4.3 Dividing Polynomials

Essential Question

How can you use the factors of a cubic polynomial to solve a division problem involving the polynomial?

EXPLORATION 1 Dividing Polynomials

Work with a partner. Match each division statement with the graph of the related cubic polynomial $f(x)$. Explain your reasoning. Use a graphing calculator to verify your answers.

a. $\frac{f(x)}{x} = (x - 1)(x + 2)$

b. $\frac{f(x)}{x - 1} = (x - 1)(x + 2)$

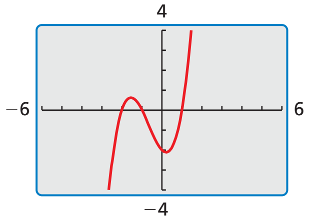
c. $\frac{f(x)}{x + 1} = (x - 1)(x + 2)$

d. $\frac{f(x)}{x - 2} = (x - 1)(x + 2)$

e. $\frac{f(x)}{x + 2} = (x - 1)(x + 2)$

f. $\frac{f(x)}{x - 3} = (x - 1)(x + 2)$

A.



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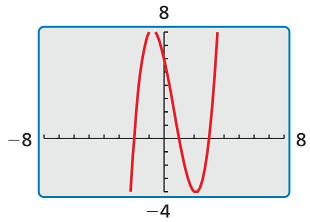

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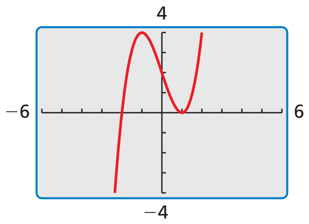

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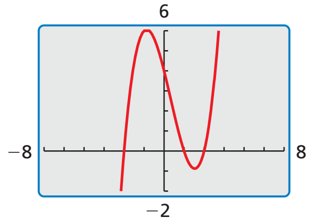

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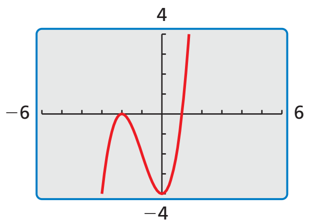
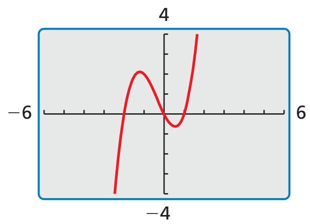


F.



REASONING ABSTRACTLY

To be proficient in math, you need to understand a situation abstractly and represent it symbolically.

EXPLORATION 2 Dividing Polynomials

Work with a partner. Use the results of Exploration 1 to find each quotient. Write your answers in standard form. Check your answers by multiplying.

a. $(x^3 + x^2 - 2x) \div x$

b. $(x^3 - 3x + 2) \div (x - 1)$

c. $(x^3 + 2x^2 - x - 2) \div (x + 1)$

d. $(x^3 - x^2 - 4x + 4) \div (x - 2)$

e. $(x^3 + 3x^2 - 4) \div (x + 2)$

f. $(x^3 - 2x^2 - 5x + 6) \div (x - 3)$

Communicate Your Answer

3. How can you use the factors of a cubic polynomial to solve a division problem involving the polynomial?

4.3 Lesson

Core Vocabulary

polynomial long division, *p. 174*
synthetic division, *p. 175*

Previous
long division
divisor
quotient
remainder
dividend

What You Will Learn

- Use long division to divide polynomials by other polynomials.
- Use synthetic division to divide polynomials by binomials of the form $x - k$.
- Use the Remainder Theorem.

Long Division of Polynomials

When you divide a polynomial $f(x)$ by a nonzero polynomial divisor $d(x)$, you get a quotient polynomial $q(x)$ and a remainder polynomial $r(x)$.

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

The degree of the remainder must be less than the degree of the divisor. When the remainder is 0, the divisor *divides evenly* into the dividend. Also, the degree of the divisor is less than or equal to the degree of the dividend $f(x)$. One way to divide polynomials is called **polynomial long division**.

EXAMPLE 1 Using Polynomial Long Division

Divide $2x^4 + 3x^3 + 5x - 1$ by $x^2 + 3x + 2$.

