
Work with a partner. The graph of the cubic function

$$f(x) = x^3$$



is shown. The graph of each cubic function g represents a transformation of the graph of f. Write a rule for g. Use a graphing calculator to verify your answers.

a.



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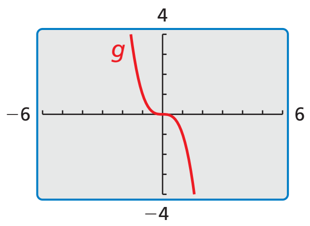

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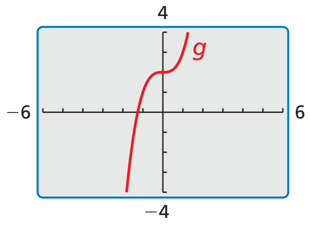

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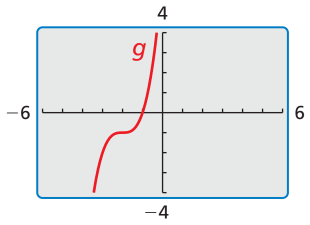
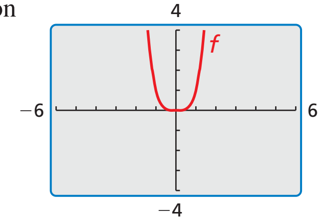


EXPLORATION 2 Transforming the Graph of a Quartic Function

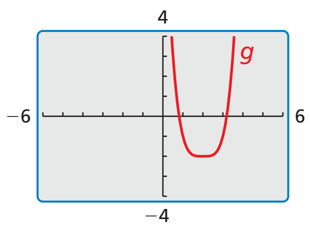
Work with a partner. The graph of the quartic function

$$f(x) = x^4$$



is shown. The graph of each quartic function g represents a transformation of the graph of f. Write a rule for g. Use a graphing calculator to verify your answers.

a.



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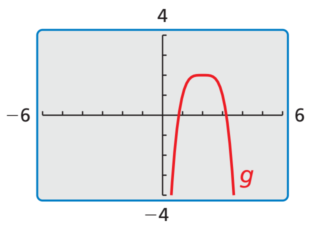


LOOKING FOR STRUCTURE

To be proficient in math, you need to see complicated things, such as some algebraic expressions, as being single objects or as being composed of several objects.

Communicate Your Answer

3. How can you transform the graph of a polynomial function?

4. Describe the transformation of $f(x) = x^4$ represented by $g(x) = (x + 1)^4 + 3$. Then graph g.

4.7 Lesson

Core Vocabulary

Previous
polynomial function
transformations

What You Will Learn

- Describe transformations of polynomial functions.
- Write transformations of polynomial functions.

Describing Transformations of Polynomial Functions

You can transform graphs of polynomial functions in the same way you transformed graphs of linear functions, absolute value functions, and quadratic functions. Examples of transformations of the graph of $f(x) = x^4$ are shown below.

Core Concept

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x - h)$	$g(x) = (x - 5)^4$ $g(x) = (x + 2)^4$	5 units right 2 units left
Vertical Translation Graph shifts up or down.	$f(x) + k$	$g(x) = x^4 + 1$ $g(x) = x^4 - 4$	1 unit up 4 units down
Reflection Graph flips over x- or y-axis.	$f(-x)$ $-f(x)$	$g(x) = (-x)^4 = x^4$ $g(x) = -x^4$	over y-axis over x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis.	$f(ax)$	$g(x) = (2x)^4$ $g(x) = \left(\frac{1}{2}x\right)^4$	shrink by a factor of $\frac{1}{2}$ stretch by a factor of 2
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis.	$a \cdot f(x)$	$g(x) = 8x^4$ $g(x) = \frac{1}{4}x^4$	stretch by a factor of 8 shrink by a factor of $\frac{1}{4}$

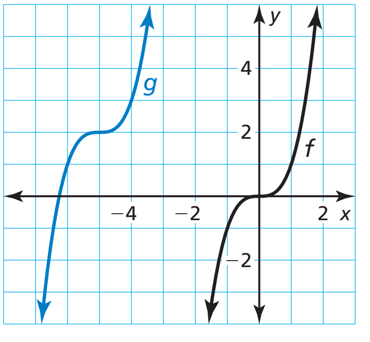
EXAMPLE 1 Translating a Polynomial Function

Describe the transformation of $f(x) = x^3$ represented by $g(x) = (x + 5)^3 + 2$. Then graph each function.

SOLUTION

Notice that the function is of the form $g(x) = (x - h)^3 + k$. Rewrite the function to identify h and k.

$$g(x) = (x - (\underset{h}{\underset{\uparrow}{-5}}))^3 + \underset{k}{\underset{\uparrow}{2}}$$



▶ Because $h = -5$ and $k = 2$, the graph of g is a translation 5 units left and 2 units up of the graph of f.