SOLUTION

Write polynomial division in the same format you use when dividing numbers. Include a "0" as the coefficient of x^2 in the dividend. At each stage, divide the term with the highest power in what is left of the dividend by the first term of the divisor. This gives the next term of the quotient.

$$\begin{array}{r l}
2x^2 - 3x + 5 & \leftarrow \text{quotient} \\
x^2 + 3x + 2 \,\overline{)\, 2x^4 + 3x^3 + 0x^2 + 5x - 1} & \\
\underline{2x^4 + 6x^3 + 4x^2} \qquad\qquad & \text{Multiply divisor by } \frac{2x^4}{x^2} = 2x^2. \\
-3x^3 - 4x^2 + 5x \qquad & \text{Subtract. Bring down next term.} \\
\underline{-3x^3 - 9x^2 - 6x} \qquad & \text{Multiply divisor by } \frac{-3x^3}{x^2} = -3x. \\
5x^2 + 11x - 1 & \text{Subtract. Bring down next term.} \\
\underline{5x^2 + 15x + 10} & \text{Multiply divisor by } \frac{5x^2}{x^2} = 5. \\
-4x - 11 & \leftarrow \text{remainder}
\end{array}$$

COMMON ERROR

The expression added to the quotient in the result of a long division problem is $\frac{r(x)}{d(x)}$, not $r(x)$.

$$\frac{2x^4 + 3x^3 + 5x - 1}{x^2 + 3x + 2} = 2x^2 - 3x + 5 + \frac{-4x - 11}{x^2 + 3x + 2}$$

Check

You can check the result of a division problem by multiplying the quotient by the divisor and adding the remainder. The result should be the dividend.

$$(2x^2 - 3x + 5)(x^2 + 3x + 2) + (-4x - 11)$$
$$= (2x^2)(x^2 + 3x + 2) - (3x)(x^2 + 3x + 2) + (5)(x^2 + 3x + 2) - 4x - 11$$
$$= 2x^4 + 6x^3 + 4x^2 - 3x^3 - 9x^2 - 6x + 5x^2 + 15x + 10 - 4x - 11$$
$$= 2x^4 + 3x^3 + 5x - 1 \checkmark$$

Monitoring Progress

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Divide using polynomial long division.

1. $(x^3 - x^2 - 2x + 8) \div (x - 1)$

2. $(x^4 + 2x^2 - x + 5) \div (x^2 - x + 1)$

Synthetic Division

There is a shortcut for dividing polynomials by binomials of the form $x - k$. This shortcut is called **synthetic division**. This method is shown in the next example.

EXAMPLE 2 Using Synthetic Division

Divide $-x^3 + 4x^2 + 9$ by $x - 3$.

SOLUTION

Step 1 Write the coefficients of the dividend in order of descending exponents. Include a "0" for the missing x-term. Because the divisor is $x - 3$, use $k = 3$. Write the k-value to the left of the vertical bar.

$$\text{k-value} \rightarrow \begin{array}{r|rrrr} 3 & -1 & 4 & 0 & 9 \\ & & & & \\ \hline \end{array} \leftarrow \text{coefficients of } -x^3 + 4x^2 + 9$$

Step 2 Bring down the leading coefficient. Multiply the leading coefficient by the k-value. Write the product under the second coefficient. Add.

$$\begin{array}{r|rrrr} 3 & -1 & 4 & 0 & 9 \\ & & -3 & & \\ \hline & -1 & 1 & & \end{array}$$

Step 3 Multiply the previous sum by the k-value. Write the product under the third coefficient. Add. Repeat this process for the remaining coefficient. The first three numbers in the bottom row are the coefficients of the quotient, and the last number is the remainder.

$$\begin{array}{r|rrrr} 3 & -1 & 4 & 0 & 9 \\ & & -3 & 3 & 9 \\ \hline \text{coefficients of quotient} \rightarrow & -1 & 1 & 3 & 18 \end{array} \leftarrow \text{remainder}$$

$$\frac{-x^3 + 4x^2 + 9}{x - 3} = -x^2 + x + 3 + \frac{18}{x - 3}$$

EXAMPLE 3 Using Synthetic Division

Divide $3x^3 - 2x^2 + 2x - 5$ by $x + 1$.

SOLUTION

STUDY TIP

Note that dividing polynomials does not always result in a polynomial. This means that the set of polynomials is *not* closed under division.