Monitoring Progress

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1. Describe the transformation of $f(x) = x^4$ represented by $g(x) = (x - 3)^4 - 1$. Then graph each function.

EXAMPLE 2 Transforming Polynomial Functions

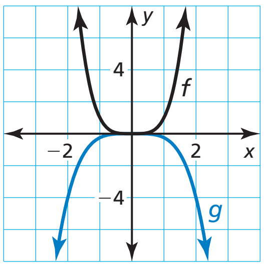
Describe the transformation of f represented by g. Then graph each function.

a. $f(x) = x^4$, $g(x) = -\frac{1}{4}x^4$ **b.** $f(x) = x^5$, $g(x) = (2x)^5 - 3$

SOLUTION

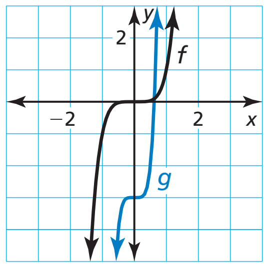
a. Notice that the function is of the form $g(x) = -ax^4$, where $a = \frac{1}{4}$.

▶ So, the graph of g is a reflection in the x-axis and a vertical shrink by a factor of $\frac{1}{4}$ of the graph of f.



b. Notice that the function is of the form $g(x) = (ax)^5 + k$, where $a = 2$ and $k = -3$.

▶ So, the graph of g is a horizontal shrink by a factor of $\frac{1}{2}$ and a translation 3 units down of the graph of f.



Monitoring Progress

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2. Describe the transformation of $f(x) = x^3$ represented by $g(x) = 4(x + 2)^3$. Then graph each function.

Writing Transformations of Polynomial Functions

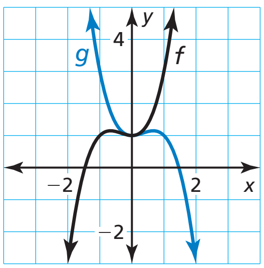
EXAMPLE 3 Writing Transformed Polynomial Functions

Let $f(x) = x^3 + x^2 + 1$. Write a rule for g and then graph each function. Describe the graph of g as a transformation of the graph of f.

a. $g(x) = f(-x)$ **b.** $g(x) = 3f(x)$

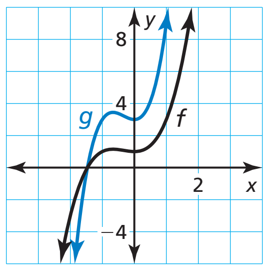
SOLUTION

a.
$$\begin{aligned} g(x) &= f(-x) \\ &= (-x)^3 + (-x)^2 + 1 \\ &= -x^3 + x^2 + 1 \end{aligned}$$



▶ The graph of g is a reflection in the y-axis of the graph of f.

b.
$$\begin{aligned} g(x) &= 3f(x) \\ &= 3(x^3 + x^2 + 1) \\ &= 3x^3 + 3x^2 + 3 \end{aligned}$$



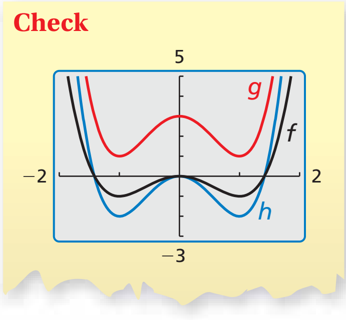
▶ The graph of g is a vertical stretch by a factor of 3 of the graph of f.

REMEMBER

Vertical stretches and shrinks do not change the x-intercept(s) of a graph. You can observe this using the graph in Example 3(b).

EXAMPLE 4 Writing a Transformed Polynomial Function

Let the graph of g be a vertical stretch by a factor of 2, followed by a translation 3 units up of the graph of $f(x) = x^4 - 2x^2$. Write a rule for g.



SOLUTION

Step 1 First write a function h that represents the vertical stretch of f.

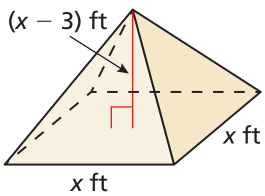
$$h(x) = 2 \cdot f(x) \qquad \text{Multiply the output by 2.}$$
$$= 2(x^4 - 2x^2) \qquad \text{Substitute } x^4 - 2x^2 \text{ for } f(x).$$
$$= 2x^4 - 4x^2 \qquad \text{Distributive Property}$$

Step 2 Then write a function g that represents the translation of h.

$$g(x) = h(x) + 3 \qquad \text{Add 3 to the output.}$$
$$= 2x^4 - 4x^2 + 3 \qquad \text{Substitute } 2x^4 - 4x^2 \text{ for } h(x).$$

▶ The transformed function is $g(x) = 2x^4 - 4x^2 + 3$.

EXAMPLE 5 Modeling with Mathematics



The function $V(x) = \frac{1}{3}x^3 - x^2$ represents the volume (in cubic feet) of the square pyramid shown. The function $W(x) = V(3x)$ represents the volume (in cubic feet) when x is measured in yards. Write a rule for W. Find and interpret $W(10)$.

SOLUTION

1. **Understand the Problem** You are given a function V whose inputs are in feet and whose outputs are in cubic feet. You are given another function W whose inputs are in yards and whose outputs are in cubic feet. The horizontal shrink shown by $W(x) = V(3x)$ makes sense because there are 3 feet in 1 yard. You are asked to write a rule for W and interpret the output for a given input.
2. **Make a Plan** Write the transformed function $W(x)$ and then find $W(10)$.
3. **Solve the Problem** $W(x) = V(3x)$

$$= \frac{1}{3}(3x)^3 - (3x)^2 \qquad \text{Replace } x \text{ with } 3x \text{ in } V(x).$$
$$= 9x^3 - 9x^2 \qquad \text{Simplify.}$$

Next, find $W(10)$.

$$W(10) = 9(10)^3 - 9(10)^2 = 9000 - 900 = 8100$$

▶ When x is 10 yards, the volume of the pyramid is 8100 cubic feet.

4. **Look Back** Because $W(10) = V(30)$, you can check that your solution is correct by verifying that $V(30) = 8100$.

$$V(30) = \frac{1}{3}(30)^3 - (30)^2 = 9000 - 900 = 8100 \checkmark$$

Monitoring Progress

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3. Let $f(x) = x^5 - 4x + 6$ and $g(x) = -f(x)$. Write a rule for g and then graph each function. Describe the graph of g as a transformation of the graph of f.
4. Let the graph of g be a horizontal stretch by a factor of 2, followed by a translation 3 units to the right of the graph of $f(x) = 8x^3 + 3$. Write a rule for g.
5. **WHAT IF?** In Example 5, the height of the pyramid is $6x$, and the volume (in cubic feet) is represented by $V(x) = 2x^3$. Write a rule for W. Find and interpret $W(7)$.

4.7 Exercises

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Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The graph of $f(x) = (x + 2)^3$ is a ____________ translation of the graph of $f(x) = x^3$.

2. **VOCABULARY** Describe how the vertex form of quadratic functions is similar to the form $f(x) = a(x - h)^3 + k$ for cubic functions.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, describe the transformation of f represented by g. Then graph each function. *(See Example 1.)*

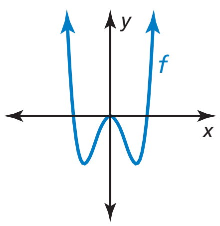
3. $f(x) = x^4,\ g(x) = x^4 + 3$

4. $f(x) = x^4,\ g(x) = (x - 5)^4$

5. $f(x) = x^5,\ g(x) = (x - 2)^5 - 1$

6. $f(x) = x^6,\ g(x) = (x + 1)^6 - 4$

ANALYZING RELATIONSHIPS **In Exercises 7–10, match the function with the correct transformation of the graph of f. Explain your reasoning.**



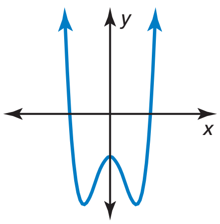
7. $y = f(x - 2)$

8. $y = f(x + 2) + 2$

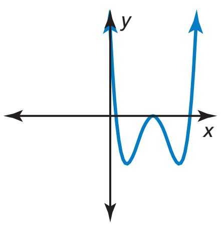
9. $y = f(x - 2) + 2$

10. $y = f(x) - 2$

A.



B.



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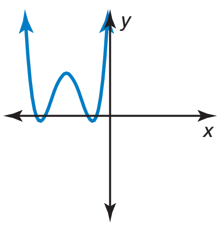

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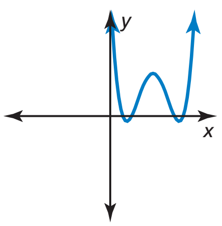


In Exercises 11–16, describe the transformation of f represented by g. Then graph each function. *(See Example 2.)*

11. $f(x) = x^4,\ g(x) = -2x^4$

12. $f(x) = x^6,\ g(x) = -3x^6$

13. $f(x) = x^3,\ g(x) = 5x^3 + 1$

14. $f(x) = x^4,\ g(x) = \frac{1}{2}x^4 + 1$

15. $f(x) = x^5,\ g(x) = \frac{3}{4}(x + 4)^5$

16. $f(x) = x^4,\ g(x) = (2x)^4 - 3$

In Exercises 17–20, write a rule for g and then graph each function. Describe the graph of g as a transformation of the graph of f. *(See Example 3.)*