Use synthetic division. Because the divisor is $x + 1 = x - (-1)$, $k = -1$.

$$\begin{array}{r|rrrr} -1 & 3 & -2 & 2 & -5 \\ & & -3 & 5 & -7 \\ \hline & 3 & -5 & 7 & -12 \end{array}$$

$$\frac{3x^3 - 2x^2 + 2x - 5}{x + 1} = 3x^2 - 5x + 7 - \frac{12}{x + 1}$$

Monitoring Progress

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Divide using synthetic division.

3. $(x^3 - 3x^2 - 7x + 6) \div (x - 2)$

4. $(2x^3 - x - 7) \div (x + 3)$

The Remainder Theorem

The remainder in the synthetic division process has an important interpretation. When you divide a polynomial $f(x)$ by $d(x) = x - k$, the result is

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$ Polynomial division

$$\frac{f(x)}{x - k} = q(x) + \frac{r(x)}{x - k}$$ Substitute $x - k$ for $d(x)$.

$$f(x) = (x - k)q(x) + r(x).$$ Multiply both sides by $x - k$.

Because either $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $x - k$, you know that $r(x)$ is a constant function. So, let $r(x) = r$, where r is a real number, and evaluate $f(x)$ when $x = k$.

$$f(k) = (k - k)q(k) + r$$ Substitute k for x and r for $r(x)$.

$$f(k) = r$$ Simplify.

This result is stated in the *Remainder Theorem*.

Core Concept

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. So, to evaluate $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$.

EXAMPLE 4 Evaluating a Polynomial

Use synthetic division to evaluate $f(x) = 5x^3 - x^2 + 13x + 29$ when $x = -4$.

SOLUTION

−4	5	−1	13	29
		−20	84	−388
	5	−21	97	−359

▶ The remainder is −359. So, you can conclude from the Remainder Theorem that $f(-4) = -359$.

Check

Check this by substituting $x = -4$ in the original function.

$$f(-4) = 5(-4)^3 - (-4)^2 + 13(-4) + 29$$
$$= -320 - 16 - 52 + 29$$
$$= -359 \checkmark$$

Monitoring Progress

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Use synthetic division to evaluate the function for the indicated value of x.

5. $f(x) = 4x^2 - 10x - 21; x = 5$

6. $f(x) = 5x^4 + 2x^3 - 20x - 6; x = 2$

4.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Explain the Remainder Theorem in your own words. Use an example in your explanation.

2. **VOCABULARY** What form must the divisor have to make synthetic division an appropriate method for dividing a polynomial? Provide examples to support your claim.

3. **VOCABULARY** Write the polynomial divisor, dividend, and quotient functions represented by the synthetic division shown at the right.

−3	1	−2	−9	18
		−3	15	−18
	1	−5	6	0

4. **WRITING** Explain what the colored numbers represent in the synthetic division in Exercise 3.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, divide using polynomial long division. *(See Example 1.)*

5. $(x^2 + x - 17) \div (x - 4)$

6. $(3x^2 - 14x - 5) \div (x - 5)$

7. $(x^3 + x^2 + x + 2) \div (x^2 - 1)$

8. $(7x^3 + x^2 + x) \div (x^2 + 1)$

9. $(5x^4 - 2x^3 - 7x^2 - 39) \div (x^2 + 2x - 4)$

10. $(4x^4 + 5x - 4) \div (x^2 - 3x - 2)$

In Exercises 11–18, divide using synthetic division. *(See Examples 2 and 3.)*

11. $(x^2 + 8x + 1) \div (x - 4)$

12. $(4x^2 - 13x - 5) \div (x - 2)$

13. $(2x^2 - x + 7) \div (x + 5)$

14. $(x^3 - 4x + 6) \div (x + 3)$

15. $(x^2 + 9) \div (x - 3)$

16. $(3x^3 - 5x^2 - 2) \div (x - 1)$

17. $(x^4 - 5x^3 - 8x^2 + 13x - 12) \div (x - 6)$

18. $(x^4 + 4x^3 + 16x - 35) \div (x + 5)$

ANALYZING RELATIONSHIPS In Exercises 19–22, match the equivalent expressions. Justify your answers.

19. $(x^2 + x - 3) \div (x - 2)$

20. $(x^2 - x - 3) \div (x - 2)$

21. $(x^2 - x + 3) \div (x - 2)$

22. $(x^2 + x + 3) \div (x - 2)$

A. $x + 1 - \dfrac{1}{x - 2}$ **B.** $x + 3 + \dfrac{9}{x - 2}$

C. $x + 1 + \dfrac{5}{x - 2}$ **D.** $x + 3 + \dfrac{3}{x - 2}$

ERROR ANALYSIS In Exercises 23 and 24, describe and correct the error in using synthetic division to divide $x^3 - 5x + 3$ by $x - 2$.