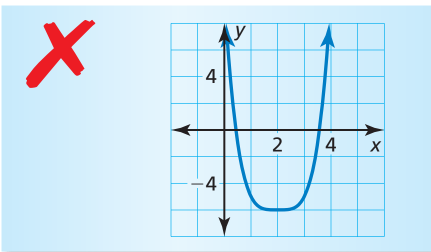
17. $f(x) = x^4 + 1,\ g(x) = f(x + 2)$

18. $f(x) = x^5 - 2x + 3,\ g(x) = 3f(x)$

19. $f(x) = 2x^3 - 2x^2 + 6,\ g(x) = -\frac{1}{2}f(x)$

20. $f(x) = x^4 + x^3 - 1,\ g(x) = f(-x) - 5$

21. **ERROR ANALYSIS** Describe and correct the error in graphing the function $g(x) = (x + 2)^4 - 6$.



22. **ERROR ANALYSIS** Describe and correct the error in describing the transformation of the graph of $f(x) = x^5$ represented by the graph of $g(x) = (3x)^5 - 4$.

> ✗ The graph of g is a horizontal shrink by a factor of 3, followed by a translation 4 units down of the graph of f.

In Exercises 23–26, write a rule for g that represents the indicated transformations of the graph of f. *(See Example 4.)*

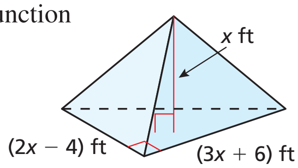
23. $f(x) = x^3 - 6$; translation 3 units left, followed by a reflection in the y-axis

24. $f(x) = x^4 + 2x + 6$; vertical stretch by a factor of 2, followed by a translation 4 units right

25. $f(x) = x^3 + 2x^2 - 9$; horizontal shrink by a factor of $\frac{1}{3}$ and a translation 2 units up, followed by a reflection in the x-axis

26. $f(x) = 2x^5 - x^3 + x^2 + 4$; reflection in the y-axis and a vertical stretch by a factor of 3, followed by a translation 1 unit down

27. **MODELING WITH MATHEMATICS** The volume V (in cubic feet) of the pyramid is given by $V(x) = x^3 - 4x$. The function $W(x) = V(3x)$ gives the volume (in cubic feet) of the pyramid when x is measured in yards. Write a rule for W. Find and interpret $W(5)$. *(See Example 5.)*



28. **MAKING AN ARGUMENT** The volume of a cube with side length x is given by $V(x) = x^3$. Your friend claims that when you divide the volume in half, the volume decreases by a greater amount than when you divide each side length in half. Is your friend correct? Justify your answer.

29. **OPEN-ENDED** Describe two transformations of the graph of $f(x) = x^5$ where the order in which the transformations are performed is important. Then describe two transformations where the order is *not* important. Explain your reasoning.

30. **THOUGHT PROVOKING** Write and graph a transformation of the graph of $f(x) = x^5 - 3x^4 + 2x - 4$ that results in a graph with a y-intercept of -2.

31. **PROBLEM SOLVING** A portion of the path that a hummingbird flies while feeding can be modeled by the function

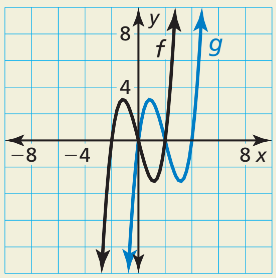
$$f(x) = -\frac{1}{5}x(x-4)^2(x-7),\ 0 \le x \le 7$$

where x is the horizontal distance (in meters) and $f(x)$ is the height (in meters). The hummingbird feeds each time it is at ground level.

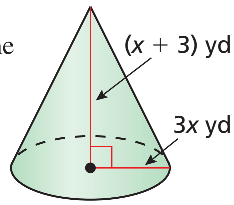
a. At what distances does the hummingbird feed?

b. A second hummingbird feeds 2 meters farther away than the first hummingbird and flies twice as high. Write a function to model the path of the second hummingbird.

32. **HOW DO YOU SEE IT?** Determine the real zeros of each function. Then describe the transformation of the graph of f that results in the graph of g.



33. **MATHEMATICAL CONNECTIONS** Write a function V for the volume (in cubic yards) of the right circular cone shown. Then write a function W that gives the volume (in cubic yards) of the cone when x is measured in feet. Find and interpret $W(3)$.



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the minimum value or maximum value of the function. Describe the domain and range of the function, and where the function is increasing and decreasing. *(Section 2.2)*

34. $h(x) = (x + 5)^2 - 7$

35. $f(x) = 4 - x^2$

36. $f(x) = 3(x - 10)(x + 4)$

37. $g(x) = -(x + 2)(x + 8)$

38. $h(x) = \frac{1}{2}(x - 1)^2 - 3$

39. $f(x) = -2x^2 + 4x - 1$