23.

2	1	0	−5	3
		2	4	−2
	1	2	−1	1

$$\frac{x^3 - 5x + 3}{x - 2} = x^3 + 2x^2 - x + 1$$

24.

2	1	−5	3
		2	−6
	1	−3	−3

$$\frac{x^3 - 5x + 3}{x - 2} = x^2 - 3x - \frac{3}{x - 2}$$

In Exercises 25–32, use synthetic division to evaluate the function for the indicated value of *x*. *(See Example 4.)*

25. $f(x) = -x^2 - 8x + 30; x = -1$

26. $f(x) = 3x^2 + 2x - 20; x = 3$

27. $f(x) = x^3 - 2x^2 + 4x + 3; x = 2$

28. $f(x) = x^3 + x^2 - 3x + 9; x = -4$

29. $f(x) = x^3 - 6x + 1; x = 6$

30. $f(x) = x^3 - 9x - 7; x = 10$

31. $f(x) = x^4 + 6x^2 - 7x + 1; x = 3$

32. $f(x) = -x^4 - x^3 - 2; x = 5$

33. **MAKING AN ARGUMENT** You use synthetic division to divide $f(x)$ by $(x - a)$ and find that the remainder equals 15. Your friend concludes that $f(15) = a$. Is your friend correct? Explain your reasoning.

34. **THOUGHT PROVOKING** A polygon has an area represented by $A = 4x^2 + 8x + 4$. The figure has at least one dimension equal to $2x + 2$. Draw the figure and label its dimensions.

35. **USING TOOLS** The total attendance A (in thousands) at NCAA women's basketball games and the number T of NCAA women's basketball teams over a period of time can be modeled by

$$A = -1.95x^3 + 70.1x^2 - 188x + 2150$$
$$T = 14.8x + 725$$

where x is in years and $0 < x < 18$. Write a function for the average attendance per team over this period of time.

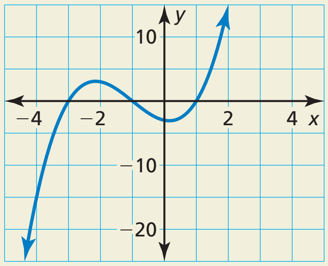
36. **COMPARING METHODS** The profit P (in millions of dollars) for a DVD manufacturer can be modeled by $P = -6x^3 + 72x$, where x is the number (in millions) of DVDs produced. Use synthetic division to show that the company yields a profit of \$96 million when 2 million DVDs are produced. Is there an easier method? Explain.

37. **CRITICAL THINKING** What is the value of k such that $(x^3 - x^2 + kx - 30) \div (x - 5)$ has a remainder of zero?

Ⓐ -14 Ⓑ -2

Ⓒ 26 Ⓓ 32

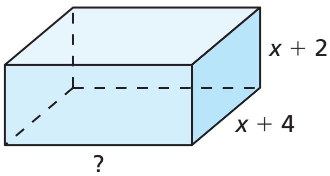
38. **HOW DO YOU SEE IT?** The graph represents the polynomial function $f(x) = x^3 + 3x^2 - x - 3$.



a. The expression $f(x) \div (x - k)$ has a remainder of -15. What is the value of k?

b. Use the graph to compare the remainders of $(x^3 + 3x^2 - x - 3) \div (x + 3)$ and $(x^3 + 3x^2 - x - 3) \div (x + 1)$.

39. **MATHEMATICAL CONNECTIONS** The volume V of the rectangular prism is given by $V = 2x^3 + 17x^2 + 46x + 40$. Find an expression for the missing dimension.



40. **USING STRUCTURE** You divide two polynomials and obtain the result $5x^2 - 13x + 47 - \frac{102}{x + 2}$. What is the dividend? How did you find it?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the zero(s) of the function. *(Sections 3.1 and 3.2)*

41. $f(x) = x^2 - 6x + 9$

42. $g(x) = 3(x + 6)(x - 2)$

43. $g(x) = x^2 + 14x + 49$

44. $h(x) = 4x^2 + 36$